Remarks on modification of Helgason's support theorem. II

By Takashi Takiguchi

Department of Mathematics, National Defense Academy,
1-10-20, Hashirimizu, Yokosuka, Kanagawa 239-8686
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Abstract: In this paper, we discuss modification of Helgason's support theorem for the Radon transform. It is essential in this theorem to assume that the function decays rapidly towards infinity. We restrict this condition to an open cone.

Key words: Radon transform; support theorem; approximation by entire functions.

1. Introduction. In this paper, we study unique solvability of the exterior problem for the Radon transform. For a function f defined on $\mathbf{R}^n$, its Radon transform Rf is defined by

$$Rf(\theta, s) := \int_{\theta^\perp} f(s\theta + y)dy, \tag{1.1}$$

where $\theta \in S^{n-1}$, $s \in \mathbf{R}$, $\theta^\perp := \{x \in \mathbf{R}^n \ ; \ x \perp \theta\}$, $y \in \theta^\perp$. (θ, s) is identified with the hyperplane

$$\xi = \xi(\theta, s) = \{x \in \mathbf{R}^n \ ; \ x \cdot \theta = s\}. \tag{1.2}$$

Uniqueness of the exterior problem was first studied by S. Helgason, who established the support theorem in 1965.

Theorem 1.1 (cf. [He]). *Let K be a compact convex set in $\mathbf{R}^n$ and $f \in C(\mathbf{R}^n \setminus K)$. Assume that $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ and that*

$$|x|^k f(x) \to 0 \ \text{as} \ |x| \to \infty, \quad \text{for } \forall k \in \mathbf{N}. \tag{1.3}$$

Then $f(x) = 0$ for $x \notin K$.

In this theorem, the condition (1.3) of rapid decay is indispensable. There is a famous counterexample also by S. Helgason [He]. Let $n = 2$ and

$$f(x_1, x_2) := \frac{1}{(x_1 + ix_2)^\alpha}, \tag{1.4}$$

where $\alpha > 1$. Change the values of f in a small neighborhood K of the origin so that f is smooth in $\mathbf{R}^2$. Consider the integrals along lines which do not intersect K, whose values are zero by Cauchy's integral theorem. By this argument, we conclude that the condition (1.3) is essential for Theorem 1.1 to hold. It is interesting to consider the case where the rapid decay condition is restricted to some subset of $\mathbf{R}^n$. In 1993, J. Boman [B2] tried this modification.

Claim (cf. Cor. 4 in [B2]). *Let $f \in C(\mathbf{R}^n \setminus K)$, K be a compact convex set in $\mathbf{R}^n$, Γ be an open convex cone in $\mathbf{R}^n$ and*

$$K_\Gamma := \bigcap_{x \in K} (x + (\Gamma \cup (-\Gamma))). \tag{1.5}$$

Assume that

$$Rf(\xi) = 0 \quad \text{for } \forall \xi \cap K = \emptyset, \tag{1.6}$$

$$|x|^k f(x) \to 0 \quad \text{uniformly as } |x| \to \infty \text{ in } \Gamma, \quad \text{for } \forall k \in \mathbf{N}, \tag{1.7}$$

and that

$$f \text{ decays enough at infinity to be integrable on hyperplanes,} \tag{1.8}$$

for instance, $f(x) = O(|x|^{-n})$ as $|x| \to \infty$. Then $f(x) = 0$ in K_Γ.

Though the condition (1.8) is assumed in this claim, in Boman's argument [B2], the condition utilized was that $Rf(\xi)$ converges absolutely and is 0 for $\xi \cap K = \emptyset$. The author [T] proved that this condition is not sufficient for $f(x) = 0$ in K_Γ. He also modified Boman's claim and proved it.

Theorem 1.2 (cf. [T]). *Let $f \in C(\mathbf{R}^n \setminus K)$, K be a compact convex set in $\mathbf{R}^n$ and Γ be an open convex cone in $\mathbf{R}^n$. Assume* (1.7),

$$Rf(\xi) = 0 \quad \text{for } \forall \xi \cap K_\Gamma \neq \emptyset \quad \text{and} \tag{1.9}$$

$$f(x) = o(|x|^{-n}) \quad \text{uniformly in } x \text{ as } |x| \to \infty. \tag{1.10}$$

Then $f(x) = 0$ in K_Γ.

