The abc conjecture and the fundamental system of units of certain real bicyclic biquadratic fields

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Let k_1 be a real quadratic field and

$$\eta_1 = (M + \sqrt{M^2 \pm 4})/2$$

be a fixed unit of k_1 with a positive integer M. Let $\bar{\eta}_1$ be the field conjugate of η_1.

Put

$$g_n(M) = \eta_1^n + \bar{\eta}_1^n, \qquad h_n(M) = \frac{\eta_1^n - \bar{\eta}_1^n}{\sqrt{M^2 \pm 4}}.$$

Then the sequences $\{g_n(M)\}_{n\in\mathbf{N}}$ and $\{h_n(M)\}_{n\in\mathbf{N}}$ are the non-degenerate binary recurrence sequences defined by

$$g_{n+2}(M) = Mg_{n+1}(M) \pm g_n(M),$$
$$h_{n+2}(M) = Mh_{n+1}(M) \pm h_n(M),$$

with the initial terms $g_0(M) = 2$, $g_1(M) = M$ and $h_0(M) = 0$, $h_1(M) = 1$. If there is no fear of confusion, we simply write $h_n(M)$ and $g_n(M)$ for h_n and g_n, respectively.

In our previous paper [1], we have investigated Hasse's unit indices Q_K of the real bicyclic biquadratic fields $K = \boldsymbol{Q}(\sqrt{M^2 \pm 4}, \sqrt{h_{2n+1}^2(M) - 1})$ and shown that $Q_K = 1$ except for finitely many indices n. In this note, assuming the *abc conjecture*, we shall determine the fundamental system of units of almost all K explicitly. First of all, we shall quote the following:

The abc conjecture. For any $\varepsilon > 0$, there exists a constant $K_0 > 0$ (depending on ε) such that if a, b, c are non-zero relatively prime integers with $a + b + c = 0$, then

$$\max\{|a|, |b|, |c|\} \le K_0 r^{1+\varepsilon},$$

where $r = rad(abc) = \prod_{p|abc} p$ (p: prime integer).

Any positive integer m can be written in the form $m = s(m)q^2(m)$, where $s(m)$ is the square free part of m. The following proposition is a corollary of a more general result of P. Ribenboim and G. Walsh [5, Theorem 2]:

Proposition 1 (Assuming the abc conjecture). *For any $\varepsilon > 0$,*

$$q(h_n) \le h_n^\varepsilon \quad \textit{and} \quad q(g_n) \le g_n^\varepsilon$$

except for finitely many indices n.

Since $h_n = s(h_n)q^2(h_n)$, we have $q(h_n) \le h_n^\varepsilon$ if and only if $(q(h_n))^{1/\varepsilon - 2} \le s(h_n)$. Hence the abc conjecture and the fact $1/\varepsilon - 2 \to \infty$ as $\varepsilon \to +0$ imply that for any $m > 0$,

$$q^m(h_n) \le s(h_n) \tag{1}$$

except for finitely many indices n. From the case $m = 2$ of the above (1), we have $h_n = s(h_n)q^2(h_n) \le s^2(h_n)$. The fact $h_n \to \infty$ as $n \to \infty$ implies the following proposition.

Proposition 2 (Assuming the abc conjecture). *For any constant $C > 0$,*

$$C \le s(h_n)$$

except for finitely many indices n.

It is easy to show that for any positive integers x and y,

$$s(xy) = s(x)s(y)/(s(x), s(y))^2 \ge s(x)s(y)/(x, y)^2,$$
$$q(xy) = q(x)q(y)(s(x), s(y)) \le q(x)q(y)(x, y).$$

