Coefficient bounds and convolution properties for certain classes of close-to-convex functions

By Yong Chan KIM,*) Jae Ho CHOI,*) and Toshiyuki SUGAWA**)

(Communicated by Shigefumi MORI, M. J. A., June 13, 2000)

Abstract: A number of authors (*cf.* Koepf [4], Ma and Minda [6]) have been studying the sharp upper bound on the coefficient functional $|a_3 - \mu a_2^2|$ for certain classes of univalent functions. In this paper, we consider the class $\mathcal{C}(\varphi, \psi)$ of normalized close-to-convex functions which is defined by using subordination for analytic functions φ and ψ on the unit disk. Our main object is to provide bounds of the quantity $a_3 - \mu a_2^2$ for functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathcal{C}(\varphi, \psi)$ in terms of φ and ψ, where μ is a real constant. We also show that the class $\mathcal{C}(\varphi, \psi)$ is closed under the convolution operation by convex functions, or starlike functions of order 1/2 when φ and ψ satisfy some mild conditions.

Key words: Univalent function; convolution; coefficient bound.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order α and convex of order α in $\mathbf{D}$. In particular, the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the familiar ones of starlike and convex functions in $\mathbf{D}$, respectively. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function ω on $\mathbf{D}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = h(\omega(z))$ for $z \in \mathbf{D}$. The subordination will be denoted by

$$g \prec h \qquad \text{or} \qquad g(z) \prec h(z) \quad \text{in } \mathbf{D}.$$

Note that $g \prec h$ if and only if $g(0) = h(0)$ and $g(\mathbf{D}) \subset h(\mathbf{D})$ when h is univalent in $\mathbf{D}$.

Let $\mathcal{M}$ be the class of analytic functions φ in $\mathbf{D}$ normalized by $\varphi(0) = 1$, and let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of those functions φ which are univalent in $\mathbf{D}$ and for which $\varphi(\mathbf{D})$ is convex. Also, for a constant $\alpha \geq 0$, set $\mathcal{N}(\alpha) = \{\varphi \in \mathcal{N} : \operatorname{Re} \varphi > \alpha\}$.

Ma and Minda [6] and the authors [3] defined the subclasses $\mathcal{K}(\varphi)$, $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi, \psi)$ of $\mathcal{A}$ by

$$\mathcal{K}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ in } \mathbf{D} \right\},$$

$$(1.1) \quad \mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \text{ in } \mathbf{D} \right\},$$

and

$$\mathcal{C}(\varphi, \psi) = \left\{ f \in \mathcal{A} : \exists h \in \mathcal{K}(\varphi) \text{ s.t. } \frac{f'(z)}{h'(z)} \prec \psi(z) \text{ in } \mathbf{D} \right\}$$

for φ, $\psi \in \mathcal{M}$. Note that $f \in \mathcal{K}(\varphi)$ if and only if $zf' \in \mathcal{S}^*(\varphi)$. Hence $f \in \mathcal{C}(\varphi, \psi)$ if and only if

$$(1.2) \quad \exists g \in \mathcal{S}^*(\varphi) \text{ such that } zf'(z)/g(z) \prec \psi(z) \text{ in } \mathbf{D}.$$

For functions φ, $\psi \in \mathcal{M}$, if φ and $e^{-i\beta}\psi$ have positive real part in $\mathbf{D}$, where β is some constant in $(-\pi/2, \pi/2)$, then the class $\mathcal{C}(\varphi, \psi)$ is obviously a subclass of close-to-convex functions, in particular, consists of univalent functions in $\mathbf{D}$. Now we recall that if $f \in \mathcal{A}$ satisfies

$$(1.3) \qquad \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \qquad (z \in \mathbf{D})$$

for a constant α $(0 < \alpha \leq 1)$, then $f(z)$ is said to be *strongly starlike of order* α in $\mathbf{D}$, and we write $f \in \mathcal{S}^*_\alpha$. If we set $\varphi_\alpha(z) = ((1+z)/(1-z))^\alpha$ $(0 < \alpha \leq 1)$, then, from (1.1) and (1.3), we can easily see

Mathematics Subject Classification. 30C45, 30C50.

