Generalizations of Coefficient Estimates for Certain Classes of Analytic Functions

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Abstract: Let E_r be the elliptical domain

$$E_r = \left\{(x, y) \in \boldsymbol{R}^2 : \frac{x^2}{\left(1+\frac{1}{r^2}\right)^2} + \frac{y^2}{\left(1-\frac{1}{r^2}\right)^2} < 1\right\}$$

where $r > 1$. Let $S(E_r)$ denote the class of functions $F(z)$ which are analytic and univalent in E_r with $F(0) = 0$ and $F'(0) = 1$. In this paper, we obtain sharp bounds for the Faber coefficients of functions $F(z)$ in certain related classes and subclasses of $S(E_r)$. The case $r \to \infty$ gives standard coefficient estimates for the corresponding classes of functions defined on the unit disc.

1. Introduction. Let S denote the class of functions $f(z)$ which are analytic and univalent in the unit disc $\boldsymbol{D} = \{z : |z| < 1\}$ with the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Coefficient problems for functions in the class S and certain subclasses and related classes of S have been attacked by several authors (see e.g., [1], [2], [5], [6], [10], and [12]). However, very little attention has been paid to the corresponding problems for functions analytic in domains Ω other than $\boldsymbol{D}$. For functions analytic in Ω, it is natural to use the Faber expansion as a generalization of the Taylor expansion.

In [7], we found sharp bounds for the Faber coefficients of certain classes of analytic functions in the elliptical domain

$$E = \left\{(x, y) \in \boldsymbol{R}^2 : \frac{x^2}{(5/4)^2} + \frac{y^2}{(3/4)^2} < 1\right\}.$$

In this paper, we generalize the results of [7] to the elliptical domain

$$E_r = \left\{(x, y) \in \boldsymbol{R}^2 : \frac{x^2}{\left(1+\frac{1}{r^2}\right)^2} + \frac{y^2}{\left(1-\frac{1}{r^2}\right)^2} < 1\right\}$$

where $r > 1$, in which the case $r = 2$ gives the results of [7]. Here it is important that the case $r \to \infty$ yields the classical coefficient estimates for the corresponding classes of functions in $\boldsymbol{D}$. As known for $r \to \infty$, there are infinitely many extremal functions. However, it is interesting that for each $r > 1$ there are two extremal functions in E_r corresponding to the number of invariant rotations of E_r.

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2. Preliminaries. Let Ω be a bounded, simply connected domain in $\boldsymbol{C}$ containing the origin. Let $g(z)$ be the unique, one-to-one and analytic mapping of $\Delta = \{z : |z| > 1\}$ onto $\boldsymbol{C} \setminus \bar{\Omega}$ with

$$g(z) = cz + \sum_{n=0}^{\infty} \frac{c_n}{z^n},\ (c > 0,\ z \in \Delta). \tag{2}$$

Assume that Ω has capacity 1 so that $c = 1$ in (2). The Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with Ω (or $g(z)$) are defined by the generating function relation [6, p.118]

$$\frac{\eta g'(\eta)}{g(\eta) - z} = \sum_{n=0}^{\infty} \Phi_n(z)\eta^{-n}. \tag{3}$$

If $\partial\Omega$ is analytic and $F(z)$ is analytic in IntΩ, then $F(z)$ can be expanded into a series of the form

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z),\ z \in \mathrm{Int}\Omega \tag{4}$$

where

$$A_n = \frac{1}{2\pi i}\int_{|z|=\rho} F(g(z))z^{-n-1}\,dz$$

with $\rho < 1$ and close to 1. In addition, the series in (4), called the *Faber series*, converges uniformly on compact subsets of IntΩ (see e.g., [13, p.42]).

