q-deformation of the group algebra $k[\widehat{W}]$ associated to the elliptic root system $A_l^{(1,1)}$ $(l \geq 2)$

By Tadayoshi TAKEBAYASHI

College of Industrial Technology, Nihon University, 1-2-1, Izumicho, Narashino, Chiba 275-8575

(Communicated by Heisuke HIRONAKA, M. J. A., March 13, 2000)

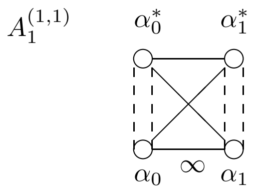
Abstract: In the case of elliptic root system $A_l^{(1,1)}$ $(l \geq 2)$, a q-deformation algebra of the hyperbolic extension of the elliptic Weyl group is constructed by using a representation according to Kazhdan-Lusztig.

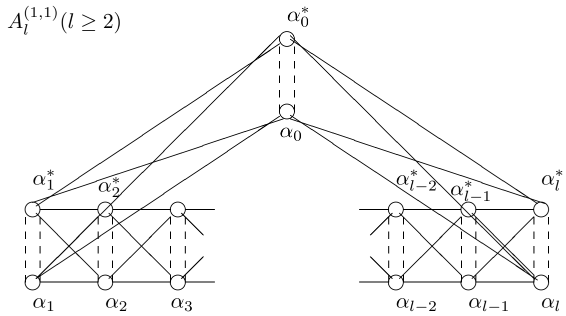
Key word: q-deformation of the hyperbolic extension of the elliptic Weyl group.

1. Introduction. Elliptic root systems (extended affine root systems), have been introduced and studied by K. Saito ([1], [2]). The Weyl group associated to the elliptic root system is called elliptic Weyl group, and the relations of the generators of the elliptic Weyl group have been described by the author and K. Saito from the view point of a generalization of Coxeter relations ([5], [6]). In the cases of finite or affine root systems, the Hecke algebras and their representation theory are well known ([3], [4]), and one can consider the Hecke algebra as a q-deformation of the group algebra $k[\widehat{W}]$. In this paper, in the case of the elliptic root system $A_l^{(1,1)}$ $(l \geq 2)$, we give a similar q-deformation $k[\widehat{W}](q)$ of the group algebra $k[\widehat{W}]$, where $\widehat{W}$ is a hyperbolic extension (a central extension) of the elliptic Weyl group W ([6]).

2. Elliptic Weyl group and elliptic root system of $A_l^{(1,1)}$. Let F be a real vector space of finite rank with a symmetric bilinear form $I : F \times F \longrightarrow \mathbf{R}$. A subset Φ of F is called an elliptic root system, if I is positive semi definite and the radical $\mathrm{rad}(I) = \{x \in F \mid I(x,y) = 0 \text{ for } \forall y \in F\}$ is of rank 2 over $\mathbf{R}$ and satisfies a system of an axiom for a generalized root system belonging to I ([1]). In the sequel, we restrict in the case of $A_l^{(1,1)}$. Let $\epsilon_1, \epsilon_2, \dots, \epsilon_{l+1}$ be orthonormal vectors belonging to the real vector space F, and $\mathrm{rad}(I) = \mathbf{R}a \oplus \mathbf{R}b$. Then the root system Φ and the Dynkin diagram of $A_l^{(1,1)}$ are given as follows ([1]):

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) + nb + ma \ (1 \leq i < j \leq l+1), \quad (n, m \in \mathbf{Z})\},$$

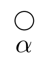




where $\alpha_0 := \epsilon_{l+1} - \epsilon_1 + b$, $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq l)$, and $\alpha_i^* := \alpha_i + a$ $(0 \leq i \leq l)$. Let $w_i := w_{\alpha_i}$ and $w_i^* := w_{\alpha_i^*}$ be the reflections corresponding to the roots α_i and α_i^*, respectively. Then w_i and w_i^* $(0 \leq i \leq l)$ generate the elliptic Weyl group of $A_l^{(1,1)}$, and the following has been established.

