Limiting Profiles of Blow-up Solutions of the Nonlinear Schrödinger Equation with Critical Power Nonlinearity

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1. Introduction and results. This paper concerns the following Cauchy problem for the nonlinear Schrödinger equation (NSC):

$$\begin{cases} \text{(NSC)}\ 2i\dfrac{\partial u}{\partial t} + \Delta u + |u|^{4/N}u = 0, & (t, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^N, \\ \text{(IV)}\ u(0, x) = u_0(x), & x \in \boldsymbol{R}^N. \end{cases}$$

Here $i = \sqrt{-1}$, and Δ is the Laplace operator on $\boldsymbol{R}^N$.

The author reviews his recent results on the asymptotic behavior of blow-up solutions of (NSC)-(IV) investigated in the series of papers [9], [10], and [11](see also [6], [7], and [8]). So, the references of this paper are not intended to be complete. For further references, see those cited in [9], [10], and [11].

We summarize here the basic properties of this Cauchy problem (NSC)-(IV) (see, *e.g.*,[3]). The unique local existence of solutions is well known: for any $u_0 \in H^1(\boldsymbol{R}^N)$, there exists a unique solution $u(t, x)$ in $C([0, T_m); H^1(\boldsymbol{R}^N))$ for some $T_m \in (0, \infty]$, (maximal existence time; for simplicity, we shall consider the forward problem only), and $u(t)$ satisfies the following three conservation laws of L^2, the energy E and the momentum P_l $(l = 1, 2, \ldots, N)$ in this order:

$$\text{(1.1)} \qquad \| u(t) \| = \| u_0 \|,$$

$$\text{(1.2)} \qquad E(u(t)) \equiv \| \nabla u(t) \|^2 - \frac{2}{\sigma} \| u(t) \|_\sigma^\sigma = E(u_0),$$

$$\text{(1.3)} \qquad P_l(u(t)) \equiv \Im \int_{\boldsymbol{R}^N} u(t, x) \frac{\partial}{\partial x_l} \overline{u(t, x)}\, dx = P_l(u_0), \quad l = 1, 2, \ldots, N,$$

for $t \in [0, T_m)$, where $\sigma = 2 + \dfrac{4}{N}$, $\|\cdot\|$ and $\|\cdot\|_\sigma$ denote the L^2 norm and the L^σ norm respectively. If, in addition, $|x|u_0 \in L^2(\boldsymbol{R}^N)$, then the solution $u(t)$ also enjoys $|x|u(\cdot) \in C([0, T_m); L^2(\boldsymbol{R}^N))$, and satisfies the following virial identity (see, *e.g.*, [12] and [15]):

$$\text{(1.4)} \qquad \| |x - a| u(t) \|^2 = \| |x - a| u_0 \|^2 + 2t \Im \langle u_0, (x - a) \cdot \nabla u_0 \rangle + t^2 E(u_0),$$

where we have used the notation: $\langle f, g \rangle = \int_{\boldsymbol{R}^N} \overline{f(x)}\, g(x)\, dx$. Furthermore we have the following alternatives: $T_m = \infty$ or $T_m < \infty$ and $\lim_{t \to T_m} \| \nabla u(t) \| = \infty$ (blow-up).

If we replace the nonlinear term by $|u|^{p-1}u$, it is known that the exponent $p = p_c = 1 + \dfrac{4}{N}$ in dimension N is the critical value for the nonexistence of global solutions (see, *e.g.*,[2] and [15]): If $p < p_c$, every solution exists globally in time; If $p \geqq p_c$, there is a class of initial data leading to blow-up solutions.

In the previous papers [6], [7], and [8] (see also [9] and [11]), we studied the asymptotic profiles of general blow-up solutions to (NSC) and obtained the following theorem.