The function

$$g(z) = z + \frac{1}{r^2 z},\ r > 1$$

is the one-to-one, analytic mapping of Δ onto

$\boldsymbol{C}\setminus\overline{\boldsymbol{E}_r}$. One obtains from (3) that the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with $\boldsymbol{E}_r$ are given by

$$\Phi_n(z) = 2^n r^{-n} P_n\left(\frac{rz}{2}\right), \ (n = 0,1,2,\cdots).$$

Here $\{P_n(z)\}_{n=0}^{\infty}$ are the monic Chebyshev polynomials of degree n, which are given by

$$P_n(z) = 2^{-n}\{[z + \sqrt{z^2-1}]^n + [z - \sqrt{z^2-1}]^n\}, \quad (n = 1,2,3,\cdots)$$

and

$$P_0(z) = 1.$$

Let $\mathrm{sn}(z\,;q)$ be the Jacobi elliptic sine function with nome q, and modulus k_0, and let

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k_0^2 t^2}}$$

(see [9, Chap. 2]). Then the function

$$\varphi(z) = \sqrt{k_0}\ \mathrm{sn}\left(\frac{2K}{\pi}\sin^{-1}\frac{rz}{2}\,;\frac{1}{r^4}\right)$$

is the one-to-one, analytic mapping of $\boldsymbol{E}_r$ onto $\boldsymbol{D}$ with $\varphi(0) = 0$ and $\varphi'(0) = \frac{r\sqrt{k_0}\,K}{\pi}$ (see [11, p.296]).

Let $S(\boldsymbol{E}_r)$ denote the class of functions $F(z)$ which are analytic and univalent in $\boldsymbol{E}_r$ and satisfying the conditions $F(0) = 0$ and $F'(0) = 1$. Also, let the class $C(\boldsymbol{E}_r)$ be defined as

$$C(\boldsymbol{E}_r) = \{F(z) \in S(\boldsymbol{E}_r) : F(\boldsymbol{E}_r) \text{ is convex}\}.$$

In addition, let $T(\boldsymbol{E}_r)$ denote the class of functions $F(z)$ analytic in $\boldsymbol{E}_r$, satisfying the conditions $F(0) = 0$ and $F'(0) = 1$ and having real values for $-1-\frac{1}{r^2} < z < 1+\frac{1}{r^2}$ and nonreal values elsewhere. Finally, let $P(\boldsymbol{E}_r)$ denote the class of functions $P(z)$ analytic in $\boldsymbol{E}_r$ with $P(0) = \frac{1}{\varphi'(0)} = \frac{r\sqrt{k_0}\,K}{\pi}$ and $Re\{P(z)\} > 0$. (The condition $F(0) = \frac{1}{\varphi'(0)}$ is imposed for convenience.)

If $f(z) \in S$, then the function $F(z)$ defined by

$$(5) \qquad F(z) = \frac{f(\varphi(z))}{\varphi'(0)}$$

is in $S(\boldsymbol{E}_r)$ and conversly every function $F(z) \in S(\boldsymbol{E}_r)$ has such a representation. In a similar way, if $F(z)$ is in one of the classes $C(\boldsymbol{E}_r)$, $P(\boldsymbol{E}_r)$ or $T(\boldsymbol{E}_r)$ then $F(z)$ may be written as in (5) for some $f(z)$ in the classes of convex functions C, functions with positive real part P or typically real functions T defined for $\boldsymbol{D}$ (see e.g., [6, Chap. 2]), respectively. Because of representation, denote the *Faber coefficients* $\{A_n\}_{n=0}^{\infty}$ of $F(z)$ in the classes defined above by $\{A_n(f)\}_{n=0}^{\infty}$, where $f(z)$ is the corresponding function in $\boldsymbol{D}$, given by (5).