*) Department of Mathematics, College of Education, Yeungnam University, Gyongsan 712-749, Korea.

**) Department of Mathematics, Kyoto University, Kitaashirakawa-Oiwakecho, Sakyo-ku, Kyoto, 606-8502.

The first named author was supported by Korea Research Foundation Grant (KRF-99-041-D00036). The third author was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, Grant-in-Aid for Encouragement of Young Scientists, 11740088.

the inclusion

$$\mathcal{S}^*_\alpha = \mathcal{S}^*(\varphi_\alpha) \subset \mathcal{C}(\varphi_\alpha, \varphi_\alpha). \tag{1.4}$$

For constants $\beta \in (-\pi/2, \pi/2)$ and γ with $0 \le \gamma < \cos\beta$, we set

$$\psi_{\beta,\gamma}(z) = \frac{1 + (e^{i\beta} - 2\gamma)e^{i\beta}z}{1 - z}.$$

The function $\psi_{\beta,\gamma}$ maps the unit disk onto the half-plane $\{z : \mathrm{Re}(e^{-i\beta}z) > \gamma\}$. Note that $\mathcal{S}^*(\alpha) \equiv \mathcal{S}^*(\psi_{0,\alpha})$ and $\mathcal{K}(\alpha) \equiv \mathcal{K}(\psi_{0,\alpha})$ for $0 \le \alpha < 1$. Note also that a function in $\mathcal{S}^*(\psi_{\beta,0})$ is usually called β-spirallike. We set

$$\mathcal{C}_{\alpha,\gamma} = \bigcup_{|\beta|<\arccos\gamma} \mathcal{C}(\psi_{0,\alpha}, \psi_{\beta,\gamma}) \tag{1.5}$$

for $0 \le \alpha < 1$ and $0 \le \gamma < 1$. A function in $\mathcal{C}_{\alpha,\gamma}$ is called *close-to-convex of order* (γ, α) (cf. [2, II, p. 89]). In particular, $\mathcal{C} \equiv \mathcal{C}_{0,0}$ is the class of usual close-to-convex functions.

In [3], the second and third authors investigated the norm estimate of the pre-Schwarzian derivatives for the class $\mathcal{C}(\varphi, \psi)$. In this paper, we shall investigate the coefficient bounds of the class $\mathcal{C}(\varphi, \psi)$ and also give convolution properties of functions in $\mathcal{C}(\varphi, \psi)$. Here, the convolution or the Hadamard product $f * g$ of two analytic functions

$$f(z) = \sum_{n=0}^\infty a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^\infty b_n z^n$$

on $\mathbf{D}$ is defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{n=0}^\infty a_n b_n z^n.$$

2. Preliminary results. The following lemmas will be required in our investigation.

Lemma 2.1. Assume that $\eta(z) = e_1 + e_2 z + \cdots$ is analytic in $\mathbf{D}$ with $|\eta(z)| \le 1$. Then $|e_1|^2 + |e_2| \le 1$.

Proof. By Schwarz-Pick's Lemma, we obtain

$$\frac{|\eta'(z)|}{1 - |\eta(z)|^2} \le \frac{1}{1 - |z|^2},$$

so that $|\eta(0)|^2 + |\eta'(0)| \le 1$. Hence $|e_1|^2 + |e_2| \le 1$. □

Lemma 2.2 (Ma and Minda [6]). Let $\varphi(z) = 1 + A_1 z + A_2 z^2 + \cdots$ be univalent in $\mathbf{D}$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{K}(\varphi)$, then $|a_3 - \mu a_2^2| \le K(\mu, A_1, A_2)$, where

$$K(\mu, A_1, A_2) = \begin{cases} (A_2 - (3\mu/2)A_1^2 + A_1^2)/6 & \text{if } 3A_1^2\mu \le 2(A_2 + A_1^2 - A_1), \\ A_1/6 & \text{if } 2(A_2 + A_1^2 - A_1) \le 3A_1^2\mu \le 2(A_2 + A_1^2 + A_1), \\ ((3\mu/2)A_1^2 - A_1^2 - A_2)/6 & \text{if } 2(A_2 + A_1^2 + A_1) \le 3A_1^2\mu. \end{cases} \tag{2.6}$$