In this paper, we give a counterexample for the condition (1.8), which shows that this condition is not correct for Boman's claim. We also make some

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remarks on Theorem 1.2.

2. A counterexample. In this section, we construct an entire function $f \not\equiv 0$ on $\mathbf{C}$ satisfying the following conditions. $f(z)$ decays rapidly uniformly outside $\{1/4 < (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 < 4,\ \operatorname{Re} z < 0,\ \operatorname{Im} z > 0\}$, as $|z| \to \infty$. The Radon transform $Rf(l)$ of f converges absolutely for any line l in $\mathbf{C}$ and $Rf(l) = 0$ for any line l. Let $n = 2$ and we regard $\mathbf{R}^2 \cong \mathbf{C}$. We construct an entire function $g(z) \not\equiv 0$ defined on $\mathbf{C}$ satisfying the following conditions.

$$\int_l |g'(z)||dz| < \infty, \quad \text{for } \forall l, \tag{2.1}$$

(2.2)

$$|z|^k g(z) \to 0 \quad \text{uniformly in } z \text{ as } |z| \to \infty,\ \forall k > 0,$$

$$\text{for } z \in \mathbf{C} \setminus \left\{ \frac{1}{4} < (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 < 4, \right.$$

$$\left. \operatorname{Re} z < 0,\ \operatorname{Im} z > 0 \right\}.$$

Assume that we obtain an entire function g satisfying (2.1) and (2.2). Let $f(z) := g'(z)$. By (2.1), $\int_l f(z)dz$ converges absolutely for any l. By (2.2) and Cauchy's integral theorem we obtain

$$\int_l f(z)dz = 0, \quad \text{for any } l. \tag{2.3}$$

Let, for example,

$$\Gamma := \left\{ \frac{\pi}{3} < \arg z < \frac{2\pi}{3} \right\}, \tag{2.4}$$

then f decreases rapidly in Γ, but f does not vanish in K_Γ. Therefore, what we have to do is the construction of an entire function g satisfying (2.1) and (2.2). Let

$$K := \{ z \in \mathbf{C}\ ;\ |z| < 5 \}, \tag{2.5}$$

$$S := \left\{ \frac{1}{4} < (\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 < 4, \right. \tag{2.6}$$

$$\left. \operatorname{Re} z < 0,\ \operatorname{Im} z > 0 \right\},$$

$$M := \mathbf{C} \setminus (K \cup S). \tag{2.7}$$

Note that M is a closed subset in $\mathbf{C}$ and $\hat{\mathbf{C}} \setminus M$ is connected and arcwise connected at infinity, where $\hat{\mathbf{C}}$ is the one-point compactification of $\mathbf{C}$. We put

$$\varphi(z) := iz^2 - i. \tag{2.8}$$

Note that we can define $0 < \arg \varphi(z) < 4\pi$ on M and that $\log \varphi(z)$ is defined as a single-valued holomorphic function on M. Consider

$$h(z) := \frac{1}{\varphi(z)^{\log \varphi(z)}} = e^{-(\log \varphi(z))^2} \in C(M) \cap \mathcal{A}(M^{\text{int}}), \tag{2.9}$$

$$\varepsilon(t) := \frac{1}{(t^2 - 1)^{\log(t^2-1)}}. \tag{2.10}$$

Lemma 2.1 (cf. [A], [F]). *Let M be such a closed set in $\mathbf{C}$ that $\hat{\mathbf{C}} \setminus M$ is connected and arcwise connected at infinity in $\hat{\mathbf{C}}$. Assume that $\varepsilon(t) \geq 0$ is a decreasing function in t satisfying*

$$\int_1^\infty \frac{\log \varepsilon(t)}{t^{3/2}} dt > -\infty. \tag{2.11}$$

Then for any $h(z) \in C(M) \cap \mathcal{A}(M^{\text{int}})$ there exists such an entire function $g(z)$ that

$$|h(z) - g(z)| < \varepsilon(|z|), \quad \text{for } \forall z \in M. \tag{2.12}$$