Theorem A. *Let $u(t)$ be a singular solution of* (NSC)-(IV) *such that*

$$\text{(A.1)} \qquad \limsup_{t \to T_m} \| \nabla u(t) \| = \limsup_{t \to T_m} \| u(t) \|_\sigma = \infty$$

for some $T_m \in (0, \infty]$. Let $\{t_n\}$ be any sequence such that, as $n \to \infty$,

$$\text{(A.2)} \qquad t_n \uparrow T_m, \quad \sup_{t \in [0, t_n)} \| u(t) \|_\sigma = \| u(t_n) \|_\sigma.$$

For this $\{t_n\}$, we put

$$\text{(A.3)} \qquad \lambda_n = \frac{1}{\| u(t_n) \|_\sigma^{\sigma/2}}$$

and, we consider the scaled functions

$$\text{(A.4)} \qquad u_n(t, x) = \overline{\lambda_n^{\frac{N}{2}}\, u(t_n - \lambda_n^2 t, \lambda_n x)}$$

for $t \in (-(T_m - t_n)/\lambda_n^2, t_n/\lambda_n^2]$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exist (i) a finite number of nontrivial solutions $u^1, u^2, \ldots, u^L$ of (NSC) *in the space $C_b(\boldsymbol{R}_+; H^1(\boldsymbol{R}^N))$ with*

$$E(u^j) = 0 \text{ and } \Im \int_{\boldsymbol{R}^N} \nabla u^j(t, x) \overline{u^j(t, x)}\, dx = 0$$

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for $j = 1, 2, \ldots, L$, *and (ii) sequences* $\{\gamma_n^1\}, \{\gamma_n^2\}, \ldots, \{\gamma_n^L\}$ *in* $\boldsymbol{R}^N$ *with* $\lim_{n\to\infty} |\gamma_n^j - \gamma_n^k| = \infty$ $(j \neq k)$, *such that, for any* $T > 0$,

(A.5) $$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| u_n(t,\cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \gamma_n^j) \right\|_\sigma = 0,$$

(A.6) $$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \nabla u_n(t,\cdot) - \sum_{j=1}^{L} \nabla u^j(t, \cdot - \gamma_n^j) \right\| = 0,$$

(A.7) $$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| u_n(t,\cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \gamma_n^j) - \varphi_n(t,\cdot) \right\| = 0,$$

where

(A.8) $$\begin{cases} 2i\dfrac{\partial \varphi_n}{\partial t} + \triangle \varphi_n = 0, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^N, \\ \varphi_n(0,x) = u_n(0,x) - \sum_{j=1}^{L} u^j(0, x - \gamma_n^j), & x \in \boldsymbol{R}^N. \end{cases}$$

Furthermore we have

(A.9) $$\|u_0\|^2 \geqq \sum_{j=1}^{L} \|u^j(t)\|^2 \geqq L\|Q_g\|^2,$$

where Q_g *is a nontrivial solution of*

(A.10) $$\triangle Q - Q + |Q|^{\frac{4}{N}} Q = 0$$

such that

(A.11) $$\frac{2}{\sigma}\|Q_g\|^{\frac{4}{N}} = \inf_{\substack{v\in H^1(\boldsymbol{R}^N)\\ v\neq 0}} \frac{\|v\|^{\frac{4}{N}} \|\nabla v\|^2}{\|v\|_\sigma^\sigma}$$
$$= \inf_{\substack{v\in H^1(\boldsymbol{R}^N)\\ v\neq 0}} \left\{ \frac{2}{\sigma}\|v\|^{\frac{4}{N}} \,\middle|\, \|\nabla v\|^2 - \frac{2}{\sigma}\|v\|_\sigma^\sigma \leqq 0 \right\}.$$

Remark 1.1. (1) The solution Q_g of (A.10) and (A.11) is called the *ground state,* since it is a solution of the second minimization problem in (A.11). $Q_g(x)\exp(i\frac{t}{2})$ is an example of zero-energy, zero-momentum, H^1-bounded, global-in-time solution. For these facts, see, *e.g.,* [8] and [15].
(2) If the initial datum u_0 is radially symmetric, then so is the corresponding solution, and we have, in this case, the above theorems with $L = 1$ and $\gamma_n^1 \equiv 0$. That is, the origin is always a "blow-up point", *i.e.,* L^2 concentration point, for radially symmetric blow-up solutions.