3. Main Results. Let $F(z)$ be analytic in $\boldsymbol{E}_r$ and have the Faber coefficients $\{A_n\}_{n=0}^{\infty}$. Then from the orthogonality of the Chebyshev polynomials we see at once that $\{A_n\}_{n=0}^{\infty}$ are given by the formula

$$A_n = \frac{r^n}{\pi}\int_0^{\pi} F\left(\frac{2\cos\theta}{r}\right)\cos n\theta\, d\theta, \ (n = 0,1,2,\cdots).$$

As a result, if $F(z)$ is in the one of the classes $S(\boldsymbol{E}_r)$, $C(\boldsymbol{E}_r)$, $P(\boldsymbol{E}_r)$ and $T(\boldsymbol{E}_r)$, then the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, of $F(z)$ are given by

$$(6) \quad A_n(f) = \frac{r^{n-1}}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\cos n\theta\, d\theta, \quad (n = 0,1,2,\cdots).$$

Let $\mathscr{F}$ denote one of the sets C, P, and T. Then $\mathscr{F}$ is a compact set. Hence the closed convex hull of $\mathscr{F}$, $\overline{\mathrm{co}}\,\mathscr{F}$, is also compact and since $A_n(f)$ is a continuous linear functional

$$M = \max_{f\in\overline{\mathrm{co}}(\mathscr{F})} |A_n(f)|$$

exists. In addition, we have

$$(7) \qquad \max_{f\in\mathscr{F}} |A_n(f)| = \max_{\mathrm{ext}(\overline{\mathrm{co}}(\mathscr{F}))} |A_n(f)|,$$

where $\mathrm{ext}(\overline{\mathrm{co}}(\mathscr{F}))$ is the set of extreme points of $\overline{\mathrm{co}}(\mathscr{F})$.

The extreme points of $\overline{\mathrm{co}}(C)$ and $\overline{\mathrm{co}}(T)$ are determined in [4] as follows:

$$(8) \quad \mathrm{ext}(\overline{\mathrm{co}}(C)) = \{f : f(z) = c_\theta(z),\ 0 \le \theta < 2\pi\}$$

and

$$(9) \quad \mathrm{ext}(\overline{\mathrm{co}}(T)) = \{f : f(z) = t_\theta(z),\ 0 \le \theta \le \pi\}$$

where $c_\theta(z)$ and $t_\theta(z)$ are given by

$$(10) \qquad c_\theta(z) = \frac{z}{1-e^{i\theta}z},$$

$$(11) \qquad t_\theta(z) = \frac{z}{1-2z\cos\theta + z^2},$$

respectively. The extreme points of $\overline{\mathrm{co}}(P)$ [3] are given by

$$(12) \quad \mathrm{ext}(\overline{\mathrm{co}}(P)) = \{f : f(z) = p_\theta(z),\ 0 \le \theta < 2\pi\}$$

where

$$(13) \qquad p_\theta(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}.$$

Using (7) with (8), (9), and (12) we see that the problem of maximizing $|A_n(f)|$ over the classes C, P, and T reduces to the problem of maximizing the values of $|A_n(c_\theta)|$ $(\theta \in [0, 2\pi))$,

$|A_n(p_\theta)|$ $(\theta \in 0,2\pi))$, and $|A_n(t_\theta)|$ $(\theta \in [0, \pi])$ over θ, respectively.

The method of [7] is used to evaluate the values of $A_n(c_\theta)$, $A_n(t_\theta)$, and $A_n(p_\theta)$, where $A_n(f)$ is given by (6). Since manipulations are the same as in [7], we give only the evaluation of $A_n(c_\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and state the other results without proof.

Theorem 1. *If* $c_\theta(z)$ *is given by* (10), *then*

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta}(e^{in\alpha(\theta)} - r^{-2n}e^{-in\alpha(\theta)}}{2rK^2\sqrt{k_0}(1-r^{-4n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}},$$

$$0 \le \theta \le \frac{\pi}{2}, \ (n = 0,1,2,\cdots)$$

where $0 \le \alpha(\theta) \le \frac{\pi}{2}$ *is given by*

$$\varphi\left[\frac{2}{r}\cos\left(\alpha(\theta)+\frac{\pi\tau}{4}\right)\right] = e^{-i\theta},$$

$$0 \le \theta \le \frac{\pi}{2} \text{ with } \tau = \frac{4i \operatorname{In} r}{\pi}.$$

