On Classification of Elliptic Fibrations with Small Number of Singular Fibres Over a Base of Genus 0 and 1

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Fix an algebraically closed field k of characteristic $p \neq 2,3$. Let $f: X \to C$ be a non-trivial Jacobian elliptic fibration defined over k with a base, a smooth projective curve C. In our consideration we always assume that f is relatively minimal and "Jacobian" means that f has a global section. Recall Shioda's formula as a special case of the Ogg-Shafarevich formula (cf. [6]).

(1) $$r + \rho_2 = 4g(C) - 4 + 2s - s_1$$

where ρ_2 denotes the so-called Lefschetz number (the difference between the 2-nd Betti number b_2 and the Picard number ρ), r is the Mordell-Weil rank, s the number of singular fibres and s_1 denotes the number of semi-stable singular fibres (*i.e.* of type I_n in the Kodaira-Néron classification).

Since $\rho_2 \geq 0$ (Igusa's inequality), so if $C \simeq \boldsymbol{P}^1$ then from (1) it is clear that $s \geq 2$. On the other hand a non-trivial elliptic fibration over any base must have at least one singular fibres, because the moduli space of elliptic curves defined over k is $\boldsymbol{A}^1_k$. It is known also that the case $s = 1$ over an elliptic base is in fact realized. Note one more fact: if $C \simeq \boldsymbol{P}^1$ and f is non-isotrivial then $s \geq 3$. This fact should be thought in a different context and in a more general situation (cf. [3]). From the classification below one obtains another proof of this fact: in other words, one sees that elliptic fibrations over $\boldsymbol{P}^1$ with $s = 2$ are isotrivial.

Theorem 1. In the situation above assume that $K\text{-}S(f) \neq 0$. Then we have:

A. In the case $C \simeq \boldsymbol{P}^1$ and $s \leq 3 : X$ is a rational or $K3$ surface. Furthermore one has the following complete list (for completeness isotrivial fibrations are also included).

1. Rational surfaces ($s = 2$, $r = 0$) :
$X_{22}(II^*, II)$, $X_{33}(III^*, III)$, $X_{44}(IV^*, IV)$, $X_{11}(j)(I_0^*, I_0^*)$ with $j \in k$.

2. Rational surfaces ($s = 3$)
 1) $(r = 0)$: $X_{141}(I_1^*, I_4, I_1)$, $X_{222}(I_2^*, I_2, I_2)$, $X_{431}(IV^*, I_3, I_1)$, $X_{411}(I_4^*, I_1, I_1)$, $X_{321}(III^*, I_2, I_1)$, $X_{211}(II^*, I_1, I_1)$;
 2) $(r = 1)$: $X^2_{321}(I_2^*, III, I_1)$, $X^3_{321}(I_1^*, III, I_2)$, $X^1_{211}(I_1^*, IV, I_1)$, $X^1_{341}(III^*, II, I_1)$, $X^2_{341}(IV^*, III, I_1)$, $X^2_{431}(I_3^*, II, I_1)$, $X^3_{431}(I_1^*, I_3, II)$, $X^1_{442}(IV^*, I_2, II)$;
 3) $(r = 2)$: $X_{444}(IV, IV, IV)$, $X^1_{33}(I_0^*, III, III)$, $X^3_{341}(I_1^*, III, II)$, $X^2_{442}(I_2^*, II, II)$, $X^1_{11}(0)(I_0^*, IV, II)$, $X^1_{444}(IV^*, II, II)$.

3. $K3$ surfaces ($s = 3$) :
$X^*_{411}(I_4^*, I_1^*, I_1^*)$, $X^*_{222}(I_2^*, I_2^*, I_2^*)$, $X^*_{431}(I_3^*, IV^*, I_1^*)$, $X^*_{321}(III^*, I_2^*, I_1^*)$, $X^*_{211}(II^*, I_1^*, I_1^*)$, $X^*_{11}(0)(II^*, IV^*, I_0^*)$, $X^*_{33}(III^*, III^*, I_0^*)$, $X^*_{341*}(III^*, IV^*, I_1^*)$, $X^*_{444}(II^*, II^*, IV)$, $X_{442*}(IV^*, IV^*, I_2^*)$, $X_{444*}(IV^*, IV^*, IV^*)$.

