A Yang-Mills-Higgs gradient flow on $\boldsymbol{R}^3$ blowing up at infinity

By Hideo KOZONO,*) Yoshiaki MAEDA, **) and Hisashi NAITO*)

(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1998)

1. Yang-Mills-Higgs functional. We prove long time existence of the Yang-Mills-Higgs gradient flow on Euclidean 3-space $\boldsymbol{R}^3$, with a geometric characterization at the singular points. Since a solution of the Yang-Mills-Higgs gradient flow constructed in this paper has geometrically reasonable properties at the ideal boundary of $\boldsymbol{R}^3$, we are motivated to propose our definition of a global solution for the gradient flow.

Let P be the trivial bundle $\boldsymbol{R}^3 \times SU(2)$ over $\boldsymbol{R}^3$ and let $\tilde{C}$ be the set of pairs of connections A on the principal bundle P and Higgs field Φ on $\boldsymbol{R}^3$; an $\mathfrak{su}(2)$-valued map on $\boldsymbol{R}^3$, where $\mathfrak{su}$ (2) is the Lie algebra of $SU(2)$. The Yang-Mills-Higgs functional is a functional on C defined by the following: for $(A, \Phi) \in \tilde{C}$,

$$E(A, \Phi) = \int_{R^3} (|F_A|^2 + |d_A \Phi|^2) dV \tag{1}$$

where d_A is the covariant exterior differentiation on the bundle P and F_A denotes the curvature 2-form of A. Critical points of the functional (1) are called *Yang-Mills-Higgs configurations.*

2. Yang-Mills-Higgs gradient flow. We define the following compactified configuration space (cf. Groissor [2]):

$$C = \{(A, \Phi) : E(A, \Phi) < \infty, \\ |\Phi(x)| \to 1 \text{ as } |x| \to \infty\}.$$

The configuration space C has a geometric invariant, $N(A, \Phi)$, defined by

$$N(A, \Phi) = \frac{1}{4\pi} \int_{R^3} F_A \wedge d_A \Phi. \tag{2}$$

$N(A, \Phi)$ is called the *monopole number* (or *magnetic charge*) of (A, Φ). Groissor [2] showed that if $(A, \Phi) \in C$, then $N(A, \Phi)$ is an integer and the functional $N : C \to \boldsymbol{Z}$ gives a path component decomposition on C. Restricting Φ to a sufficiently large 2-shpere S^2 in $\boldsymbol{R}^3$ determines a homotopy class of maps on S^2. Let S_∞ be the ideal boundary of $\boldsymbol{R}^3$. We can identify S_∞ with the unit 2-sphere $S^2(1)$ canonically: given Φ, we define a map $\hat{\Phi} : S_\infty \to S^2$ by

$$\hat{\Phi}(\omega) = \lim_{r \to \infty} \frac{\Phi(r, \omega)}{|\Phi(r, \omega)|}, \tag{3}$$

where $\Phi(r, \omega) = \Phi(x)$, $r = |x|$, $\omega = \frac{x}{|x|}$. Then, we have $N(A, \Phi) = -\deg(\hat{\Phi})$. Furthermore, $2N(A, \Phi)$ gives the first Chern number of some bundle over S^2. Thus, in constructing a solution of (4), it is reasonable to take its behavior at the ideal bundary S_∞ into account.

We consider the following heat flow associated with the Yang-Mills-Higgs functional (1):

$$\begin{cases} \partial_t A = -d_A^* F_A - [\Phi, d_A \Phi], \\ \partial_t \Phi = \Delta_A \Phi, \end{cases} \tag{4}$$

with the initial condition $(A(0), \Phi(0)) = (A_0, \Phi_0)$.

We call a curve $(A(t), \Phi(t))$ in the configuration space C a *smooth solution* of (4) if $(A(t), \Phi(t))$ satisfies (4) in the classical sense. To fix the geometrical meaning for solutions of (4), we introduce the following notion:

Definition 1. A smooth solution $(A(t), \Phi(t))$ of (4) is called *extendable* on $(0, T]$ if the following conditions are satisfied:

(i) For each $t \in (0, T]$, there exists a gauge transformation $g(t)$ such that $g^*(t)A(t)$ extends to a smooth connection over $S_\infty \cong S^2(1)$.

(ii) $N(A(t), \Phi(t))$ of (4) is independent of $t \in (0, T]$.

Let ε be a positive constant. For $\omega_0 \in S^2(1)$, let $B_\tau(\omega_0)$ be the geodesic ball centered at ω_0 with the radius τ.