Proof. The function $\frac{2\cos z}{r}$ maps the rectangle R with vertices at the points $-\frac{\pi\tau}{4}$, $\pi - \frac{\pi\tau}{4}$, $\pi + \frac{\pi\tau}{4}$, and $\frac{\pi\tau}{4}$ onto E_r. Therefore the function $\varphi\left(\frac{2\cos z}{r}\right)$ maps R onto $\boldsymbol{D}$ with

$$\varphi\left[\frac{2}{r}\cos\left(\alpha(t)+\frac{\pi\tau}{4}\right)\right] = e^{-it},\ 0 \le t \le \frac{\pi}{2} \tag{14}$$

where $\alpha(t)$ increases from 0 to $\frac{\pi}{2}$ as tincreases from 0 to $\frac{\pi}{2}$.

Integrate the function $h(z) = c_\theta\left(\varphi\left(\frac{2\cos z}{r}\right)\right)$ e^{inz} over the parallelogram $ABCD$ with vertices at the points $-\pi$, π, $\pi\tau$, and $\pi\tau - 2\pi$, respectively. From (14) we see that $\alpha(\theta) + \frac{\pi\tau}{4}$ is a pole of $h(z)$ inside $ABCD$.

Let

$$iK' = K\tau$$

and refer to $\operatorname{sn}\left(z;\frac{1}{r^4}\right)$ as $\operatorname{sn} z$ for convenience. Then

$$\varphi\left(\frac{2\cos(\pi\tau - z)}{r}\right) = \sqrt{k_0}\operatorname{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2}-\pi\tau+z\right)\right)$$
$$= \sqrt{k_0}\operatorname{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2}+z\right)\right),$$

since $\operatorname{sn} z$ is doubly periodic with periods $2iK'$ and $4K$. Thus

$$\varphi\left(\frac{2\cos z}{r}\right) = \varphi\left(\frac{2\cos(-z)}{r}\right) = \varphi\left(\frac{2\cos(\pi\tau - z)}{r}\right). \tag{15}$$

It follows from (15) that $-\alpha(\theta) + \frac{3\pi\tau}{4}$ is the other pole of $h(z)$ inside $ABCD$. So by the residue theorem,

$$\oint_{ABCD} h(z)\,dz = 2\pi i(\operatorname{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} + \operatorname{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}}) \tag{16}$$

where Res_{z_0} denotes the residue of the function $h(z)$ at the point $z = z_0$.

The contribution of the integrals on BC and DA cancel each other because $h(z)$ is a periodic function with period 2π. Now

$$\int_{AB} h(z)\,dz = \int_{-\pi}^{\pi} h(x)\,dx = 2\int_0^{\pi} c_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nx\,dx \tag{17}$$

and

$$\int_{CD} h(z)\,dz = \int_{2\pi}^{\alpha} h(x+\pi\tau-2\pi)\,dx = -\int_0^{2\pi} h(x+\pi\tau)\,dx.$$

From (15) we obtain

$$\int_{CD} h(z)\,dz = -\int_0^{2\pi} e^{in(x+\pi\tau)} c_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)dx = -2\cdot r^{-4n}\int_0^{\pi} c_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nx\,dx. \tag{18}$$

Then adding (17) and (18) results in

$$\oint_{ABCD} h(z)\,dz = 2(1-r^{-4n})\int_0^{\pi} c_\theta\left(\varphi\left(\frac{2\cos x}{r}\right)\right)\cos nx\,dx. \tag{19}$$

To evaluate $\operatorname{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}}$, expand the function $c_\theta(\sqrt{k_0}\operatorname{sn}(u+u_0))$ about $u = 0$, where

$$u_0 = \frac{2K}{\pi}\left(\frac{\pi}{2} - \alpha(\theta) - \frac{\pi\tau}{4}\right). \tag{20}$$

