On the Polynomial Hamiltonian Structure Associated with the $L(1, g+2; g)$ Type

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1. Introduction. In the paper [3], we considered the differential equation

$$\frac{d}{dx}Y = \frac{1}{x}\mathcal{A}(x, t)Y, \quad \mathcal{A}(x, t) := \sum_{k=0}^{g+1}\mathcal{A}_k x^k, \tag{1.1}$$

which satisfies the conditions:

(i) $\mathcal{A}_k$ are 2×2 matrices,

(ii) the eigenvalues of $\mathcal{A}_0$ are distinct up to additive integers,

(iii) the eigenvalues of $\mathcal{A}_{g+1}$ are distinct.

This equation can be reduced to the equation

$$\frac{d^2}{dx^2}y + p_1(x, t)\frac{d}{dx}y + p_2(x, t)y = 0, \tag{1.2}$$

which satisfies the three conditions:

(iv) The Riemann scheme of (1,1) is

$$\left\{\begin{matrix} x=0 & x=\lambda_1 & \cdots & x=\lambda_g & \multicolumn{5}{c}{x=\infty} \\ 0 & 0 & \cdots & 0 & 0\;\; 0\;\; 0 & \cdots & 0 & -\kappa_\infty \\ \kappa_0 & 2 & & 2 & \frac{1}{g+1}\;\; \frac{t_g}{g}\;\; \frac{t_{g-1}}{g-1} & \cdots & t_1 & \kappa_\infty - \kappa_0 + 1 \end{matrix}\right\},$$

(v) κ_0 and κ_∞ are not integer,

(vi) $x = \lambda_k$ $(k = 1, \ldots, g)$ are non-logarithmic singular points.

The equation (1.2) is called $L(1, g+2; g)$ type.

Let μ_k $(k = 1, \ldots, g)$ be the residue of $p_2(x, t)$ at $x = \lambda_k$ and let

$$h_j := \frac{\partial^{g-j}}{\partial x^{g-j}}\left(xp_2 - \sum_{k=1}^{g}\frac{\lambda_k\mu_k}{x-\lambda_k}\right)\Bigg|_{x=0}.$$

By the assumtion (vi), we remark that these h_j are uniquely determined as rational functions in $\lambda_k, \mu_k, t_k (k = 1, \ldots, g)$.

Using these notations λ_k, μ_k and h_k, we state that the holonomic deformation of the linear equation (1.2) is governed by the Hamiltonian system:

$$\frac{\partial\lambda_i}{\partial t_j} = \frac{\partial\tilde{K}_j}{\partial\mu_i}, \quad \frac{\partial\mu_i}{\partial t_j} = -\frac{\partial\tilde{K}_j}{\partial\lambda_i} \quad (i, j = 1, \ldots, g),$$

where the Hamiltonian $\tilde{K}_j$ are

$$\begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \\ \vdots \\ \vdots \\ \tilde{K}_g \end{bmatrix} = \begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & g \end{bmatrix}^{-1} \begin{bmatrix} 1 & & & & 0 \\ t_g & 1 & & & \\ \vdots & t_g & 1 & & \\ t_3 & & \ddots & \ddots & \\ t_2 & t_3 & \cdots & t_g & 1 \end{bmatrix}^{-1} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_g \end{bmatrix}.$$

It is known that, if $g = 1$ the holonomic deformation of (1.2) is governed by the fourth Painlevé equation. So the Hamiltonian system $(\lambda, \mu, \tilde{K}, t)$ is a extension of the fourth Painlevé equation.

Now, we assume that $g < 8$. The purpose of this paper is to transform the Hamiltonian system $(\lambda, \mu, \tilde{K}, t)$ into the Hamiltonian system (q, p, H, ξ) with the conditions:

(C_1) H_j $(j = 1, \ldots, g)$ are polynomial in q_k, p_k $(k = 1, \ldots, g)$,

(C_2) $\dfrac{\partial H_k}{\partial \xi_j} = \dfrac{\partial H_j}{\partial \xi_k}$.

We will state that if $\kappa_\infty = 0$, a special solution of this system is written by the multivalue Hermite function.

By the condition (C_2), we can introduce the function $\tau_{\mathrm{IV}}^{(g)}$ as follows:

$$\frac{\partial}{\partial\xi_k}\log\tau_{\mathrm{IV}}^{(g)} = H_k.$$

In [7], Okamoto defined the τ function associate with the fourth Painlevé transcendental function. This function is equivalent to $\tau_{\mathrm{IV}}^{(1)}$.