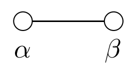
Proposition 2.1 ([5], [6])**.**

(I) The relations of w_i and w_i^* are given as follows: For any $\alpha, \beta, \in \{\alpha_0, \alpha_1, \dots, \alpha_l, \alpha_0^*, \alpha_1^*, \dots, \alpha_l^*\}$

(i) 

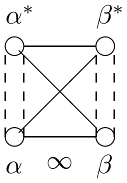
$\Longrightarrow$ $w_\alpha^2 = 1$,

$\Longrightarrow$ $(w_\alpha w_\beta)^2 = 1$,

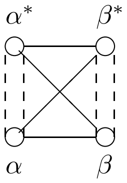


$\Longrightarrow$ $(w_\alpha w_\beta)^3 = 1$,

1991 Mathematics Subject Classification. Primary 20C15.



$$\Longrightarrow \quad w_\alpha w_\alpha^* w_\beta w_\beta^* = w_\alpha^* w_\beta w_\beta^* w_\alpha = w_\beta w_\beta^* w_\alpha w_\alpha^* = w_\beta^* w_\alpha w_\alpha^* w_\beta$$

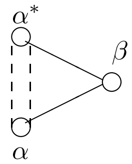


$$\Longrightarrow \quad w_\alpha w_\beta^* w_\alpha = w_\beta w_\alpha^* w_\beta,$$

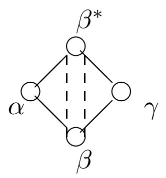
(ii) $w_0 w_0^* w_1 w_1^* \cdots\cdots w_{l-1} w_{l-1}^* w_l w_l^* = 1.$

(II) If we consider only the relation (i), then the corresponding group is the hyperbolic extension $\widehat{W}$ of the elliptic Weyl group W.

Remark. In the case of $A_l^{(1,1)}$, the following relations can be obtained from (i):



$$\Longrightarrow \quad (w_\alpha w_\beta w_\alpha^* w_\beta)^3 = 1,$$

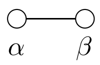


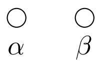
$$\Longrightarrow \quad (w_\alpha w_\beta^* w_\alpha w_\beta w_\gamma w_\beta)^2 = 1.$$

3. q-deformation algebra $k[\widehat{W}](q)$ of the group algebra $k[\widehat{W}]$. In this section, let k be a field, and we define a q-deformation algebra $k[\widehat{W}](q)$ of the hyperbolic extension $\widehat{W}$ of the elliptic Weyl group W of $A_l^{(1,1)}$ ($l \geq 2$), according to Kazhdan-Lusztig [4].

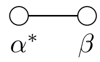
Definition 3.1. For each simple root $\alpha \in \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$, we define the elements $T_\alpha, T_\alpha^* \in \mathrm{End}(F \otimes k(q^{1/2}, q^{-1/2}))$ by the following actions:

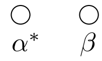
$$\begin{cases} T_\alpha(\alpha) = -\alpha, \\ T_\alpha(\alpha^*) = q\alpha^* - (q+1)\alpha, \\ T_\alpha(\beta) = q\beta + q^{1/2}\alpha \quad (\text{if } \ \text{[diagram 5]}\ ), \\ T_\alpha(\beta) = q\beta \quad (\text{if } \ \text{[diagram 6]}\ ), \end{cases}$$





$$\begin{cases} T_\alpha^*(\alpha) = q\alpha - (q+1)\alpha^*, \\ T_\alpha^*(\alpha^*) = -\alpha^*, \\ T_\alpha^*(\beta) = q\beta + q^{1/2}\alpha^* \quad (\text{if } \ \text{[diagram 7]}\ ), \\ T_\alpha^*(\beta) = q\beta \quad (\text{if } \ \text{[diagram 8]}\ ). \end{cases}$$





Under these actions, we examine the relations of the generators $T_i := T_{\alpha_i}$ and $T_i^* := T_{\alpha_i^*}$ $(0 \leq i \leq l)$, then we obtain the following.

Proposition 3.2. For any $\alpha, \beta, \in \{\alpha_0, \alpha_1, \ldots, \alpha_l, \alpha_0^*, \alpha_1^*, \ldots, \alpha_l^*\}$

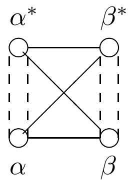
(I) 

$\Longrightarrow \quad T_\alpha^2 = (q-1)T_\alpha + q,$

(II) 

$\Longrightarrow \quad T_\alpha T_\beta = T_\beta T_\alpha,$

$\Longrightarrow \quad T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta,$

(III) 

$$\Longrightarrow \quad \begin{aligned} & T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta \\ &= Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]), \\ & T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* \\ &= Q[T_\beta, T_\beta^*], \\ & T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta^* \\ &= Q[T_\alpha, T_\alpha^*], \end{aligned}$$

where, $Q = \frac{q^{3/2} - q^2}{1+q}$.

By this proposition, we define the $k[\widehat{W}](q)$ as follows.