Definition 2. A smooth solution $(A(t), \Phi(t))$ of (4) of has the *ε-property* if

$$\liminf_{r \to \infty} \int_{B_r(\omega_0)} r^2 (|F_A(t, r, \omega)| \\ + |d_A \Phi(t, r, \omega)|) d\omega \le \varepsilon, \tag{5}$$

for sufficiently small τ, for all $t \in (0, T]$ and for all $\omega_0 \in S^2$.

This definition gives a criterion for obtaining an extendable solution, and is one of the fundamental observations for constructing a global

*) Graduate School of Mathematics, Nagoya University.
**) Department of Mathematics, Faculty of Science and Technology, Keio University.

solution of (4). Namely, we have:

Theorem 3. *There exists a universal constant* $\varepsilon > 0$ *such that if the smooth solution* $(A(t), \Phi(t))$ *on* $(0, T]$ *of* (4) *satisfies the* ε*-property with the following initial conditions,* $(A(0), \Phi(0)) = (A_0, \Phi_0)$:

(A1) $|\nabla_A^n F_A(0, x)| + |\nabla_A^n d_A \Phi(0, x)| \leq C|x|^{-n-2}$, *for all* $n \in \boldsymbol{N} \cup \{0\}$,

(A2) $(A(0), \Phi(0)) \in C$,

(A3) $|1 - |\Phi(0, x)|^2| \leq C|x|^{-1}$,

then $(A(t), \Phi(t))$ *is extendable on* $(0, T]$.

Hessel (see [3]) showed that for any initial value satisfying (A1)-(A3), there exists a global smooth solution of (4). However, he did not investigate the asymptotic behavior of $(A(t), \Phi(t))$ as $|x| \to \infty$. The above theorem shows that if a smooth solution satisfies the condition (5) on $(0, T]$, then the solution can be continued to an *extendable* solution beyond T.

We remark that the Prasad-Sommerfield monopole (cf. Jaffe-Taubes [5, IV, 1]) satisfies the assumptions (A1)-(A3). It is also known that for any integer N, there exists at least one monopole solution (A, Φ) with the monopole number $N(A, \Phi) = N$ and such that (A1) is fulfilled for $n = 0$ (see [5, p. 109]).

3. Main theorem. We first show the existence of the time local solution of (4).

Theorem 4. *Assume that the initial condition* (A_0, Φ_0) *satisfies the conditions* (A1)-(A3). *Then there exists a positive constant* T^* *such that* (4) *has an extendable solution* $(A(t), \Phi(t))$ *for* $(0, T^*)$ *with the initial condition* $(A(0), \Phi(0)) = (A_0, \Phi_0)$.

To state our main result, we introduce the following notion:

Definition 5. A smooth solution $(A(t), \Phi(t))$ of (4) is called *weakly extendable* on $(0, T)$, $0 < T \leq \infty$ if the following conditions are satisfied:

(i) $(A(t), \Phi(t))$ is a smooth solution of (4) for $(0, T) \times \boldsymbol{R}^3$.

(ii) There exist $\{T_i\}_{i=0}^L \subset [0, T]$, $T_0 = 0$, $T_L = T$ such that the solution $(A(t), \Phi(t))$ of (4) is extendable on the interval (T_i, T_{i+1}), $i = 1, \cdots, L-1$.

By Definition 5, the solution of Theorem 4 is weakly extendable. Moreover, by the fact that the L^2-norm is lower semi-continuous, so is the map $t \in (0, T) \mapsto E(t) \in \boldsymbol{R}$. In this sense, we call the solution obtained in (4) is *weak*.

The following is the main result:

Theorem 6. *Assume that the initial conditions* (A_0, Φ_0) *satisfies* (A1)-(A3). *Then, we have the following:*

(i) *There exists an extendable weak solution of* (4) *on* $(0, \infty)$ *with initial condition* $(A(0), \Phi(0)) = (A_0, \Phi_0)$.

(ii) *The set of times* $t \in (0, \infty]$ *where the solution* $(A(t), \Phi(t))$ *given by* (i) *is not extendable is a finite set of* $[0, \infty]$.

(iii) *If* $(A(t), \Phi(t))$ *is not extendable at* $t = T_0$, *then there exists a finite set points* $\{\omega_k\} \in S^2$ *such that at each point* ω_k, *there exists a neighborhood* U *of* ω_k *such that on* U, *a renormalized sequence of* $A(t)$ *can be extended to a non-flat* $U(1)$-*Yang-Mills connection with finite energy, and the renormalized sequence of* $\Phi(t)$ *also has finite energy.*

As an easy consequence of Theorem 6, we obtain:

Corollary 7. *Under the same assumptions and notations as in Theorem* 4, *assume that* $E(A_0, \Phi_0) < C_\varepsilon$. *Then there exists a classical extendable solution on* $(0, \infty)$.