Lemma 2.3 (Ruscheweyh and Sheil-Small [8]). Suppose either $g \in \mathcal{K}$, $h \in \mathcal{S}^*$ or else g, $h \in \mathcal{S}^*(1/2)$. Then for any analytic function G in $\mathbf{D}$, we have

$$\frac{(g * hG)(z)}{(g * h)(z)} \in \overline{\mathrm{co}}G(\mathbf{D}) \qquad (z \in \mathbf{D}),$$

where $\overline{\mathrm{co}}G(\mathbf{D})$ is the closed convex hull of $G(\mathbf{D})$.

3. Main results. We begin by proving

Theorem 3.1. Let $\varphi(z) = 1 + A_1 z + A_2 z^2 + \cdots$ be univalent in $\mathbf{D}$ and let $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be analytic in $\mathbf{D}$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{C}(\varphi, \psi)$, then

$$|a_3 - \mu a_2^2| \le K(\mu, A_1, A_2) + M(\mu, A_1, B_1, B_2),$$

where $K(\mu, A_1, A_2)$ is given by (2.6) and

$$M(\mu, A_1, B_1, B_2) = \begin{cases} (1/3)(|B_2 - (3\mu/4)B_1^2| + A_1|B_1||1 - 3\mu/2|) & \text{if } A_1|B_1||1 - 3\mu/2| \ge 2(|B_1| - |B_2 - (3\mu/4)B_1^2|), \\ \dfrac{|B_1|}{3} + \dfrac{(A_1|B_1||1 - 3\mu/2|)^2}{12(|B_1| - |B_2 - (3\mu/4)B_1^2|)} & \text{otherwise.} \end{cases}$$

Proof. If $f \in \mathcal{C}(\varphi, \psi)$, from the definition of the class $\mathcal{C}(\varphi, \psi)$ there exists a function $h \in \mathcal{K}(\varphi)$ such that $f'/h' \prec \psi$. We set

$$h(z) = z + d_2 z^2 + d_3 z^3 + \cdots$$

and

$$g(z) = \frac{f'(z)}{h'(z)} = 1 + b_1 z + b_2 z^2 + \cdots = \psi(\omega(z)), \tag{3.7}$$

where ω is an analytic function on $\mathbf{D}$ such that $|\omega(z)| \le |z|$ for $z \in \mathbf{D}$. Then a simple calculation shows $b_1 = 2(a_2 - d_2)$ and $b_2 = 3(a_3 - d_3) - 4d_2(a_2 - d_2)$, so that $a_2 = b_1/2 + d_2$ and $a_3 = d_3 + b_2/3 + (2/3)b_1 d_2$. Thus we have

$$a_3 - \mu a_2^2 = (d_3 - \mu d_2^2) + \frac{1}{3}\left(b_2 - \frac{3\mu}{4}b_1^2\right) + \left(\frac{2}{3} - \mu\right) b_1 d_2. \tag{3.8}$$