2. Polynomial Hamiltonian structure.

Theorem 2.1. *Put σ_k, ρ_k $(k = 1, \ldots, g)$ as follows:*

$\sigma_k =$ the k-th elementary symmetric function of $\lambda_1, \lambda_2, \ldots, \lambda_g$,

$$\rho_k = (-1)^{k-1}\sum_{l=1}^{g}\lambda_l^{g-k}\mu_l\prod_{\substack{j=1\\ j\neq 1}}(\lambda_l - \lambda_j)^{-1}.$$

Then the transformation $(\lambda, \mu, \tilde{K}, t) \to (\sigma, \rho, \tilde{K}, t)$ *is canonical. Moreover, the Hamiltonians* $\tilde{K}_j$ $(j = 1, \dots, g)$ *are polynomial in* σ_k, ρ_k $(k = 1, \dots, g)$.

Remark 1. Changing the variables λ_k, μ_k into the σ_k, ρ_k, the rational functions h_{g+1-j} $(j = 1, \dots, g)$ are expressed as follows:

$$h_{g+1-j} = \sum_{r=1}^{g}\sum_{s=1}^{g} E_{r,s}^{(j)}\rho_r\rho_s + \sum_{k=1}^{g} F_k^{(j)}\rho_k + (-1)^{g+j}\sigma_{g-j+1}\cdot\kappa_\infty,$$

$$E_{r,s}^{(j)} = (-1)^j\Big[\sum_{\alpha=0}^{g-j}\sigma_\alpha\sigma_{r+s-j-\alpha} - \sum_{\alpha=0}^{s-j}\sigma_\alpha\sigma_{r+s-j-\alpha} - \sum_{\alpha=0}^{r-j}\sigma_\alpha\sigma_{r+s-j-\alpha}\Big],$$

$$F_k^{(j)} = (-1)^{g-j+1}\Big[\sigma_k\sigma_{g+1-j} + \sum_{r=1}^{g}(-1)^{g+r}c_{k,r}^{(j)}t_r\sigma_{k+r-j} - \sigma_{k+g+1-j} + (-1)^{g-1}\{(\kappa_0 - 1) + g + 1 - k\}\sigma_{k-j}\Big],$$

$$c_{k,r}^{(j)} := \begin{cases} +1 & (g+1 \le k + r \le g + j \text{ and } r \ge j) \\ -1 & (j \le k + r \le g \text{ and } r \le j - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

To state the following theorems, we prepare some notations. Let Δ_d be the set of all partitions of a natural number d, and $Y = (Y_1, Y_2, \dots, Y_r)$ be an element of Δ_d. $m_0(Y)$ stand for the depth of Y. For $t = 1, \dots, g$, $m_l(Y)$ stands for the multiplicity of Y i.e.

$$m_l(Y) = \mathrm{Card}\ \{i \in \boldsymbol{Z} \mid Y_i = l\}.$$

We define the polynomial ξ_k $(k = 1, \dots, g)$ as follows:

$$\xi_{g+1-d} := \sum_{Y\in\Delta_d}\Big[\prod_{k=1}^{m_0(Y)-1}(-1)^g\{d - k(g+1)\}\Big]\Big\{\prod_{k=1}^{d}\frac{1}{m_k(Y)!}s_{g+1-k}{}^{m_k(Y)}\Big\},$$

where $s_k = (-1)^{k-1}t_k/k$.

Remark 2. The following relation holds for $g < 8$.

$$t_{g+1-d} = \frac{(-1)^d}{d}\cdot\det M(g + 1 - d\,;\ \xi_g, \xi_{g-1}, \dots, \xi_{g+1-d})\ (d = 1, 2, \dots, g),$$

where

$$D_d := \begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & 1 & 0 \end{bmatrix},$$

$$M(c\,; x_1, \dots, x_d) := -\operatorname{diag}(1, 2, \dots, d)\cdot{}^tD_d + (-1)^g\cdot c\cdot\sum_{j=1}^{d} j\cdot x_j\cdot D_d{}^{j-1}$$

Using the above notations, the transformation $(\sigma, \rho, \tilde{K}, t) \to (q, p, H, \xi)$ is given by the following theorem.

Theorem 2.2. *We assume that* $g < 8$. *Put* q_k, p_k *and* H_k $(k = 1, \dots, g)$ *as follows:*

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_g \end{bmatrix} = \begin{bmatrix} \psi_{1,1} & & & 0 \\ \psi_{2,1} & \psi_{2,2} & & \\ \vdots & & \ddots & \\ \psi_{g,1} & \cdots & \psi_{g,g-1} & \psi_{g,g} \end{bmatrix}\begin{bmatrix} \sigma_1 - \varphi_{1,0} \\ \sigma_2 - \varphi_{2,0} \\ \vdots \\ \sigma_g - \varphi_{g,0} \end{bmatrix},$$