Since Corollary 7 provides the global existence under a small initial data, it gives a global solution with the monopole number zero.

Here, we note the outline of proof of Theorem 6. Let $T_0 > 0$ be a maximal existence time on which an extendable solution exists. To prove Theorem 6, we may show that the energy $E(t) = E(A(t), \Phi(t))$ satisfies

$$E(T_0) \leq E(0) - C, \tag{6}$$

where the constant C depends only on ε_1 and the number of singularities at T_0. From the singular time T_0, we solve (4) as an initial condition $(A(T_0), \Phi(T_0))$, so we may find a next singular time T_1. Iterating this procedure, if the solution does not extend to time infinity, by (6), the energy of the solution is going to negative. This is a contradiction to positivity of the energy. Therefore, we may construct a global weak extendable solution.

Let $T_0 > 0$ be a singular time, that is, the solution $(A(t), \Phi(t))$ of (4) is not extendable at $t = T_0$. Let $\omega_0 \in S^2(1)$ be a singular point. Then, there exists sequences $\{t_i\}$, $\{r_i\}$, $\{\tau_i\}$ and $\{\omega_i\}$ such that $t_i \uparrow T_0$, $r_i \to \infty$, $\tau_i \to 0$ and $\omega_i \to$

ω_0. We may take the renormalized sequence (A_i, Φ_i) in (iii) of Theorem 6 of the extendable solution as

$$\Phi_i(t, r, \omega) = \Phi(t_i + \tau_i^2 t, \tau_i r, \omega),$$
$$A_i(t, r, \omega) = A(t_i + \tau_i^2 t, \tau_i r, \omega).$$

Since $(A(t), \Phi(t))$ satisfies the equation (4), the renormalized sequence $(A_i(t), \Phi_i(t))$ satisfies

$$\begin{cases} \partial_t \Phi_i = \Delta_{A_i} \Phi_i, \\ \partial_t A_i = -d_{A_i}^* F_A - \tau_i^2 [\Phi_i, d_{A_i} \Phi_i]. \end{cases}$$

We can apply similar discussion to renormalized sequence with the extendable solution. Hence $(\tilde{A}_i(\omega), \tilde{\Phi}_i(\omega)) = (A_i(0, r_i\tau_i, \omega), \Phi_i(0, r_i\tau_i, \omega))$ converges to $(\tilde{A}_\infty, \tilde{\Phi}_\infty)$ in the smooth topology under the suitable gauge transformation. Moreover $\tilde{A}_\infty$ extends to a non-flat $U(1)$-Yang-Mills connection with finite energy on S^2, and $\tilde{\Phi}_\infty$ has finite energy.

In contrast to harmonic maps and Yang-Mills fields, we characterize the singularity by means of local concentration not of the energy functional but of the L^1-norms of the curvature tensor and the first derivative of the Higgs field. From an analytical point of view, such a characterization should take place in the L^p-space whose norm is invariant under the change of scaling. Unfortunately, for the Yang-Mills-Higgs functional, the norm for which bounds imply the smoothness of solutions does not coincide with the norm defining the energy functional. This makes global regularity results for weak solutions of Yang-Mills-Higgs gradient flow diffcult to obtain. These problems must be overcome by a diffierent technique.

Added in proof. After accepted this paper, the authors show that for any $T < \infty$, the extendable weak solution is extendable on $(0, T)$. That is to say, any weakly extendable solution has not singular point in finite time.

References

[1] S. Dostoglou: On the asymptotics of finite energy solutions of the Yang-Mills-Higgs equations. J. Math. Phys., **31**, 2490–2496 (1990).

[2] D. Groisser: Integrality of the monopole number in $SU(2)$ Yang-Mills-Higgs gauge theory on $\boldsymbol{R}^3$. Commun. Math. Phys., **93**, 367–378 (1984).

[3] A. Hassel: The Yang-Mills-Higgs heat flow on $\boldsymbol{R}^3$. J. Funct. Analy., **111**, 431–448 (1993).

[4] N. J. Hitchin: Monopoles and geodesics. Commun. Math. Phys., **83**, 579–602 (1982).

[5] A. Jaffe and C. H. Taubes: Vortices and Monopoles. Birkhäuser, Boston (1980).

[6] H. Kozono, Y. Maeda, and H. Naito: Global solutions for that Yang-Mills gradient flow on 4-manifolds. Nagoya Math. J., **139**, 93–128 (1995).

[7] M. Struwe: On the evolution of harmonic mappings of Riemann surfaces. Comm. Math. Helv., **4**, 558–581 (1985).

[8] M. Struwe: The Yang-Mills flow in four dimensions. Calc. Var., **2**, 123–150 (1994).