Moreover these surfaces are unique.

B. In the case $C \simeq E$, an elliptic curve, and $s = 1$, the fibration $f: X \to E$ has a unique configuration (I_6^*).

In characteristic zero, formula (1) is sufficient to conclude: $p_g(X) \leq 1$. In the general case it requires involving the so-called function field analog of Szpiro's conjecture which we formulate below.

Theorem ([1, Theorem 3]). Let $f: X \to C$ be a non-isotrivial family of elliptic curves (*i.e.* j-invariant is non-constant) with conductor of degree m. Then

(2) $$\deg(\Delta) \leq 6p^e(2g(C) - 2 + m)$$

where Δ is the discriminant divisor on C and e is the inseparability exponent of the induced j-map: $C \to \boldsymbol{P}^1$.

First of all we remark that isotrivial case

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can be easily treated. Next in view of the well-known theory of Ogg-Shafarevich the condition $p \neq 2,3$ implies that we are dealing with the "tamely" ramified case. So that $m \leq 2s$. Hence in case A: $p_g(X) \leq 1$, and $p_g(X) = 1$ in case B. It remains to classify case A. In the rational subcase, the theory of Mordell-Weil lattices ([7]) is applied. Subcase of $K3$ surfaces is reduced to the above because of the following argument. In this case $m = 6$, so that $s_1 = 0$, *i.e.* all singular fibres are non-semi-stable. Next since $c_2(X) = 24$ and from the Kodaira-Néron classification it can be shown that there exist at least two singular fibres with upper star (in the Kodaira notation). So producing inverse twist transforms at the corresponding critical points one obtains rational surfaces with 3 singular fibres.

It is not difficult to write down normal Weierstrass forms for the surfaces above and one can see that they are in fact unique.

Remarks. 1. Theorem 1 is a generalization of the corresponding situation in characteristic zero ([5]). It turns out that the classification in the general case is the same as in characteristic 0. In other words their normal Weierstrass forms can be obtained from the corresponding in characteristic zero by taking "reduction" $mod\, p$. For the defining equations we refer to [2].

2. $K3$ surfaces with $s = 3$ in characteristic 0 are singular and their binary quadratic forms (or discriminants) of transcendental sublattices are completely determined in [2]. In particular one can see for which p they are supersingular and the density of such p is equal to $1/2$.

3. **Example.** $C \simeq E$, $s = 1$ ([6, §5]). Let Γ be the commutator subgroup of $SL(2, \boldsymbol{Z})$. Then $-1_2 \notin \Gamma \supset \Gamma(6)$ and the corresponding elliptic modular surface has the configuration (I_6^*) over an elliptic base.

Theorem 2. Let $f : X \rightarrow C$ be a non-trivial Jacobian family of elliptic curves in characteristic $p \neq 2,3$. Assume that

1) either $C \simeq \boldsymbol{P}^1$, $s \leq 3$,

2) or $C \simeq E$, $s = 1$.

Then X is pull-back of a surface described in Theorem 1 via a base change of some power of the absolute Frobenius morphism.

First note that one can take a base change, say of degree 12, after which our fibration becomes semi-stable. Obviously this semi-stable reduction base change does not affect to the inseparability degree of f. Furthermore the proof in the semi-stable case is similar to that in [4].

Thus one obtains a class of unirational surfaces from the classification above. Also note that in this classification the action of the absolute Frobenius reduces isotrivial fibrations to isotrivial ones. In particular fibrations over $\boldsymbol{P}^1$ with $s = 2$ are rational.

Remark. The following interesting case over $\boldsymbol{P}^1$ with $s = 4$ can be also treated by means of methods exposed here. We do hope to come back to this situation in a future publication.

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