The addition formula for $\operatorname{sn} u$ [9, p.33] yields

$$\sqrt{k_0}\operatorname{sn}(u+u_0) = \frac{\sqrt{k_0}\operatorname{sn} u \operatorname{cn} u_0 \operatorname{dn} u_0 + \sqrt{k_0}\operatorname{sn} u_0 \operatorname{cn} u \operatorname{dn} u}{1 - k_0^2 \operatorname{sn}^2 u_0 \operatorname{sn}^2 u} \tag{21}$$

where $\operatorname{cn} z$ and $\operatorname{dn} z$ refer to $\operatorname{cn}\left(z;\frac{1}{r^4}\right)$ and $\operatorname{dn}\left(z;\frac{1}{r^4}\right)$, respectively. It follows from (14) that

$$\sqrt{k_0}\operatorname{sn} u_0 = e^{-i\theta}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

To evaluate $\operatorname{cn} u_0$ and $\operatorname{dn} u_0$ employ the identities

(22) $$\operatorname{sn}^2 z + \operatorname{cn}^2 z = 1$$

and

(23) $$k_0^2 \operatorname{sn}^2 z + \operatorname{dn}^2 z = 1$$

[9, p.25]. To determine whether to use $+$ or $-$ sign for $\operatorname{cn} u_0$ and $\operatorname{cn} u_0$ check the signs of $Re\left\{\operatorname{cn}\left(x - \frac{iK'}{2}\right)\right\}$ and $Re\left\{\operatorname{dn}\left(x - \frac{iK'}{2}\right)\right\}$, respectively. Deduce from the addition formulas for $\operatorname{cn} u$ and $\operatorname{dn} u$ [9, p.34] that

$$\operatorname{cn}\left(x - \frac{iK'}{2}\right) = \sqrt{\frac{1+k_0}{k_0}}\,\frac{\operatorname{cn} x + i \operatorname{sn} x \operatorname{dn} x}{1 + k_0 \operatorname{sn}^2 x}$$

and

$$\operatorname{dn}\left(x - \frac{iK'}{2}\right) = \frac{\sqrt{1+k_0}(\operatorname{dn} x + i k_0 \operatorname{sn} x \operatorname{cn} x)}{1 + k_0 \operatorname{sn}^2 x}.$$

Thus $Re\left\{\operatorname{cn}\left(x - \frac{iK'}{2}\right)\right\} \geq 0$ and $Re\left\{\operatorname{dn}\left(x - \frac{iK'}{2}\right)\right\} \geq 0$ for $x \in [0, K]$ since $\operatorname{cn} x$ decreases from 1 to 0 and $\operatorname{dn} x$ decreases from 1 to $\sqrt{1-k_0^2}$ for $x \in [0, K]$. Hence using (22) and (23) we obtain

$$\operatorname{cn} u_0 = \sqrt{1 - \frac{e^{-2i\theta}}{k_0}}$$

and

$$\operatorname{dn} u_0 = \sqrt{1 - k_0 e^{-2i\theta}}.$$

Choosing the principal branch as $-\pi < \arg z \leq \pi$ we obtain

$$0 \leq \arg(\operatorname{cn} u_0) \leq \frac{\pi}{2}$$

and

$$0 \leq \arg(\operatorname{dn} u_0) \leq \frac{\pi}{4}.$$

Therefore

$$0 \leq \arg(\operatorname{cn} u_0 \operatorname{dn} u_0) \leq \frac{3\pi}{4}$$

which implies

(24) $$\sqrt{k_0} \operatorname{cn} u_0 \operatorname{dn} u_0 = ie^{-i\theta}(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}.$$

Using

(25) $$\operatorname{sn} u = u - \frac{1}{3!}(1 + k_0^2)u^3 + \cdots$$

(26) $$\operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \cdots$$

(27) $$\operatorname{dn} u = 1 - \frac{1}{2!}k_0^2 u^2 + \cdots$$

[9, p.37], and (24) in (21) and doing necessary calculations result in

(28) $$\sqrt{k_0} \operatorname{sn}(u + u_0) = e^{-i\theta} + ie^{-i\theta}(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2} u + \cdots.$$