Definition 3.3. The q-deformation algebra $k[\widehat{W}](q)$ is the algebra presented by the following generators and relations:

Generators: T_α, T_α^* $\forall \alpha \in \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$.

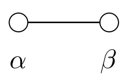
Relations:

(I) 

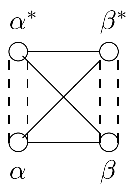
$\Longrightarrow \quad T_\alpha^2 = (q-1)T_\alpha + q,$

(II) 

$\Longrightarrow \quad T_\alpha T_\beta = T_\beta T_\alpha,$



$\Longrightarrow \quad T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta,$

(III) 

$$\Longrightarrow \quad \begin{aligned} & T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta \\ &= Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]), \\ & T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* \\ &= Q[T_\beta, T_\beta^*], \\ & T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta^* \\ &= Q[T_\alpha, T_\alpha^*], \end{aligned}$$

where, $Q = \frac{q^{3/2} - q^2}{1+q}$.

Remark. The subalgebra generated by the T_α, $\forall \alpha \in \{\alpha_0, \ldots, \alpha_l\}$ (or T_α^*) is isomorphic to the affine Hecke algebra of type A_l.

In the following, we prove Proposition 3.2. Let $\tilde{T}_i$ be the representation matrix on the basis $(\alpha_0, \alpha_0^*, \ldots, \alpha_l, \alpha_l^*)$, i.e. $T_i(\alpha_0, \alpha_0^*, \ldots, \alpha_l, \alpha_l^*) = (\alpha_0, \alpha_0^*, \ldots, \alpha_l, \alpha_l^*)\tilde{T}_i$, then from Definition 3.1, we see that

$$\begin{aligned}\tilde{T}_0 &= q^{1/2}(E_{0,2} + E_{0,3} + E_{0,2l} + E_{0,2l+1}) \\ &\quad -(q+1)(E_{0,0} + E_{0,1}) + qI, \\ \tilde{T}_i &= q^{1/2}(E_{2i,2i-2} + E_{2i,2i-1} + E_{2i,2i+2} + E_{2i,2i+3}) \\ &\quad -(q+1)(E_{2i,2i} + E_{2i,2i+1}) + qI, \\ &\qquad\qquad (1 \leq i \leq l-1), \\ \tilde{T}_l &= q^{1/2}(E_{2l,0} + E_{2l,1} + E_{2l,2l-2} + E_{2l,2l-1}) \\ &\quad -(q+1)(E_{2l,2l} + E_{2l,2l+1}) + qI,\end{aligned}$$

where $E_{i,j}$ is the $(2l+2) \times (2l+2)$ matrix whose (i,j) component is 1 and all others are 0, and I is the identity matrix. We set $\tilde{T}_i := \tilde{T}_i{}' + qI$, and identify $\tilde{T}_i$, $\tilde{T}_i{}'$ with T_i, T_i', respectively. In the same way, $\tilde{T}_i^* = \tilde{T}_i^{*\prime} + qI$ are given by

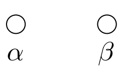
$$\begin{aligned}\tilde{T}_0^* &= q^{1/2}(E_{1,2} + E_{1,3} + E_{1,2l} + E_{1,2l+1}) \\ &\quad -(q+1)(E_{1,0} + E_{1,1}) + qI, \\ \tilde{T}_i^* &= q^{1/2}(E_{2i+1,2i-2} + E_{2i+1,2i-1} + E_{2i+1,2i+2} \\ &\quad +E_{2i+1,2i+3}) - (q+1)(E_{2i+1,2i} + E_{2i+1,2i+1}) \\ &\quad +qI, \\ \tilde{T}_l^* &= q^{1/2}(E_{2l+1,0} + E_{2l+1,1} + E_{2l+1,2l-2} \\ &\quad +E_{2l+1,2l-1}) - (q+1)(E_{2l+1,2l} + E_{2l+1,2l+1}) \\ &\quad +qI.\end{aligned}$$

Then there hold the following relations.