M, h and ε defined by (2.7), (2.9) and (2.10) satisfy the assumption of Lemma 2.1, therefore there exists an entire function $g(z)$ satisfying

$$\left| \frac{1}{\varphi(z)^{\log \varphi(z)}} - g(z) \right| < \frac{1}{(|z|^2 - 1)^{\log(|z|^2-1)}} \quad \text{for } z \in M. \tag{2.13}$$

Since

$$\left| \frac{1}{\varphi(z)^{\log \varphi(z)}} \right| = \frac{e^{(\arg \varphi(z))^2}}{|\varphi(z)|^{\log |\varphi(z)|}}, \tag{2.14}$$

we have $g(z) \not\equiv 0$. In fact, assume $g \equiv 0$. Taking $z \in \mathbf{R}$ yields contradiction. Also by (2.14) we have

$$|g(z)| \leq \frac{e^{16\pi^2} + 1}{|\varphi(z)|^{\log |\varphi(z)|}} \quad \text{for } z \in M. \tag{2.15}$$

Therefore $g(z)$ is rapidly decreasing in M, which implies (2.2). Let $z \in M^{\text{int}}$ and

$$d = d(z) := \frac{1}{2} \operatorname{dist}(z, \partial M), \tag{2.16}$$

where ∂M is the boundary of M. Put $L(z) := \max_{|\zeta - z| = d} |g(\zeta)|$ then we have

$$\frac{1}{2}|z| \leq |z| - d \leq |\zeta|, \quad \text{for } |\zeta - z| = d, \tag{2.17}$$

since $d(z) \leq (1/2)|z|$. Hence it holds by (2.15), (2.17) and $|z| \geq 5$ that

$$L(z) \leq \max_{|\zeta - z| = d} \frac{e^{16\pi^2} + 1}{(|\zeta|^2 - 1)^{\log(|\zeta|^2-1)}} \leq (e^{16\pi^2} + 1) e^{-(\log(|z|^2/4 - 1))^2} \tag{2.18}$$

Cauchy's integral formula yields

$$(2.19)\ |g'(z)| \le \frac{L(z)}{d(z)} \le \frac{e^{16\pi^2}+1}{d(z)} e^{-(\log(|z|^2/4-1))^2}.$$

Since $d(z) = O(1/|z|)$ on the most critical line $\{\operatorname{Im} z = -\operatorname{Re} z\}$ as $-\operatorname{Re} z = \operatorname{Im} z \to \infty$, $|g'(z)|$ is integrable on all lines in $\mathbf{C}$ by (2.19). Thus we have (2.1). Therefore, we have proved the following theorem, which proves that Claim in Section 1 is not true.

Theorem 2.2. *There exists an analytic function $f \not\equiv 0$ on $\mathbf{R}^2$ satisfying* (1.6), (1.7) *and* (1.8).

3. Conclusion and remarks. In this section, we make several remarks on Theorem 1.2 and on our counterexample established in Section 2. First, note that our counterexample is stronger than the one established in [T]. Our counterexample also suggests that uniqueness of the Radon transform does not hold without any global decay condition even if integrals along any hyperplanes absolutely converge. This fact was first proved by L. Zalcman [Z]. Our argument in Section 2 is a modification of Zalcman's argument in [Z]. For uniqueness to hold, it is sufficient to assume that $f \in L^1 \cap C$, which was also mentioned in [Z]. It is interesting that similar situation arises when we study uniqueness of the exterior problem.

Let us consider the global decay condition (1.10).

Definition 3.1. We imbed $\mathbf{R}^n$ in $S^n/2$ by

$$(3.1)\qquad I : x \to \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}\right),$$

where $x \in \mathbf{R}^n$. By this imbedding, a function $f(x)$ defined on $\mathbf{R}^n$ is regarded as the one $F(s)$ defined on $S^n/2$, that is,

$$(3.2)\quad F(s) := f(I^{-1}(s)), \quad s = I(x) \in S^n/2.$$

We extend F defined on $S^n/2$ to a function defined on S^n by identifying the antipodal points; i.e., $F(s) = F(-s)$, except on the equator $\{s_{n+1} = 0\}$, where $s = (s_1, \dots, s_{n+1})$ is the coordinate for S^n. We define $n-1$ form $d\mu(I(\xi), s)$ on $I(\xi)$ by

$$(3.3)\quad Rf(\xi) = 2\int_{x\in\xi} F(I(x))d\mu(I(\xi), I(x)),$$

for f for which $Rf(\xi)$ is well-defined.