Thus

$$c_\theta(\sqrt{k_0} \operatorname{sn}(u + u_0)) = \frac{ie^{-i\theta}}{(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2} u} + \cdots.$$

or

$$c_\theta\left(\sqrt{k_0} \operatorname{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2} - z\right)\right)\right) = -\frac{\pi i e^{-i\theta}}{2K(1 + k_0^2 - 2k_0 \cos 2\theta^{1/2}\left(z - \alpha(\theta) - \frac{\pi\tau}{4}\right)} + \cdots.$$

Hence we obtain

(29) $$\operatorname{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} = -\frac{\pi i e^{-i\theta} r^{-n} e^{in\alpha(\theta)}}{2K(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}}.$$

In a similar way, residue of $h(z)$ at the point $-\alpha(\theta) + \frac{3\pi\tau}{4}$ may be obtained as

(30) $$\operatorname{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}} = \frac{\pi i e^{-i\theta} r^{-3n} e^{-in\alpha(\theta)}}{2K(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}}.$$

Substituting (29) and (30) into (16) yields

(31) $$\oint_{ABCD} h(z)\,dz = \frac{\pi^2 e^{-i\theta} r^{-n}}{K(1 + k_0^2 - 2k_0 \cos 2\theta)}\left(e^{in\alpha(\theta)} - r^{-2n} e^{-in\alpha(\theta)}\right).$$

Comparing (19) and (31) gives the desired result.

Theorem 2. *If $c_\theta(z)$ is given by* (10), *then*

$$A_n(c_\theta) = \frac{(-1)^n \pi^2 e^{-i\theta}\left(e^{-in\alpha(\pi-\theta)} - r^{-2n} e^{in\alpha(\pi-\theta)}\right)}{2rK^2\sqrt{k_0}(1 - r^{-4n})(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}},$$
$$\frac{\pi}{2} \leq \theta \leq \pi, \ (n = 0,1,2,\cdots),$$

where $\alpha(\theta)$ is as in Theorem 1.

Theorem 3. *If $c_\theta(z)$ is given by* (10), *then*

$$A_n(c_\theta) = \frac{(-1)^n \pi^2 e^{-i\theta}\left(e^{in\alpha(\theta-\pi)} - r^{-2n} e^{-in\alpha(\theta-\pi)}\right)}{2rK^2\sqrt{k_0}(1 - r^{-4n})(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}},$$
$$\pi \leq \theta \leq \frac{3\pi}{2}, \ (n = 0,1,2,\cdots),$$

where $\alpha(\theta)$ is as in Theorem 1.

Theorem 4. *If $c_\theta(z)$ is given by* (10), *then*

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta}\left(e^{-in\alpha(2\pi-\theta)} - r^{-2n} e^{in\alpha(2\pi-\theta)}\right)}{2rK^2\sqrt{k_0}(1 - r^{-4n})(1 + k_0^2 - 2k_0 \cos 2\theta)^{1/2}},$$
$$\frac{3\pi}{2} \leq \theta \leq 2\pi, \ (n = 0,1,2,\cdots),$$

where $\alpha(\theta)$ is as in Theorem 1.

Theorem 5. *If $p_\theta(z)$ is given by* (13), *then*
$$A_n(p_\theta) = 2A_n(c_\theta), \quad 0 \leq \theta < 2\pi, \quad (n = 0,1,2,\cdots).$$

Theorem 6. *If $t_\theta(z)$ is given by* (11), *and $k(z)$ is the Koebe function given by*

$$k(z) = t_0(z) = \frac{z}{(1 - z)^2},$$

then

$$A_n(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}(1-k_0)^2(1-r^{-2n})}, \quad (n = 1,2,\cdots).$$

Theorem 7. *If $t_\theta(z)$ is given by* (11), *then*

$$A_n(t_\pi) = \frac{(-1)^{n-1}\pi^3 n}{4rK^3\sqrt{k_0}(1-k_0)^2(1-r^{-2n})}, \quad (n = 1,2,\cdots).$$

Theorem 8. *If $t_\theta(z)$ is given by* (11), *then*

$$A_n(t_\theta) = \frac{\pi^2 \sin n\alpha(\theta)}{2rK^2\sqrt{k_0}(1-r^{-2n})\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad 0 < \theta \le \frac{\pi}{2}, \ (n = 1,2,\cdots)$$

where $\alpha(\theta)$ is as in Theorem 1.