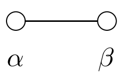
Lemma 3.4. For any $\alpha, \beta \in \{\alpha_0, \alpha_1, \ldots, \alpha_l, \alpha_0^*, \alpha_1^*, \ldots, \alpha_l^*\}$

(I) 

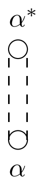
$\Longrightarrow$ $T_\alpha'^2 = -(q+1)T_\alpha'$,

(II) 

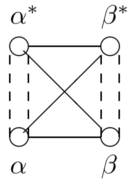
$\Longrightarrow$ $T_\alpha' T_\beta' = T_\beta' T_\alpha'$,



$\Longrightarrow$ $T_\alpha' T_\beta' T_\alpha' = q T_\beta'$,



$\Longrightarrow$
$$\begin{aligned}&[T_\alpha', T_\alpha^{*\prime}] = -(q+1)(T_\alpha' - T_\alpha^{*\prime}), \\ &T_\alpha^{*\prime} T_\alpha' = -(q+1)T_\alpha^{*\prime}, \\ &T_\alpha' T_\alpha^{*\prime} = -(q+1)T_\alpha',\end{aligned}$$

(III) 

$\Longrightarrow$
$$\begin{aligned}&T_\alpha^{*\prime} T_\beta' - T_\alpha' T_\beta^{*\prime} = q^{1/2}(T_\beta^{*\prime} - T_\beta'), \\ &T_\alpha^{*\prime} T_\beta^{*\prime} T_\alpha' = T_\alpha^{*\prime} T_\beta' T_\alpha', \\ &T_\alpha' T_\beta' T_\alpha^{*\prime} = T_\alpha' T_\beta^{*\prime} T_\alpha^{*\prime}, \\ &T_\alpha' T_\beta' T_\alpha^{*\prime} + T_\alpha' T_\alpha^{*\prime} + T_\alpha' = 0, \\ &T_\alpha^{*\prime} T_\beta' T_\alpha' + T_\alpha^{*\prime} T_\alpha' + T_\alpha^{*\prime} = 0.\end{aligned}$$

Proof. It is proved by direct calculations by using the relation, in the representation matrix $E_{2i+1,k} - E_{2i,k} = E_{2j+1,k} - E_{2j,k}$ for any i, j, k, which is obtained from the property $\alpha_i^* - \alpha_i = \alpha_j^* - \alpha_j$. □

Proof of Proposition 3.2.

((I), $T_\alpha^2 = (q-1)T_\alpha + q$)

$$\begin{aligned}T_\alpha^2 &= (T_\alpha' + qI)^2 \\ &= T_\alpha'^2 + 2qT_\alpha' + q^2 I \\ &= -(q+1)T_\alpha' + 2qT_\alpha' + q^2 \ (\Leftarrow \text{Lemma 3.4 (I)}) \\ &= (q-1)(T_\alpha - qI) + q^2 I \\ &= (q-1)T_\alpha + qI\end{aligned}$$

((II), $T_\alpha T_\beta = T_\beta T_\alpha$)

$$\begin{aligned}T_\alpha T_\beta &= (T_\alpha' + qI)(T_\beta' + qI) \\ &= T_\alpha' T_\beta' + q(T_\alpha' + T_\beta') + q^2 I \\ \text{and} \quad T_\beta T_\alpha &= T_\beta' T_\alpha' + q(T_\beta' + T_\alpha') + q^2 I\end{aligned}$$

so from Lemma 3.4 (II), we obtain the result.

((II), $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$)

$$\begin{aligned}T_\alpha T_\beta T_\alpha &= T_\alpha' T_\beta' T_\alpha' + q(T_\alpha' T_\beta' + T_\beta' T_\alpha') + q^2(T_\alpha' + T_\beta') \\ &\quad +qT_\alpha'^2 + q^2 T_\alpha' + q^3 I \\ &= T_\alpha' T_\beta' T_\alpha' + q(T_\alpha' T_\beta' + T_\beta' T_\alpha') + q^2(T_\alpha' + T_\beta') \\ &\quad -qT_\alpha' + q^3 I \ (\Leftarrow \text{Lemma 3.4 (I)}) \\ &= q(T_\alpha' T_\beta' + T_\beta' T_\alpha') \\ &\quad +q^2(T_\alpha' + T_\beta') + q^3 I \ (\Leftarrow \text{Lemma 3.4 (II)})\end{aligned}$$

in the same way,

$$T_\beta T_\alpha T_\beta = q(T_\beta' T_\alpha' + T_\alpha' T_\beta') + q^2(T_\beta' + T_\alpha') + q^3 I$$

so we get the result.