Let $\xi_t = \{x_n = t\}$ then

$$(3.4)\qquad Rf(\xi_t) = 2\int_{I(\xi_t)} F(s)\frac{1}{s_{n+1}^n}ds,$$

where ds is the $n-1$-dimensional surface measure on $\overline{I(\xi_t) \cup (-I(\xi_t))}$. By (3.4), the measure $d\mu(I(\xi), s)$ has singularity of the type $1/s_{n+1}^n$ at $s_{n+1} = 0$ for any $I(\xi)$.

Proposition 3.2. *Assume that $f(x)$ defined on $\mathbf{R}^n$ satisfy* (1.10) *then*

$$(3.5)\qquad \lim_{s_{n+1}\to 0} \frac{F(s)}{s_{n+1}^n} = 0.$$

The proof is obtained by easy calculation. Take f satisfying (1.10) and we have $F(s) = F_1(s)s_{n+1}^n$ with $F_1 \in C(S^n/2)$. Since $s_{n+1}^n d\mu(I(\xi), s)$ is an analytic measure in $(I(\xi), s)$ we have

$$(3.6)\quad \int_{s\in\{s_{n+1}=0\}} F(s)d\mu(\{s_{n+1}=0\}, s) = 0$$

and

$$(3.7)\qquad Rf(\xi) = \int_{s\in S_\xi} F(s)d\mu(S_\xi, s),$$

where $S_\xi = \overline{I(\xi) \cup (-I(\xi))}$. Note that we identified $d\mu(I(\xi), s) = d\mu(-I(\xi), -s)$. The condition (1.10) implies that $F(s)$ is not singular on $\{s_{n+1} = 0\}$, which gives a sufficient condition for Theorem 1.2 to hold.

J. Boman claimed that $f(x) = O(|x|^{-n})$, as $|x| \to \infty$, would be sufficient for Theorem 1.2 (cf. Claim in Section 1), however, we are not able to judge whether it is true by our argument in this paper.

In Theorem 1.2, we have assumed that Γ is an open convex cone, however, this assumption of convexity is not necessary. More precisely, the following theorem holds.

Theorem 3.3. *Let $f \in C(\mathbf{R}^n \setminus K)$, K be a compact convex set in $\mathbf{R}^n$ and Γ is an open cone in $\mathbf{R}^n$. Assume* (1.7), (1.9) *and* (1.10). *Then $f(x) = 0$ in $K_{\widehat{\Gamma}}$, where $\widehat{\Gamma}$ is the convex hull of Γ.*

By Theorem 1.2, $f(x) = 0$ in K_Γ. (1.9) and Holmgren's uniqueness theorem [Hö] yields that $f(x) = 0$ in $K_{\widehat{\Gamma}}$ by the sweeping out method.

Theorem 1.1 accompanies Theorem 1.2, moreover, by virtue of Theorem 1.2, it is sufficient to assume (1.3) in an open cone whose convex hull is the whole space.

Theorem 3.4. *Assume that Γ be an open cone whose convex hull $\widehat{\Gamma} = \mathbf{R}^n$. Let K be a compact convex set in $\mathbf{R}^n$, $f \in C(\mathbf{R}^n \setminus K)$. If* (1.7), (1.10) *and $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ hold then $f(x) = 0$ for $x \notin K$.*

This is a generalization of Helgason's support theorem.

Theorem 1.1 is extended for any singular functions [TK], however, we have to assume $f \in C$ for Theorems 1.2 and 3.3. If we assume (1.7) in $\Gamma\cup(-\Gamma)$, not in Γ, then Theorems 1.2 and 3.3 also extends for functions with singularities with a little modification (cf. Theorem 3.6 below). We study the relation between the regularity of functions and the sufficient decay condition for uniqueness of the exterior problem. This relation is closely related to uniqueness of functions with microlocal analyticity.