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_g \end{bmatrix} = {}^t\begin{bmatrix} \varphi_{1,1} & & & 0 \\ \varphi_{2,1} & \varphi_{2,2} & & \\ \vdots & & \ddots & \\ \varphi_{g,1} & \cdots & \varphi_{g,g-1} & \varphi_{g,g} \end{bmatrix}\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_g \end{bmatrix},$$

$$\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_g \end{bmatrix} = \begin{bmatrix} \psi_{1,1} & & & 0 \\ \psi_{2,1} & \psi_{2,2} & & \\ \vdots & & \ddots & \\ \psi_{g,1} & \cdots & \psi_{g,g-1} & \psi_{g,g} \end{bmatrix}\begin{bmatrix} \hat{h}_1 - \kappa_\infty\varphi_{1,0} \\ \hat{h}_2 - \kappa_\infty\varphi_{2,0} \\ \vdots \\ \hat{h}_g - \kappa_\infty\varphi_{g,0} \end{bmatrix},$$

where $\varphi_{k,k-d}$, $\psi_{k,k-d}$ *and* $\hat{h}_j$ *are*

$$\varphi_{k,k} = \psi_{k,k} = 1,$$

$$\varphi_{k,k-d} = \frac{1}{d!}\cdot\det M(g + 1 - k\,;\ \xi_g, \xi_{g-1}, \dots, \xi_{g+1-d}),$$

$$\varphi_{k,k-d} = \frac{1}{d!}\cdot\frac{g+1-k}{g+1-k+d}\cdot\det M(-g - 1 + k - d\,;\ \xi_g, \xi_{g-1}, \dots, \xi_{g+1-d})\quad (d = 1, \dots, k-1),$$

$$\hat{h}_j := (-1)^{j-1}\Big\{h_j + (-1)^{g-j}\sum_{k=1}^{g}(g + 1 - k)\sigma_{k-g-1+j}\cdot\rho_k\Big\}.$$

Then the Hamiltonian system:

$$\frac{\partial\lambda_i}{\partial t_j} = \frac{\partial\tilde{K}_j}{\partial\mu_i},\ \frac{\partial\mu_i}{\partial t_j} = -\frac{\partial\tilde{K}_j}{\partial\lambda_i}\ (i, j = 1, \dots, g),$$

is equivalent to the Hamiltonian system

$$\frac{\partial q_i}{\partial\xi_j} = \frac{\partial H_j}{\partial p_i},\ \frac{\partial p_i}{\partial\xi_j} = -\frac{\partial H_j}{\partial p_i}\ (i, j = 1, \dots, g).$$

Theorem 2.3. *The Hamiltonian system* (q, p, H, ξ) *satisfies the properties* (C_1) and (C_2).

Remark 3. The functions $\varphi_{k,k-d}$ and $\psi_{k,k-d}$ are rewrite as follows:

$$\varphi_{k,k-d} = \sum_{Y\in\Delta_d}\{(-1)^g(g + 1 - k)\}^{m_0(Y)}\prod_{k=1}^{d}\frac{1}{m_k(Y)!}s_{g+1-k}{}^{m_k(Y)},$$

$$\psi_{k,k-d} = \sum_{Y\in\Delta_d}\{(-1)^{g-1}(g + 1 - k + d)\}^{m_0(Y)-1}\prod_{k=1}^{d}\frac{1}{m_k(Y)!}s_{g+1-k}{}^{m_k(Y)}.$$

Remark 4. We assume that $g < 8$. The functions $\varphi_{k,k-d}$ and $\psi_{k,k-d}$ have the following relation:

$$\begin{bmatrix} \varphi_{0,0} & & & 0 \\ \varphi_{1,0} & \varphi_{1,1} & & \\ \vdots & & \ddots & \\ \varphi_{g,1} & \cdots & \varphi_{g,g-1} & \varphi_{g,g} \end{bmatrix} \cdot \begin{bmatrix} \psi_{0,0} & & & 0 \\ \psi_{1,0} & \psi_{1,1} & & \\ \vdots & & \ddots & \\ \psi_{g,1} & \cdots & \psi_{g,g-1} & \psi_{g,g} \end{bmatrix} = I.$$

Remark 5. If $g = 1$, the Hamiltonian system is equivalent to the fourth Painlevé equation.

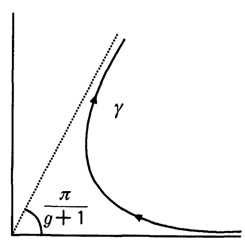
It is known that the fourth Painlevé equation has a special solution which is written by Hermite function. This fact is a special case of the following theorem.

Theorem 2.4. *Under the assumption* $\kappa_\infty = 0$, *the Hamiltonian system* (q, p, H, ξ) *has a special solution*:

$$p_k = 0,$$
$$q_k = \frac{\partial}{\partial \xi_k} \log u \quad (k = 1, \ldots, g),$$

where

$$u(t_1, \ldots, t_g) = \int_\gamma z^{-\kappa 0} \exp\left(- \sum_{k=1}^{g} \frac{t_k}{k} z^k - \frac{1}{g+1} z^{g+1} \right) dz.$$



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