Theorem 9. *If $t_\theta(z)$ is given by* (11), *then*

$$A_n(t_\theta) = \frac{(-1)^{n-1}\pi^2 \sin n[\alpha(\pi-\theta)]}{2rK^2\sqrt{k_0}(1-r^{-2n})\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad \frac{\pi}{2} \le \theta < \pi, \ (n = 1,2,\cdots)$$

where $\alpha(\theta)$ is as in Theorem 1.

In the following three theorems we obtain sharp bounds for the Faber coefficients of functions in the classes $C(E_r)$, $P(E_r)$ and $T(E_r)$. Here we give only the statements of the results since proofs are similar to the proofs given in [7].

Theorem 10. *If $f \in C$ and $c(z) = c_0(z) = \frac{z}{1-z}$, then for each $r > 1$,*

$$|A_n(f)| \le A_n(c) = \frac{\pi^2}{2rK^2\sqrt{k_0}(1-k_0)(1+r^{-2n})}, \quad (n = 0,1,2,\cdots).$$

Equality occurs only for the functions $f(z) = c(z)$ and $f(z) = -c(-z)$.

Remark 1. *In the extreme case $r \to \infty$, one obtains at once that for $f \in C$, then*

$$\lim_{r\to\infty} |A_n(f)| \le \lim_{r\to\infty} |A_n(c_\theta)| = 1, \quad (n = 0,1,2,\cdots), \ \forall\, \theta \in [0,2\pi)$$

which coincides with the standard coefficient estimate proved by Loewner [10] *in the class C as expected. (For the asymptotic behaviour of k_0 and K as $r \to \infty$, see the infinite product expansion of k_0 and K in terms of nome q given in* [9, *p.*25].)

Theorem 11. *If $f \in P$ and $c(z)$ is as in Theorem* 10, *then for each $r > 1$,*

$$|A_n(f)| \le 2A_n(c), \quad (n = 0,1,2,\cdots).$$

Equality occurs only for the functions $f(z) = p(z)$ and $f(z) = p(-z)$ where $p(z) = \frac{1+z}{1-z}$.

Remark 2. *It follows from Theorem* 10 *and Remark* 1 *that for $f \in P$,*

$$\lim_{r\to\infty} |A_n(f)| \le 2, \quad (n = 0,1,2,\cdots), \ \forall\, \theta \in [0,2\pi)$$

which coincides with the standard coefficient estimate proved by Carathéodory [5].

Theorem 12. *If $f \in T$ and $k(z)$ is the Koebe function, then for each $r > 1$,*

$$|A_n(f)| \le A_n(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}(1-k_0)^2(1-r^{-2n})}, \quad (n = 1,2,\cdots).$$

Equality occurs only for the functions $f(z) = k(z)$ and $f(z) = -k(-z)$.

Remark 3. *The proof for $n = 0$ is given in* [8].

Remark 4. *It follows from Theorem* 12 *that for $f \in T$,*

$$\lim_{r\to\infty} |A_n(f)| \le \lim_{r\to\infty} A_n(k) = n, \quad (n = 0,1,2,\cdots) \ \forall\, \theta \in [0,2\pi)$$

which coincides with the standard coefficient estimate proved by Rogosinski [12].

As a final note we make the following conjecture, whose special case for $r \to \infty$ is the famous Bieberbach Conjecture.

Conjecture. If $f \in S$, then

$$|A_n(f)| \le An(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}(1-k_0)^2(1-r^{-2n})}, \quad (n = 1,2,\cdots), \quad f \in S$$

and

$$|A_0(f)| \le A_0(k), \quad f \in S.$$

Proof of this conjecture for the cases $n = 0,1,2$ is given in [8].

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