Dirichlet Problem at Infinity for Harmonic Maps between Carnot Spaces

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If M and M' are simply connected complete Riemannian manifolds of negative curvature, one can compactify them by adding the spheres at infinity ∂M and $\partial M'$, defined by the asymptotic classes of geodesic rays in M and M', respectively. Given a continuous map $f: \partial M \to \partial M'$, the Dirichlet problem at infinity consists, roughly speaking, of finding a harmonic map $u: M \to M'$ which assumes the boundary value f continuously.

The first progress toward this problem was established by Li and Tam [6], [7], [8], Gromov [9] and Akutagawa [1] around 1990, in the case that M and M' both are real hyperbolic spaces. In particular, Li and Tam proved a number of significant results concerning uniqueness, existence and boundary regularity of solutions. Subsequently, in 1993 Donnelly [3] extended their results to the context of rank one symmetric spaces of noncompact type, namely to complex or quaternion hyperbolic spaces and the Cayley plane. Recently, Ueno [12] proved that these can be further extended to the case of Damek-Ricci spaces, which are a generalization of rank one symmetric spaces of noncompact type ([2]).

We study the problem for more general family of homogeneous spaces of negative curvature, that is, in the context of k-term Carnot spaces arising as a semidirect solvable extension of the k-step nilpotent Lie groups called Carnot groups ([13]).

More precisely, let G be a simply connected solvable Lie group satisfying the following conditions:

1. G is diffeomorphic to the product $\boldsymbol{R}_+ \times N$ of the positive real line $\boldsymbol{R}_+$ with a nilpotent Lie group N.
2. If $\mathfrak{n}$ and $\mathfrak{g} = \boldsymbol{R}\{H\} + \mathfrak{n}$ denote the Lie algebras of N and G respectively, then $\mathfrak{n}$ has a graded Lie algebra structure $\mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i$ given, for $i = 1, \dots, k$, by
$$\mathfrak{n}_i = \{X \in \mathfrak{n} \mid ad(H)X = iX\}.$$

Then G admits a left invariant metric g of negative curvature ([5]), and we call $M = (G, g)$ *a k-term Carnot space.* For example, real hyperbolic spaces are 1-term Carnot spaces, and complex or quaternion hyperbolic spaces and the Cayley plane are 2-term Carnot spaces.

We see that the realization of M as $\boldsymbol{R}_+ \times N$ defines a local coordinate chart at the boundary ∂M of the compactification $\overline{M} = M \cup \partial M$, and in this chart, the metric g is realized as a k-ply warped product metric
$$g = \frac{dy^2}{y^2} + \frac{1}{y^2} g_{\mathfrak{n}_1} + \frac{1}{y^4} g_{\mathfrak{n}_2} + \cdots + \frac{1}{y^{2k}} g_{\mathfrak{n}_k},$$
where y denotes the coordinate on $\boldsymbol{R}_+$. Denoting by ∞ the point at infinity corresponding to the geodesics in the $\boldsymbol{R}_+$ direction, $\partial M \setminus \{\infty\}$ can be identified with $\{0\} \times N$. Moreover, each subspace $\mathfrak{n}_i$ of the Lie algebra $\mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i$ of N defines by left translations a distribution on ∂M, which we denote also by $\mathfrak{n}_i$.

Let M and M' be k-term Carnot spaces and $u \in C^\infty(M, M') \cap C^1(\overline{M}, \overline{M'})$ a proper smooth map from M to M' which extends up to the boundary as a C^1 map. Set $f = u|\partial M$. We say that u or f is *nondegenerate* if, in the coordinate charts $\boldsymbol{R}_+ \times N$ and $\boldsymbol{R}_+ \times N'$ at the boundaries, f satisfies
$$df_p((\mathfrak{n}_k)_p) \not\subset \sum_{j=1}^{k-1} (\mathfrak{n}'_j)_{f(p)}$$
at any $p \in N$.

