On the Diophantine equation $x(x+1)\cdots(x+n)+1=y^2$

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Abstract: Let $\mathbf{N}$ denote the set of natural numbers $\{1, 2, 3, \ldots\}$. n being an odd natural number, we consider the Diophantine equation as mentioned in the title and solve it completely for $n \leq 15$, i.e. find all $(x, y) \in \mathbf{N}^2$ satisfying this equation.

Key word: Diophantine equation.

1. Introduction. It was shown by Erdös and Serfridge [1] that the product of consecutive integers is never a power, so that the Diophantine equation $x(x+1)\cdots(x+n) = y^2$ has no solution, but we do not know if the Diophantine equation $x(x+1)\cdots(x+n)+1 = y^2$ has hitherto been ever treated. We shall consider it in this paper for the case n is odd and solve it completely for the case $n \leq 15$. We shall put $F_n(x) = x(x+1)\cdots(x+n)+1$. This is a monic polynomial with integral coefficients of an even degree $n+1$. Put $m = (n+1)/2$. As solutions of a Diophantine equation in x, y, we shall always mean $(x, y) \in \mathbf{N}^2$ satisfying it. We have obtained the following

Theorem.

(1) *$F_1(x) = y^2$ has no solution.*

(2) *$F_3(x) = y^2$ has an infinite number of solutions: x can take any element x of $\mathbf{N}$, $y = x^2+3x+1$.*

(3) *$F_5(x) = y^2$ has only one solution $(x, y) = (2, 71)$.*

(4) *$F_n(x) = y^2$ with odd n has no solution for $7 \leq n \leq 15$.*

Remark 1. We should like to conjecture that $F_n(x) = y^2$ with odd n has no solution also for $n \geq 17$, but we could not yet prove it.

Remark 2. Our proof of this theorem for the case $n \geq 5$ is based on a principle in solving Diophantine equations of the form $F(x) = y^2$, where $F(x)$ is a monic integral polynomial of an even degree, which will be explained in the following paragraph.

2. A principle. Let $F(x)$ be a monic integral polynomial of an even degree $2m$. To find solutions $(x, y) \in \mathbf{N}^2$ of $F(x) = y^2$, one can proceed as follows:

Put $F(x) = x^{2m} + a_1x^{2m-1} + \cdots + a_{2m} \in \mathbf{Z}[x]$. We can obtain a monic polynomial $G(x) = x^m + b_1x^{m-1} + \cdots + b_m \in \mathbf{Q}[x]$ and another polynomial $R(x) \in \mathbf{Q}[x]$ whose degree $\deg R < m$, such that $F(x) = (G(x))^2 + R(x)$ (uniquely by the method of indeterminate coefficients). In fact, the denominators of the coefficients of G, R are the powers of 2. We shall denote by ε the inverse number of the maximum of these denominators when $G(x) \notin \mathbf{Z}[x]$ and put $\varepsilon = 1$ when $G(x) \in \mathbf{Z}[x]$.

Put now for $x \in \mathbf{N}$

$$Y(x) = \begin{cases} [G(x)] & \text{when } G(x) \notin \mathbf{Z}[x], \\ G(x) - 1 & \text{when } G(x) \in \mathbf{Z}[x], \end{cases}$$

so that $Y : \mathbf{Z} \to \mathbf{Z}$. Notice that $\varepsilon < 1$ or $\varepsilon = 1$ according as $G(x) \notin \mathbf{Z}[x]$ or $\in \mathbf{Z}[x]$, and in the first case

$$\varepsilon \leq G(x) - Y(x) \leq 1 - \varepsilon.$$

If we could prove the existence of some $x_0 \in \mathbf{N}$, such that

$$(*) \qquad (Y(x))^2 < F(x) < (Y(x)+1)^2$$

holds for all $x \geq x_0$, then for any possible solution (x, y) of $F(x) = y^2$, we should have $x < x_0$, and these x could be found by a computer (if x_0 is not so large). The existence of number x_0 for $F = F_n$, $5 \leq n \leq 15$ will be shown in the following paragraph for individual cases.

3. Proof of the theorem. We shall omit the proof of (1), (2) which is immediate, and describe first the proof of (3) in detail.

In that case, we obtain

$$G(x) = x^3 + \frac{15}{2}x^2 + \frac{115}{8}x + \frac{75}{16}$$

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so that $\varepsilon = 1/16$, and

$$Y(x) \leq G(x) - \varepsilon < G(x) + \varepsilon \leq Y(x) + 1.$$

By calculation, we have

$$\begin{aligned}&(G(x)+\varepsilon)^2 - F(x)\\ &= \frac{1}{8}x^3 + \frac{249}{64}x^2 + \frac{265}{16}x + \frac{345}{16} > 0,\\ &F(x) - (G(x)-\varepsilon)^2\\ &= \frac{1}{8}x^3 - \frac{129}{64}x^2 - \frac{415}{32}x - \frac{1304}{64},\end{aligned}$$

of which the last polynomial has only one root between 21 and 22 (by Descartes' rule) so that $(*)$ holds for $x \geq 22$. The rest of the proof is done by a computer.

The proof of the cases $n = 9, 11, 13$ is done in the same way, the values of ε and x_0 in each case being as follows:

n	9	11	13
ε	1/256	1/2	1/2048
x_0	20277	88	20606985

In the cases $n = 7, 15$ we obtain $G(x) \in \mathbf{Z}[x]$, $\varepsilon = 1$. The concrete forms of $G(x)$ in respective cases are:

$$x^4 + 14x^3 + 63x^2 + 98x + 28 \quad \text{if } n = 7$$

$$\begin{aligned}&x^8 + 60x^7 + 1490x^6 + 19800x^5 + 151761x^4\\ &\quad + 671580x^3 + 1609180x^2 + 1741200x + 430016\\ &\quad \text{if } \ n = 15\end{aligned}$$

and the values of x_0 are 4, 1015, respectively.

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References

[1] Erdös, P. and Selfridge, J. L.: The Product of consecutive integers is never a power. Illinois J. Math., **19**, 292–301 (1975).