A Remark on the Chern Classes of Local Complete Intersections

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§0. Introduction. For a possibly singular complex algebraic or analytic variety X there are (at least) three kinds of Chern classes available. One is the *Chern-Schwartz-MacPherson class* [3 and 17], denoted $C^{SM}(X)$. This was first constructed by M.-H. Schwartz using radial vector fields, then its existence as a natural transformation of functors was conjectured by P. Deligne and A. Grothendieck and was proved by R. MacPherson. Another is the *Chern-Mather class*, denoted $C^{M}(X)$. This is defined via the Nash blow-up and is, roughly speaking, the Chern class of the limiting tangent bundle of the smooth part of X. The relation between these two classes is another aspect of MacPherson's theory, which expresses $C^{SM}(X)$ in terms of $C^{M}(M)$ and the extra terms supported on the singular locus. This theorem is proved by introducing the *local Euler obstruction*, which also appears in the Dubson-Kashiwara index [4 and 12]. The third is the *canonical class* or *Fulton-Johnson's Chern class* [7 and 8], denoted $C^{FJ}(X)$. This is defined in terms of the Segre class of X and is relatively easy to understand when X is a local complete intersection.

These three classes are identical when the variety has no singularities, thus the differences among them are expected to be expressible in terms of certain invariants of singularities. For a (strong) local complete intersection X with isolated singularities, in [19] is proved a formula expressing $C^{SM}(X)$ in terms of $C^{FJ}(X)$ and the Milnor numbers of the singularities. The purpose of this note is to report an observation that this formula together with other already known formulas implies an interesting and possibly promising formula relating $C^{M}(X)$ and $C^{FJ}(X)$ for such varieties X (see Theorem 3.3 below).

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§1. Three Chern classes. In this section we give a brief review on the above mentioned Chern classes. First, to define the Chern-Mather class, let $\nu : \hat{X} \to X$ be the Nash blow-up of X with $\widehat{TX}$ the Nash tangent bundle over $\hat{X}$. Then the Chern-Mather class is defined by

$$C^{M}(X) := \nu_*(c(\widehat{TX}) \frown [\hat{X}]),$$

where $c(\widehat{TX})$ is the usual total Chern cohomology class of the vector bundle $\widehat{TX}$ and $[\hat{X}]$ is the fundamental class of $\hat{X}$.

Using the Chern-Mather class we can define the Chern-Schwartz-MacPherson class. Let $\mathcal{F}(X)$ be the abelian group of constructible functions on X, which is freely generated by the local Euler obstruction functions Eu_W's of reduced, irreducible subvarieties W's of X. It is proved in [17] that there exists a unique natural transformation $C_* : \mathcal{F} \to H_*(\ ;\boldsymbol{Z})$ satisfying the extra condition that, if X is smooth then $C_*(1_X) = c(TX) \frown [X]$, the Poincaré dual of the total Chern cohomology class of the tangent bundle TX. In fact, C_* is given by $C_*(\sum_W Eu_W) := \sum_W C^{M}(W)$. Then the Chern-Schwartz-MacPherson class of X is defined by:

$$C^{SM}(X) := C_*(1_X).$$

Since we may write $1_X = Eu_X + \sum_S n_S Eu_S$ for uniquely determined subvarieties S of the singular locus of X and integers n_S, we have

$$C^{SM}(X) = C^{M}(X) + \sum_S n_S C^{M}(S).$$

Let X be a local complete intersection in a complex analytic manifold M. Then the normal bundle to the smooth part of X can be extended to a vector bundle N_X over the whole X. More precisely, let $\mathcal{I}_X$ be the ideal sheaf of X in the structure sheaf $\mathcal{O}_M$ of M and $\mathcal{O}_X = \mathcal{O}_M / \mathcal{I}_X$, then the vector bundle N_X is identified with the normal sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$, which is locally free in this case. For such X, we have the virtual tangent bundle $TM|_X - N_X$, whose total Chern cohomology class is given by $c(TM|_X - N_X) = \dfrac{c(TM|_X)}{c(N_X)}$. Then Fulton-Johnson's Chern class in this case is defined by

$$C^{FJ}(X) := c(TM|_X - N_X) \frown [X].$$

Note that these three "singular" Chern classes all become $c(TM) \frown [X]$ when the variety X is non-singular.

§2. Milnor numbers and polar multiplicities. In this section X is assumed to be a complete intersection variety in $\boldsymbol{C}^N$ with an isolated singularity at the origin. We list [10, 11, 14, and 16] as general references for the Milnor number of such a singularity. Let $n = \dim X$ and suppose that the germ $(X, 0)$ is given as the zero set $f^{-1}(0)$ of an analytic map-germ $f = (f_1, \dots, f_{N-n}): (\boldsymbol{C}^N, 0) \to (\boldsymbol{C}^{N-n}, 0)$. Taking a generic set of linear functions $f_{N-n+1}, \dots, f_N$ on $\boldsymbol{C}^N$, we set (cf. [5 and 6])

$$a_j := \dim_{\boldsymbol{C}} (\mathcal{O}_{\boldsymbol{C}^N,0}/(J(f_1, \dots, f_j), f_1, \dots, f_{j-1})), \quad j = 1, \dots, N,$$

where $J(f_1, \dots, f_j)$ is the Jacobian ideal of the map $(f_1, \dots, f_j): (\boldsymbol{C}^N, 0) \to (\boldsymbol{C}^j, 0)$. We also set

$$a_{N+1} := \dim_{\boldsymbol{C}} (\mathcal{O}_{\boldsymbol{C}^N,0}/(f_1, \dots, f_N)),$$

which is equal to the multiplicity $\mathrm{mult}_0(X)$ of X at 0. Then we set

$$\mu^{(i)}(X, 0) := \sum_{j=1}^{N+1-i} (-1)^{N+1-i-j} a_j, \quad i = 0, \dots, n+1.$$

