On the Holonomic Deformation of Linear Differential Equations

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1. Introduction. Consider a linear differential equation:

$$\frac{d}{dx}Y = M(x, t)Y, \tag{1.1}$$

where $M(x, t)$ is an $m \times m$ matrix whose entries are rational functions of x, and depend on $t \in U \subset \mathbf{C}^g$ analytically. We call the following problem as extended Fuchs problem. "Give a condition under which there exist a solution whose monodromy groups and Stokes multipliers are independent of t".

When the differential equation (1.1) is of the form:

$$\frac{d}{dx}Y = \mathscr{A}(x, t)Y,$$

$$\mathscr{A}(x, t) = \sum_{j=1}^{n}\sum_{k=0}^{r_j}\frac{\mathscr{A}_{j,-k}}{(x - a_j)^{k+1}} - \sum_{k=1}^{r_\infty}\mathscr{A}_{\infty,-k}x^{k-1},$$

the Fuchs problem was studied by Jimbo-Miwa-Ueno [4 and 6]. They show that a solution of this problem is given by a nonlinear differential equation with the Painlevé property. This nonlinear differential equation is called the monodromy preserving deformation equation, called in short MPD equation.

It is known by [7, 8, and 9] that the Garnier system and the Painlevé equations are special cases of MPD equation, and that each of these equations is described as a polynomial Hamiltonian systems. By the use of these results, the contiguity relations of Painlevé equations are given by [10, 11, 12, and 13].

In this paper, we consider the Fuchs problem for the linear differential equation:

$$\frac{d}{dx}Y = \frac{1}{x}\mathscr{A}(x, t)Y, \quad \mathscr{A}(x, t) := \sum_{k=0}^{g+1}\mathscr{A}_k x^k, \tag{1.2}$$

with following assumptions

(i) $\mathscr{A}_k$ are 2×2 matrices,

(ii) the eigenvalues of $\mathscr{A}_0$ are distinct up to additive integers,

(iii) the eigenvalues of $\mathscr{A}_{g+1}$ are distinct.

We show in what follows that the MPD equation is written as a Hamiltonian system. Notice that if $g = 1$, the MPD equation is equivalent to the fourth Painlevé equation, and that if $g = 2$, the MPD equation is equivalent to the nonlinear differential equation given in [5].

2. Holonomic deformation. Theorem 2.1. *Changing suitably the variables, we can transform (1.2) to the linear differential equation:*

$$\frac{d}{dx}Y = \frac{1}{x}\sum_{k=0}^{g+1}\bar{\mathscr{A}}_k Y, \tag{2.1}$$

which satisfies the following conditions:

- $\bar{\mathscr{A}}_{g+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, • $\det \bar{\mathscr{A}}_0 = 0$,
- $\deg \sum_{k=0}^{g+1}\bar{\mathscr{A}}_k x^k \le g + 1$, • *The* (1,1) *component of* $\bar{\mathscr{A}}_g$ *is* 0.

Theorem 2.2. *The differential equation* (2.1) *is equivalent to the following equation:*

$$\frac{d^2}{dx^2}y + p_1(x, t)\frac{d}{dx}y + p_2(x, t)y = 0 \tag{2.2}$$

$$p_1(x, t) = \frac{1 - \kappa_0}{x} - \sum_{k=1}^{g}t_k x^{k-1} - x^g - \sum_{k=1}^{g}\frac{1}{x - \lambda_k},$$

$$p_2(x, t) = -\frac{1}{x}\sum_{k=1}^{g}h_{g+1-k}\cdot x^{k-1} + \kappa_\infty x^{g-1} + \sum_{k=1}^{g}\frac{\lambda_k\mu_k}{x(x - \lambda_k)},$$

where h_k $(k = 1, \dots, g)$ *are*

$$h_j = (-1)^{j-1}\sum_{l=1}^{g}\frac{1}{\Lambda'(\lambda_l)}\Big[\lambda_l\sigma_{l,j-1}\mu_l^2 - \sigma_{l,j-1}(\lambda_l^{g+1} + \sum_{k=1}^{g}t_k\lambda_l^k + \kappa_0)\mu_l + \kappa_\infty\lambda_l^g\sigma_{l,j-1}\Big] - \sum_{l=1}^{g}\sum_{k=0}^{j-2}(-1)^k\sigma_{l,k}\lambda_l^{j-1-k}\frac{\mu_l}{\Lambda'(\lambda_l)},$$

$$\Lambda(\lambda_k) = \prod_{\substack{j=1 \\ j\neq k}}^{g}(\lambda_k - \lambda_j),$$

$$\sigma_{k,j} = \frac{1}{j!}\frac{d^j}{dx^j}\prod_{\substack{i=1 \\ i\neq k}}^{g}(1 + \lambda_i x)\Big|_{x=0}.$$

The number of accessory parameters (2.2) is $2g$. (2.2) has singular points at $x = 0$ and $x =$

∞. The Poincaré rank at these singular points are 0 and $g+1$, respectively. Therefore (2.2) is called $L(1, g+2; g)$ type.