By the proof of this theorem [8], we can show (see, *e.g.,* [10] and [11]):

Corollary B. *Under the same assumptions, definitions and notations of Theorem A, we have*:

(B.1) $$\lim_{n\to\infty} \sup_{t\in[t_n-\lambda_n^2 T, t_n]} \left\| \overline{u(t,\cdot)} - \sum_{j=1}^{L} u_n^j(t,\cdot) - \tilde{\varphi}_n(t,\cdot) \right\| = 0$$

with

(B.2) $$\lim_{n\to\infty} \lambda_n^2 \sup_{t\in[0,T]} \|\tilde{\varphi}_n(t)\|_\sigma^\sigma = 0,$$

where

(B.3) $$u_n^j(t,x) = \frac{1}{\lambda_n^{N/2}} u^j\left(\frac{t_n - t}{\lambda_n^2}, \frac{x - \gamma_n^j\lambda_n}{\lambda_n}\right),$$

(B.4) $$\tilde{\varphi}(t,x) = \frac{1}{\lambda_n^{N/2}} \varphi_n\left(\frac{t_n - t}{\lambda_n^2}, \frac{x}{\lambda_n}\right).$$

Furthermore we have, for any $T > 0$ *and any* $f \in \mathfrak{B} \equiv C(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$,

(B.5) $$\lim_{n\to\infty} \sup_{t\in[t_n-\lambda_n^2 T, t_n]} \left| \int_{\boldsymbol{R}^N} \left(|u(t,x)|^2 - \sum_{j=1}^{L} |u_n^j(t,x)|^2 - |\tilde{\varphi}_n(t,x)|^2 \right) f(x)\,dx \right| = 0.$$

Theorem A tells us that the blow-up solutions of (NSC) behaves like a finite super position of *dilated* zero-energy, zero-momentum, H^1-bounded, global-in-time solutions accompanied by a *dilated* wave of the free Schrödinger equation. And finally, it loses its L^2 continuity at the blow-up time because of the concentration of its L^2 mass which amounts to $\|Q_g\|$ at least. In addition, the formula (B. 5) suggests that we might have:

(1.5) $$|u(s_n,x)|^2 dx \rightharpoonup \sum_{j=1}^{L} \|u^j(0)\|^2 \delta_{a_j}(dx) + \mu(dx)$$

in the weak topology of measures, *i.e.,* weakly* in $\mathfrak{B}'$, for some suitable sequence $\{s_n\}$ such that $s_n \to T_m$ as $n \to \infty$, provided that the following limits exist: $a^j \equiv \lim_{n\to\infty} \gamma_n^j \lambda_n$ (in $\boldsymbol{R}^N$) and $\mu(dx) = \lim_{n\to\infty} |\tilde{\varphi}_n(t_n,x)|^2 dx$. It can be considered that each u^j carries one singularity in the blow-up solution.

Fortunately, we can prove that the formula (1.5) is mathematically true under some conditions:

Theorem C. *Suppose one of the following conditions*:

(a) $N = 1$ *and*
$$E(u_0) < \frac{\left(\Im \int_{\boldsymbol{R}} \frac{d}{dx} u_0(x)\overline{u_0(x)}\,dx\right)^2}{\|u_0\|^2};$$

(b) $N \geqq 2$, $E(u_0) < 0$ *and* u_0 *is radially symmetric;*

(c) $N \geqq 1$, $|x|u_0 \in L^2(\boldsymbol{R}^N)$ *and* $T_m < \infty$.