Our first observation is that the boundary value f of a nondegenerate proper harmonic map $u \in C^\infty(M, M') \cap C^k(\overline{M}, \overline{M'})$ preserves the filtrations on the boundaries defined by distributions $\mathfrak{n}_i$ and $\mathfrak{n}_i$.

Theorem 1. *Let $u \in C^\infty(M, M') \cap C^k(\overline{M}, \overline{M'})$ be a nondegenerate proper harmonic map between k-term Carnot spaces M and M'. Then the boundary value $f = u|\partial M$ of u satisfies for each $1 \le i \le k$*

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$$(1) \qquad df_p\left(\sum_{j=1}^{i}(\mathfrak{n}_j)_p\right) \subset \sum_{j=1}^{i}(\mathfrak{n}_j')_{f(p)}$$

at any $p \in N$.

In fact, under the assumption of Theorem 1, one can deduce the asymptotic behavior of higher order derivatives, in the $\boldsymbol{R}_+$ direction, of a nondegenerate proper harmonic map f near the boundary ∂M. In particular, we observe

Theorem 2. *Let* $u \in C^\infty(M, M') \cap C^k(\overline{M}, \overline{M'})$ *be a nondegenerate proper harmonic map between k-term Carnot spaces* M *and* M'. *Then the asymptotic behavior in the* $\boldsymbol{R}_+$ *direction of* u *near the boundary* ∂M *is determined by the boundary value* $f = u|\partial M$ *of* u.

This leads, for instance, to the following uniqueness result of the Dirichlet problem at infinity for harmonic maps.

Theorem 3. *Let* $u, v \in C^\infty(M, M') \cap C^k(\overline{M}, \overline{M'})$ *be nondegenerate proper harmonic maps between k-term Carnot spaces* M *and* M'. *If* u *and* v *agree on* ∂M, *then* $u = v$ *on* M *everywhere.*

For a given nondegenerate boundary data f satisfying the necessary condition (1), Theorem 2 also enables one to construct an asymptotically harmonic map assuming f on the boundary. Then, by applying the parabolic harmonic map equation to deform these approximate solutions to desired harmonic maps ([6]), we obtain the following existence result.

Theorem 4. *Let* M *and* M' *be k-term Carnot spaces and* $f \in C^{k,\epsilon}(\partial M, \partial M')$, $0 < \epsilon < 1$, *a nondegenerate map satisfying* (1). *Then there exists a harmonic map* $u \in C^\infty(M, M') \cap C^0(\overline{M}, \overline{M'})$ *which assumes the boundary value* f *continuously.*

Let $\boldsymbol{CH}^m$ denote the m-dimensional complex hyperbolic space of holomorphic sectional curvature -1, that is, the unit ball in $\boldsymbol{C}^m$ with its Bergman metric. Then, for $m \geq 2$, $\boldsymbol{CH}^m$ is a 2-term Carnot space and the Lie algebra $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ of the nilpotent part of the realization $\boldsymbol{R}_+ \times N$ of $\boldsymbol{CH}^m$ is a Heisenberg algebra. In this case, the sphere at infinity of $\boldsymbol{CH}^m$ is identified with the unit sphere S^{2m-1}, and the necessary condition (1) for the boundary value f of a proper harmonic self-map u of $\boldsymbol{CH}^m$ means that $f: S^{2m-1} \to S^{2m-1}$ is a contact transformation.

It has been known that if, in particular, $u: \boldsymbol{CH}^m \to \boldsymbol{CH}^m$ is a proper holomorphic map, then u extends smoothly up to the boundary and $f = u|S^{2m-1}$ is a CR map ([4]). Conversely, we can prove the following.

Theorem 5. *Let* $M = \boldsymbol{CH}^m$, $M' = \boldsymbol{CH}^{m'}$ *be the complex hyperbolic spaces of dimension* $m, m' \geq 2$, *respectively. Let* $u \in C^\infty(M, M') \cap C^4(\overline{M}, \overline{M'})$ *be a nondegenerate proper harmonic map and* $f = u|S^{2m-1}$. *If* $f: S^{2m-1} \to S^{2m'-1}$ *is a CR map, then* u *is a holomorphic map.*

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