Thus we have

$$a_{N-i+1} = \mu^{(i)}(X, 0) + \mu^{(i+1)}(X, 0), \quad i = 0, \dots, n. \tag{2.1}$$

It is known ([10 and 14]) that, for $i = 1, \dots, n$, $\mu^{(i)}(X, 0)$ is equal to the Milnor number of X intersected with the linear subspace L_i of $\boldsymbol{C}^N$ (of codimension $n - i + 1$) defined by $l_i = (f_{N-n+1}, \dots, f_{N-i+1})$. In particular, $\mu^{(n+1)}(X, 0) = \mu_0(X)$, the Milnor number of X at 0 and $\mu^{(1)}(X, 0) = \mathrm{mult}_0(X) - 1$. Hence, from (2.1), we see that $\mu^{(0)}(X, 0) = 1$.

Next we recall the polar multiplicity. For a generic linear subspace L_i of $\boldsymbol{C}^N$ of codimension $n - i + 1$ as above, the polar variety $P(X, L_i)$ of X with respect to L_i is defined to be the closure of the critical set of the map $l_i|_{X_{reg}}$. Note that $P(X, L_0) = X$ and $P(X, L_n) = \emptyset$. For $i = 0, \dots, n-1$, the i-th polar multiplicity $m_i(X, 0)$ of X at 0 is defined to be the multiplicity of the polar variety $P(X, L_i)$ at the origin 0:

$$m_i(X, 0) := \mathrm{mult}_0 P(X, L_i).$$

Note that $m_i(X, 0)$ is independent of the choice of generic L_i ([20]). In [9] T. Gaffney has proved that

$$m_i(X, 0) = a_{N-i+1}, \quad i = 0, \dots, n-1.$$

Following [9], we define the n-th polar multiplicity of $(X, 0)$ by

$$m_n(X, 0) := a_{N-n+1},$$

which is also independent of the choice of generic L_n. We may summarize the above argument as:

Proposition 2.2. *For an isolated complete intersection singularity $(X, 0)$ of dimension n, we have*

$$m_i(X, 0) = \mu^{(i)}(X, 0) + \mu^{(i+1)}(X, 0), \quad i = 0, \dots, n.$$

It is also shown in [9] that the n-th polar multiplicity $m_n(X, 0)$ is equal to the Jacobian Buchsbaum-Rim multiplicity $e(X, 0)$.

§3. The Formula. Let again X be an n-dimensional local complete intersection in a complex analytic manifold M. Following [13], we say that X is a *strong local complete intersection* if the bundle N_X extends to a (C^∞) vector bundle on a neighborhood of X in M. This class of varieties include the following: (1) non-singular varieties, (2) hypersurfaces, (3) (projective algebraic) complete intersections in the projective space, (4) varieties defined as the zero set of a regular section s of a holomorphic vector bundle on M such that the ideal sheaf is locally generated by the components of s. The following is proved in [19]:

Theorem 3.1. *For an n-dimensional compact strong local complete intersection X with isolated singularities $p_1, \dots, p_r$ in M, the following equality holds:*

$$C^{SM}(X) = C^{FJ}(X) + (-1)^{n+1} \sum_{i=1}^{r} \mu_{p_i}(X),$$

where $\mu_{p_i}(X)$ denotes the Milnor number of X at p_i.

The proof uses the "adjunction formula" in [18], for which the "strongness" is necessary. For hypersurfaces with arbitrary singularities, P. Aluffi [2] has obtained a similar formula, which involves his μ-class introduced in [1].

On the other hand, from Proposition 2.2 and the Lê-Teissier formula [15]:

$$Eu_X(0) = \sum_{i=1}^{n-1} (-1)^i m_i(X, 0),$$

we have the following formula for the local Euler obstruction ([5, Corollary 2.15], see also [6, §4]), which was proved by M. Kashiwara [12] for isolated hypersurface singularities:

Theorem 3.2. *Let $(X, 0) \subset (\boldsymbol{C}^N, 0)$ be an n-dimensional isolated local complete intersection singularity. Then*

$$Eu_X(0) = 1 + (-1)^{n-1}\mu^{(n)}(X, 0).$$

Combining these two theorems we have the following:

Theorem 3.3. *Let X be an n-dimensional compact strong local complete intersection with isolated singularities $p_1, \ldots, p_r$ in a complex analytic manifold M. Then we have*

$$C^M(X) = C^{FJ}(X) + (-1)^{n+1}\sum_{i=1}^{r} m_n(X, p_i)$$

where $m_n(X, p_i)$ is the n-th polar multiplicity of (X, p_i).

Proof. By Theorem 3.2 we have

$$1_X = Eu_X + (-1)^n \sum_{i=1}^{r} \mu^{(n)}(X, p_i) 1_{p_i},$$

where 1_{p_i} is the characteristic function of one point p_i. Hence applying the transformation $C_*: \mathscr{F}(X) \to H_*(X; \boldsymbol{Z})$ of [17] to the above equality, we get

$$C^{SM}(X) = C^M(X) + (-1)^n \sum_{i=1}^{r} \mu^{(n)}(X, p_i).$$

Comparing this with the one in Theorem 3.1 and noting that $\mu_{p_i}(X) = \mu^{(n+1)}(X, p_i)$ and $m_n(X, p_i) = \mu^{(n)}(X, p_i) + \mu^{(n+1)}(X, p_i)$, we get the formula. □

It would be an interesting problem to prove Theorem 3.3 directly and also to extend it to local complete intersections with arbitrary singularities.

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