Suppose that u_0 *gives rise to a blow-up solution. Let* $\{t_n\}$ *be a time sequence as in (A.2) of Theorem A. For any* $T > 0$, *we put*

(C.1) $$s_n = t_n - \lambda_n^2 T, \quad T > 0.$$

Note that $s_n \to T_m$ *as* $n \to \infty$. *Then there exists a* *subsequence of* $\{s_n\}$ (*still denoted by the same letter*) *which satisfies the following properties*: *there is a finite number* $L \in \boldsymbol{N}$, *a family of points* $\{a^1, a^2, \ldots a^L\} \in \boldsymbol{R}^N$ *and a positive measure* $\mu \in \mathfrak{B}'$ (*the dual of* $\mathfrak{B}$) *such that we have* (1.5) *as* $n \to \infty$ *in the sense of measures. In case of* u_0 *being radially symmetric,* (1.5) *should read with* $L = 1$ *and* $a^1 = 0$. *We note that*

(C.2) $$\|u_0\|^2 = \sum_{j=1}^{L} \|u^j(0)\|^2 + \mu(\boldsymbol{R}^N).$$

Remark 1.2. (1) As we will see in Theorem D below, under the assumption of (*a*) or (*b*), the corresponding solution blows up in a finite time (see [9], [11], [13], and [14]). In the case of (*c*), if we assume, for example, $E(u_0) < 0$, then the corresponding solution blows up in a finite time (see [2] and [15]).
(2) We can reduce the condition made on the energy in (*a*) to $E(u_0) < 0$ by the Galilei transformations as in [9], [10], and [11].

We treat (NSC)-(IV) in the pure energy space $H^1(\boldsymbol{R}^N)$ in this paper, so that we shall consider the case (*a*) and (*b*) in what follows.

The key ingredient to prove the formula (1.5) is the following theorem ([9] and [11]).

Theorem D. *We suppose one of the conditions* (*a*) *and* (*b*) *of Theorem C. Then, we have*

(D.1) $$T_m < \infty \text{ and } \lim_{t \to T_m} \|\nabla u(t)\| = \infty.$$

Furthermore, we have: (*i*) *there exists a constant* m_* > 0 *for which we have that, for any* $m \in (0, m_*)$, *there exists a constant* $R_m > 0$ *such that*

(D.2) $$\int_{|x|>R_m} |u_0(x)|^2 dx < m \Rightarrow \int_{|x|>R_m} |u(t, x)|^2 dx < m \quad t \in (0, T_m);$$

and (*ii*) *we have, for sufficiently large* $R > 0$,

(D.3) $$\int_0^{T_m} (T_m - t) \left(\int_{|x|>R} |\nabla u(t, x)|^2 dx \right) dt < \infty,$$

(D.4) $$\int_0^{T_m} (T_m - t) \left(\int_{|x|>R} |u(t, x)|^\sigma dx \right) dt < \infty.$$

Remark 1.3. (1) The nonexistence part of global-in-time solutions was already proved in Ogawa-Y. Tsutsumi [13] and [14]. The novelty here is the estimates (D.2) and (D.3)-(D.4). In the papers [9] and [11], in order to prove the nonexistence of global-in-time solutions, we introduce a variational problem seeking a non-zero minimum of L^2-norm under the constraint of negative "local energy" on (NSC). The constant m_* is determined by the variational value. We shall give the definition of it in Sect. 2 of this paper (see (2.8) and (2.9)).
(2) Suppose the condition (b) of Theorem C. Then we have from (D.3)-(D.4) with the help of the radial interpolation inequality (see [10] and [11]) that

(1.6) $$\liminf_{t \to T_m} (T_m - t)^{2/N} \|u(t)\|^\sigma_{L^\sigma(|x| > R)} = 0.$$

It is worth while noting here that the following lower estimate of the blow-up rate (see Cazenave-Weissler [1]); we have, for some constant $C > 0$,

(1.7) $$\frac{C}{T_m - t} \leqq \|u(t)\|_\sigma^\sigma.$$

Comparing (1.6) and (1.7), we can safely say that the "shoulder" decouples the singularity.
(3) For the general $u_0 \in H^1(\boldsymbol{R}^N)$ with $E(u_0)$ < 0, we can show $\sup_{t \in [0, T_m)} \|\nabla u(t)\| = \infty$, *i.e.*, the corresponding solution $u(t)$ blows up in a finite time or grows up at infinity (see [6], [7], [9], and [11]).