In Proposition 1 of [1], we have shown $h_{2n+1}^2 - 1 = h_{2n}h_{2n+2}$ with $(h_{2n}, h_{2n+2}) = M$. Hence, assuming the abc conjecture, we have that, for any $m > 0$,

$$s(h_{2n})s(h_{2n+2}) \ge M^{2m+4}$$

except for finitely many indices n. The inequality (1) implies $s(h_{2n}) \ge q^{2m}(h_{2n})$ and $s(h_{2n+2}) \ge q^{2m}(h_{2n+2})$ except for finitely many indices n. Hence we have

$$\begin{aligned}
&s^2(h_{2n+1}^2 - 1) = s^2(h_{2n}h_{2n+2}) \\
&\ge s^2(h_{2n})s^2(h_{2n+2})/M^4 \\
&\ge q^{2m}(h_{2n})q^{2m}(h_{2n+2})s(h_{2n})s(h_{2n+2})/M^4 \\
&= (Mq(h_{2n})q(h_{2n+2}))^{2m}s(h_{2n})s(h_{2n+2})/M^{2m+4} \\
&\ge q^{2m}(h_{2n}h_{2n+2}) = q^{2m}(h_{2n+1}^2 - 1).
\end{aligned}$$

Thus assuming the abc conjecture, for any $m > 0$,

$$s(h_{2n+1}^2 - 1) \geq q^m (h_{2n+1}^2 - 1)$$

except for finitely many indices n.
Similarly the fact $g_{2n+1}^2 - M^2 = (M^2 \pm 4)(h_{2n+1}^2 - 1)$ implies that for any $m > 0$,

$$s((g_{2n+1}/M)^2 - 1) \geq q^m ((g_{2n+1}/M)^2 - 1)$$

except for finitely many indices n.
Combining these, we have the following proposition.

Proposition 3 (Assuming the abc conjecture)**.** *For any* $m \geq 0$*,*

$$s(h_{2n+1}^2 - 1) \geq q^m (h_{2n+1}^2 - 1),$$
$$s((g_{2n+1}/M)^2 - 1) \geq q^m ((g_{2n+1}/M)^2 - 1),$$

except for finitely many indices n*.*
In [1], we have shown that except for finitely many indices n, the unit $\eta_2 = h_{2n+1} + \sqrt{h_{2n+1}^2 - 1}$ and $\eta_3 = g_{2n+1}/M + \sqrt{(g_{2n+1}/M)^2 - 1}$ are the odd powers of the fundamental units of $k_2 = \boldsymbol{Q}(\sqrt{h_{2n+1}^2 - 1})$ and $k_3 = \boldsymbol{Q}(\sqrt{g_{2n+1}^2 - M^2})$, respectively. Suppose

$$\eta_2 = ((t + \sqrt{t^2 - 4})/2)^{2l+1}$$

with $l \geq 1$. Thus $h_{2n+1}^2 - 1 = h_{2l+1}^2(t)(t^2 - 4)/4$ implies

$$\begin{aligned} q^2(h_{2n+1}^2 - 1) &\geq ((h_{2l+1}(t)/2)^2 \geq ((h_3(t)/2)^2 \\ &= ((t^2 - 1)/2)^2 > t^2 - 4 \geq s(t^2 - 4) \\ &\geq s(h_{2n+1}^2 - 1). \end{aligned}$$

Similarly if η_3 is not the fundamental unit and an odd power of the fundamental unit of k_3, we have

$$q^2((g_{2n+1}/M)^2 - 1) > s((g_{2n+1}/M)^2 - 1).$$

Combining these inequalities and the inequalities of the case $m = 2$ of Proposition 3, we have the following theorem.

Theorem 1 (Assuming the abc conjecture)**.**

$$\eta_2 = h_{2n+1} + \sqrt{h_{2n+1}^2 - 1}$$

$$(\textit{resp. } \eta_3 = g_{2n+1}/M + \sqrt{(g_{2n+1}/M)^2 - 1})$$

is the fundamental unit of the real quadratic field k_2 (*resp.* k_3)*, except for finitely many indices* n*.*

As a corollary of this theorem, we can show the following theorem which is a refinement of Theorem 1 of our previous paper [1].

Theorem 2 (Assuming the abc conjecture)**.** *Let* $\eta_1 = (M + \sqrt{M^2 \pm 4})/2$ *be a fundamental unit of* k_1*. Then* $\{\eta_1, \eta_2, \eta_3\}$ *is a fundamental system of units of* $K = \boldsymbol{Q}(\sqrt{M^2 \pm 4}, \sqrt{h_{2n+1}^2 - 1})$ *except for finitely many indices* n*.*

Remark 1. Numerical investigations which support the above two theorems will be given in a forthcoming paper [2].

References

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