First, let us review the proof of Theorem 1.2 (equivalently, the proofs of Theorems 3.3 and 3.4) shortly. Assume (1.7). $\int_{s\in S_\xi} F(s)d\mu(S_\xi, s) = 0$ for any $\xi \cap K = \emptyset$ yields that $WF_A(F) \cap N^*(S_\xi) = \emptyset$ for any $\xi \cap K = \emptyset$ (cf. [B2]). By (1.9) and (1.10), $F(s)$ and its derivatives of all orders along normal directions to $\{s_{n+1} = 0\}$ tend to 0 as $s_{n+1} \to +0$, $s \in I(\Gamma)$. Then we apply a local vanishing theorem for distributions (cf. [B1] and [TT]).

Proposition 3.5. *Let $S \subset \mathbf{R}^n$ be a real analytic submanifold. Assume that f is a distribution satisfying*

$$N^*(S) \cap WF_A(f) = \phi, \tag{3.8}$$

where $WF_A(f)$ is the analytic wave front set of f and $N^*(S)$ is the conormal bundle of S. Assume also that the restrictions to S of f and all its derivatives along the conormal direction to S vanish, that is,

$$\partial_\xi^\alpha f|_S = 0 \quad \text{for all } \alpha, \tag{3.9}$$

where $(x, \xi) \in N^*(S)$. Then $f = 0$ in some neighborhood of S.

Remark that when f is continuous, it is sufficient for this proposition to assume that the boundary values from one side to S of f and all its derivatives vanish. Therefore, F vanishes in a neighborhood of $\{s_{n+1} = 0\} \cap \overline{(I(\Gamma) \cup (-I(\Gamma)))}$. Holmgren's uniqueness theorem (cf. [Hö]) gives the theorems.

Note that for hyperfunctions, any decay condition would not imply that they are regular at infinity. Hence we have to assume

$$\int_{s_{n+1}=0} F(s)\frac{1}{s_{n+1}^n}ds = 0 \tag{3.10}$$

for hyperfunctions.

Theorem 3.6. *Assume the same assumptions on Γ and K as Theorem 3.4. Let f be a Fourier hyperfunction with defining function of the order $o(|z|^{-n})$. Suppose that as a Fourier hyperfunction on $\Gamma \cup (-\Gamma)$, f decays exponentially. If $Rf(\xi) = 0$ for $\xi\cap K = \emptyset$ and* (3.10) *hold then* supp f *is contained in the complement of $K_{\widehat{\Gamma}}$.*

For decay conditions of hyperfunctions, confer [K2] and [TK]. We introduce the idea of the proof of Theorem 3.6. $Rf(\xi) = 0$ for $\xi \cap K = \emptyset$ and (3.10) imply that F is micro analytic at $\{s_{n+1} = 0\}$ in its conormal directions. Exponential decay of f together with the fact

$$WF_A(F) \cap N^*(\{s_{n+1} = 0\}) = \phi$$

yields that $P(D')F|_{s_{n+1}=0} = 0$, where $P(D')$ is any differential operator along conormal direction to $\{s_{n+1} = 0\}$ with constant coefficients, the symbol of P is an infra-exponential entire function. Hence uniqueness of hyperfunction with analytic parameter [K1] and Kashiwara-Kawai's theorem (Theorem 4.4.1 in [K2]) prove the theorem. Note that we have to assume exponential decay on $\Gamma \cup (-\Gamma)$, which yields the restriction to $\{s_{n+1} = 0\} \cap \overline{(I(\Gamma) \cup (-I(\Gamma)))}$ of the derivatives with any local operator vanish. This condition is very important for hyperfunctions. For Theorem 3.6, we shall give details in another paper.

Remark 3.7. i) Since Proposition 3.5 holds for non-quasi-analytic ultradistributions, Theorem 3.3 is extended for non-quasi-analytic ultradistributions with the assumptions (1.7) in $\Gamma \cup (-\Gamma)$, (1.9), (1.10) and (3.10) (cf. [TT]).

ii) Proposition 3.5 does not hold for quasi-analytic ultradistributions [B3] (then neither for hyperfunctions). For hyperfunctions it is sufficient to assume that f decays exponentially in $\Gamma \cup (-\Gamma)$ because of the uniqueness of hyperfunctions with analytic parameters established by A. Kaneko [K1]. This condition is replaced by the one expressed in the terms of associated functions when we study quasi-analytic ultradistributions. It is interesting to study whether the decay condition in $\Gamma \cup (-\Gamma)$ in Theorem 3.6 is weakened, which is left open.

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