From (D.2). we see that the family of "probability measures" $\{|u(t, x)|^2 dx\}_{t \geqq 0}$ is tight. Hence, using (B.5), we can show (1.5) along the sequence $\{s_n\}$ defined by (C. 1).

Now recall the examples of "explicit" blow-up solutions of (NSC) in [5] and [16]. These examples correspond to (B.1) with $\tilde{\varphi}_n \equiv 0$. In order words, (1.5) with $\mu \equiv 0$. However, some numerical analyses suggest that, in general, the blow-up solution consists of singularities and non-singular part called "shoulder" or "slope" (see, *e.g.*, McLaughlin *et al.* [4]).

Therefore it is an interesting question that we ask whether each blow-up solution produces a nontrivial measure $\mu \in \mathfrak{B}'$ in the formula (1.5) or not. For this question, we have ([10] and [11]):

Theorem E. *Suppose the condition* (*a*) *or* (*b*) *of Theorem C. Let* $\mu \in \mathfrak{B}'$ *be the positive measure found in Theorem C. Then we have:*

(E. 1) $$\int_{\boldsymbol{R}^N} |x|^2 \mu(dx) < \infty \Rightarrow |x| u_0 \in L^2(\boldsymbol{R}^N).$$

In other words, we have:

(E. 2) $$|x| u_0 \notin L^2(\boldsymbol{R}^N) \Rightarrow \int_{\boldsymbol{R}^N} |x|^2 \mu(dx) = \infty.$$

Therefore, we can safely say that, under the conditions (*a*) or (*b*) of Theorem C, if $|x|u_0$ does not belong to $L^2(\boldsymbol{R}^N)$, then the corresponding blow-up solution must be accompanied by the "shoulder", $\tilde{\varphi}_n$, whose square of absolute value converges to a positive measure $\mu \in \mathfrak{B}'$ (in the sense of measures) which satisfies $\int_{\boldsymbol{R}^N} |x|^2 \mu(dx) = \infty$. So, there is *no quantization effect* observed in blow-up solutions.

In the proof of this theorem ([10] and [11]), we shall use (D.3) and (D.4), and the proof is closely related to the argument performed in Nawa-M. Tsutsumi [12].

2. Basic idea of proof of Theorem D. We assume that $N = 1$ or $N \geqq 2$ and u_0 is radially symmetric. We suppose that $-E^* \equiv E(u_0) < 0$, and suppose that the corresponding solution of (NSC) exists globally in time. We note here that, if the initial datum $u_0(x)$ is radially symmetric, so is the corresponding solution $u(t,x)$ of (NSC)-(IV) with respect to $x \in \boldsymbol{R}^N$ for any $t \in [0, T_m)$.

We introduce a $W^{3,\infty}(\boldsymbol{R})$ odd function, following Ogawa-Y. Tsutsumi [13] and [14]:

$$\text{(2.1)} \quad \phi(\xi) = \begin{cases} \xi, & 0 \leqq \xi < 1, \\ \xi - (\xi - 1)^3, & 1 \leqq \xi < 1 + \dfrac{1}{\sqrt{3}}, \\ \text{smooth}, \quad (\phi' \leqq 0) & 1 + \dfrac{1}{\sqrt{3}} \leqq \xi < 2, \\ 0, & 2 \leqq \xi. \end{cases}$$

We put $r \equiv |x| = \sqrt{\sum_{k=1}^N x_k^2}$ for $x = (x_1, \ldots, x_N)$. This convention will be also applied to one dimensional case. Using $\phi(\xi)$ defined in (2.1), we define, for $R > 0$,

$$\text{(2.2)} \qquad \Psi_R(x) = \frac{x}{r}\phi_R(r) = \frac{x}{r}R\phi\left(\frac{r}{R}\right),$$

$$\text{(2.3)} \qquad \Phi_R(x) = 2\int_0^r \phi_R(s)\,ds.$$

One of our key ingredients in the proof is the following generalization of virial identity (1.4):