Remark 1. By the assumptions (ii) and (iii), κ_0 and κ_∞ are not integer.

Remark 2. (2.2) has non-logarithmic singular points at $x=\lambda_k$ $(k=1,\dots,g)$ besides the singularity of (2.1).

Remark 3. Let $f_{i,j}(x)$ denotes the (i,j) component of matrix $\mathcal{A}(x,t)$. Then λ_k are zeros of $f_{1,2}(x)$, and μ_k are given as follows:

$$\mu_k = f_{1,1}(\lambda_k)/\lambda_k.$$

As for the holonomic deformation of linear equation's the following result is known:

Proposition 2.1 ([4,7]). *A linear differential equation*

$$\text{(2.3)}\quad \frac{\partial^2}{\partial x^2}y + p_1(x,t)\frac{\partial}{\partial x}y + p_2(x,t)y = 0,$$

has a fundamental system of solutions whose monodromy is independent of $t \in U \subset \mathbf{C}^g$, if and only if there exist rational functions of x, $A_j(x,t)$, and $B_j(x,t)$ $(j=1,\dots,g)$ such that the extended system of the differential equations

$$\text{(2.4)}\quad \frac{\partial^2}{\partial x^2}y + p_1(x,t)\frac{\partial}{\partial x}y + p_2(x,t)y = 0,$$

(2.5)

$$\frac{\partial}{\partial t_j}y = A_j(x,t)\frac{\partial}{\partial x}y + B_j(x,t)y \ (j=1,2,\dots,g),$$

is completely integrable.

In the case of equation (2.2), by examing a local behavior of solutions, we can determine the function A_j and B_j as follows:

Theorem 2.3. *The rational functions A_j and B_j $(j=1,\dots,g)$ are*

$$A_j = \frac{1}{j}\prod_{l=1}^{g}(x-\lambda_l)^{-1}\sum_{k=0}^{j-1}\sum_{i=0}^{k}(-1)^i\sigma_i T_{k-i}x^{j-k},$$

$$B_j = \frac{1}{2}\left\{-\frac{\partial A_j}{\partial x} + p_1 A_j + \frac{1}{j}x^j - \sum_{k=0}^{g}\frac{1}{x-\lambda_k}\frac{\partial \tilde{K}_j}{\partial u_k}\right\},$$

where σ_i is the i-th elementary symmetric function of $\lambda_1,\dots,\lambda_g$ and T_j, $\tilde{K}_j$ are given as follows:

$$\begin{bmatrix} T_0 & & & & \\ T_1 & T_0 & & & \\ T_2 & T_1 & T_0 & & \\ \vdots & & \ddots & \ddots & \\ T_g & T_{g-1} & \cdots & T_1 & T_0 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ t_g & 1 & & & \\ t_{g-1} & t_g & 1 & & \\ \vdots & & \ddots & \ddots & \\ t_1 & t_2 & \cdots & t_g & 1 \end{bmatrix}^{-1},$$

$$\begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \\ \vdots \\ \vdots \\ \tilde{K}_g \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & g \end{bmatrix}^{-1} \begin{bmatrix} 1 & & & & \\ t_g & 1 & & & \\ \vdots & t_g & 1 & & \\ t_3 & & \ddots & \ddots & \\ t_2 & t_3 & \cdots & t_g & 1 \end{bmatrix}^{-1} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_g \end{bmatrix}.$$

In general, under the assumption that there exist rational functions $A_j(x,t)$ and $B_j(x,t)$ which satisfies the condition of Theorem 2.1 and (2.3) is said to admit the holonomic deformation.

Considering the compatibility condition of (2.4) and (2.5), we obtain the following theorem.

Main Theorem 1. *The linear differential equation* (2.2) *admits a holonomic deformation, if and only if λ_j and μ_j satisfy the following Hamiltonian system*

$$\text{(2.6)}\quad \frac{\partial \lambda_i}{\partial t_j} = \frac{\partial \tilde{K}_j}{\partial \mu_i},\ \frac{\partial \mu_i}{\partial t_j} = -\frac{\partial \tilde{K}_j}{\partial \lambda_i}\ (i,j=1,\dots,g),$$

where the Hamiltonians $\tilde{K}_j$ are the rational functions of λ_k and μ_k which are given in Theorem 2.2.

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