((III), $T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta = Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*])$)

In the same way as the previous case,

$$\begin{aligned}T_\alpha T_\beta^* T_\alpha &= q(T_\alpha' T_\beta^{*\prime} + T_\beta^{*\prime} T_\alpha' + q^2(T_\alpha' + T_\beta^{*\prime}) + q^3 I, \\ T_\beta T_\alpha^* T_\beta &= q(T_\beta' T_\alpha^{*\prime} + T_\alpha^{*\prime} T_\beta') + q^2(T_\beta' + T_\alpha^{*\prime}) + q^3 I,\end{aligned}$$

so,

$$\begin{aligned}
&T_\alpha T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta \\
&= q(T_\alpha' T_\beta^{*\prime} + T_\beta^{*\prime} T_\alpha' - T_\beta' T_\alpha^{*\prime} - T_\alpha^{*\prime} T_\beta') \\
&\quad + q^2(T_\alpha' + T_\beta^{*\prime} - T_\beta' - T_\alpha^{*\prime}) \\
&= q^{3/2}(T_\beta' - T_\beta^{*\prime} + T_\alpha^{*\prime} - T_\alpha') \\
&\quad + q^2(T_\alpha' + T_\beta^{*\prime} - T_\beta' - T_\alpha^{*\prime}) \ (\Leftarrow \text{Lemma 3.4 (III)}) \\
&= (q^2 - q^{3/2})((T_\alpha' - T_\alpha^{*\prime}) - (T_\beta' - T_\beta^{*\prime})) \\
&= Q([T_\alpha, T_\alpha^*] - [T_\beta, T_\beta^*]) \ (\Leftarrow \text{Lemma 3.4 (II)}).
\end{aligned}$$

((III), $T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* = Q[T_\beta, T_\beta^*]$)

$$\begin{aligned}
T_\alpha^* T_\beta T_\alpha &= T_\alpha^{*\prime} T_\beta' T_\alpha' + q(T_\alpha^{*\prime} T_\alpha' + T_\beta' T_\alpha' + T_\alpha^{*\prime} T_\beta') \\
&\quad + q^2(T_\alpha' + T_\alpha^{*\prime} + T_\beta') + q^3 I \\
&= (q-1) T_\alpha^{*\prime} T_\alpha' + q(T_\beta' T_\alpha' + T_\alpha^{*\prime} T_\beta') \\
&\quad + (q^2 - 1) T_\alpha^{*\prime} + q^2(T_\alpha' + T_\beta') \\
&\quad + q^3 I \ (\Leftarrow \text{Lemma 3.4 (III)}) \\
&= q(T_\beta' T_\alpha' + T_\alpha^{*\prime} T_\beta') + q^2(T_\alpha' + T_\beta') \\
&\quad + q^3 I \ (\Leftarrow \text{Lemma 3.4 (II)}),
\end{aligned}$$

and similarly,

$$T_\beta T_\alpha T_\beta^* = q(T_\alpha' T_\beta^{*\prime} + T_\beta' T_\alpha') + q^2(T_\beta^{*\prime} + T_\alpha') + q^3 I$$

so,

$$\begin{aligned}
&T_\alpha^* T_\beta T_\alpha - T_\beta T_\alpha T_\beta^* \\
&= q(T_\alpha^{*\prime} T_\beta' - T_\alpha' T_\beta^{*\prime}) + q^2(T_\beta' - T_\beta^{*\prime}) \\
&= q^{3/2}(T_\beta^{*\prime} - T_\beta') + q^2(T_\beta' - T_\beta^{*\prime}) \ (\Leftarrow \text{Lemma 3.4 (III)}) \\
&= (q^{3/2} - q^2)(T_\beta^{*\prime} - T_\beta') \\
&= Q[T_\beta, T_\beta^*] \ (\Leftarrow \text{Lemma 3.4(II)}).
\end{aligned}$$

The relation $T_\alpha^* T_\beta^* T_\alpha - T_\beta T_\alpha^* T_\beta^* = Q[T_\alpha, T_\alpha^*]$, is similarly proved as the previous. □

References

[1] Saito, K.: Extended affine root systems I. Publ. Res. Inst. Math. Sci., Kyoto Univ., **21**, 75–179 (1985).

[2] Saito, K.: Extended affine root systems II. Publ. Res. Inst. Math. Sci., Kyoto Univ., **26**, 15–78 (1990).

[3] Humphreys, J. E.: Reflection Groups and Coxeter Groups. Cambridge University Press, Cambridge-New York (1990).

[4] Kazhdan, D., and Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Invent. Math., **53**, 165–184 (1979)

[5] Takebayashi, T.: Relations of the Weyl groups of extended affine root systems $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$, $D_l^{(1,1)}$. Proc. Japan Acad., **71A**, 123–124 (1995).

[6] Saito, K., and Takebayashi, T.: Extended affine root systems III, Publ. Res. Inst. Math. Sci., Kyoto Univ., **33**, 301–329 (1997).