Lemma 2.1. *We have for* $t \in [0, T_m)$,

$$\begin{aligned} \text{(2.4)} \quad & \langle \Phi_R, |u(t)|^2 \rangle \\ &= \langle \Phi_R, |u_0|^2 \rangle + 2\Im t \langle u_0, \Psi_R \cdot \nabla u_0 \rangle \\ &\quad - t^2 E^* - 2\int_0^t ds \int_0^s d\tau E^R(u(\tau)) \\ &\quad - \frac{1}{2}\int_0^t ds \int_0^s d\tau \langle \triangle(\nabla \cdot \Psi_R), |u(\tau)|^2 \rangle \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Here the functional E^R *is defined by:*

$$\text{(2.5)} \quad E^R(v) \equiv \int_{\boldsymbol{R}^N} \rho_1(r)|\nabla v(x)|^2 - \rho_2(r)|v(x)|^\sigma dx,$$

where

$$\text{(2.6)} \quad \rho_1(r) \equiv 1 - \phi'_R(r),$$

$$\text{(2.7)} \quad \rho_2(r) \equiv \frac{2}{\sigma N}\left(N - \phi'_R(r) - \frac{N-1}{r}\phi_R(r)\right).$$

We note that we have $\rho_2 = \frac{1}{3}\rho_1$ *if* $N = 1$.

For the proof of this Proposition, see Ogawa-Y. Tsutsumi [13] and [14](see also [9] and [11]).

The third term (III) in (2.4) can be easily handled to be absorbed in the term $-E^* t^2$ of (I), if we choose $R > 0$ sufficiently large. Hence, if we manage to overcome the second term (II) to be absorbed in $-E^* t^2$ of (I) as well, the right hand side of (2.4) will be dominated by a quadratic form of t whose top term has a negative coefficient, so that we are led to a contradiction.

In [9] and [11], in order to handle the second term (II) in (2.4), we introduce the following variational value:

$$\text{(2.8)} \quad m_R \equiv \inf_{\substack{v \in \chi \\ v \neq 0}} \left\{ \int_{|x|>R} |v(x)|^2 dx \,\middle|\, E^R(v) \leqq -\frac{1}{4}E^*, \ \|v\| \leqq \|u_0\| \right\},$$

where: $\chi \equiv H_r^1(\boldsymbol{R}^N)$ the space of all radially symmetric functions in $H^1(\boldsymbol{R}^N)$ if $N \geqq 2$; $\chi \equiv H^1(\boldsymbol{R})$ if $N = 1$, we can obtain a constant $m_* > 0$ independent of $R > 0$ large enough such that we have

$$\text{(2.9)} \qquad m_R \geqq m_*$$

for sufficiently large $R > 0$.

Then we can show, by contradiction, through the generalized virial identity (2.4), that

$$\text{(2.10)} \quad \sup\left\{ t > 0 \,\middle|\, \int_{|x|>R} |u(\tau, x)|^2 dx < m_*, \ 0 \leqq \tau < t \right\} = \infty.$$

From this, we thus obtain by the definition of m_* that

$$\text{(2.11)} \qquad -\frac{1}{4}E^* \leqq E^R(u(t)) \ \text{ for } t \geqq 0.$$

Consequently, taking $R > 0$ sufficiently large, we have from (2.4) that, for $t \geqq 0$,

$$\begin{aligned} \text{(2.12)} \quad \langle \Phi_R, |u(t)|^2 \rangle \leqq & \langle \Phi_R, |u_0|^2 \rangle \\ & + 2\Im\langle u_0, \Psi_R \nabla u_0 \rangle t - \frac{1}{2}t^2 E^*, \end{aligned}$$

which leads us to a contradiction.

We have sketched the proof of the nonexistence of negative-energy global solutions. As in the same way of proving (2.10), we can show (D.2).

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