By Lemma 2.2, we have

$$(3.9)\qquad |d_3-\mu d_2^2|\le K(\mu,A_1,A_2).$$

We write $\omega(z)=e_1z+e_2z^2+\cdots$. Then, from (3.7) we have $b_1=B_1e_1$ and $b_2=B_1e_2+B_2e_1^2$. Since $1+(zh''(z))/(h'(z))\prec\varphi(z)$ in $\mathbf{D}$, Rogosinski's result [7] implies $|d_2|\le(1/2)A_1$. Therefore, we get

$$\begin{aligned}&\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right|\\&\le\frac{|B_1|}{3}|e_2|+\frac{1}{3}\left|B_2-\frac{3\mu}{4}B_1^2\right||e_1|^2\\&\quad+\left|\frac{2}{3}-\mu\right||d_2B_1||e_1|\\&\le\frac{|B_1|}{3}|e_2|+\frac{1}{3}\left|B_2-\frac{3\mu}{4}B_1^2\right||e_1|^2\\&\quad+\left|\frac{1}{3}-\frac{\mu}{2}\right|A_1|B_1||e_1|.\end{aligned}$$

Taking $\eta(z)=\omega(z)/z$ in Lemma 2.1, we obtain $|e_2|\le 1-|e_1|^2$, so that

$$\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right|\le P(|e_1|),$$

where $P(x)=ax^2+bx+c$ and $a=\frac{1}{3}(|B_2-\frac{3\mu}{4}B_1^2|-|B_1|)$, $b=A_1|B_1||\frac{1}{3}-\frac{\mu}{2}|$ and $c=|B_1|/3$. Since $b\ge 0$ and $0\le|e_1|\le 1$, we have

$$P(|e_1|)\le\begin{cases}P(-b/2a)=c-b^2/4a\\ \quad\text{if } a<0 \text{ and } -b/2a<1,\\ P(1)=a+b+c\qquad\text{otherwise.}\end{cases}$$

Thus we conclude

$$(3.10)\qquad \begin{aligned}&\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right|\\&\le M(\mu,A_1,B_1,B_2).\end{aligned}$$

Hence, making use of (3.9) and (3.10) in equality (3.8), we obtain the desired result. □

Corollary 3.2. If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{C}(\psi_{0,0},\ \psi_{0,0})$, then

$$|a_3-\mu a_2^2|\le\begin{cases}3-4\mu & \text{if } \mu\le 1/3,\\ 1/3+4/9\mu & \text{if } 1/3\le\mu\le 2/3,\\ (16-21\mu+9\mu^2)/3(4-3\mu) & \\ \qquad\text{if } 2/3\le\mu\le 1 & \\ 3\mu-5/3 & \text{if } 1\le\mu\le 4/3,\\ 4\mu-3 & \text{if } 4/3\le\mu.\end{cases}$$

Remark. From (1.5) it is clear that $\mathcal{C}(\psi_{0,0},\ \psi_{0,0})\subset\mathcal{C}$. For the cases of $0\le\mu\le 1/3$ and $1/3\le\mu\le 2/3$, the above estimates agree with those of Koepf [4].

If we take $\varphi=\psi=\varphi_\alpha=z+2\alpha z^2+2\alpha^2z^3+\cdots$ in Theorem 3.1, we obtain

Corollary 3.3. If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{C}(\varphi_\alpha,\varphi_\alpha)$, then

$$|a_3-\mu a_2^2|\le\begin{cases}(3-4\mu)\alpha^2 & \text{if } 3\alpha\mu\le 2\alpha-1\\ (1-\mu)\alpha^2+\dfrac{\alpha}{3}\left\{2+\dfrac{(2-3\mu)^2\alpha^2}{2-(2-3\mu)\alpha}\right\} & \\ \qquad\text{if } 2\alpha-1\le 3\alpha\mu\le 3\alpha-1 & \\ \alpha\left\{1+\dfrac{(2-3\mu)^2\alpha^2}{3(2-2\alpha+3\alpha\mu)}\right\} & \\ \qquad\text{if } 3\alpha-1\le 3\alpha\mu\le 2\alpha & \\ \alpha\left\{1+\dfrac{(2-3\mu)^2\alpha^2}{3(2-3\alpha\mu+2\alpha)}\right\} & \\ \qquad\text{if } 2\alpha\le 3\alpha\mu\le 2\alpha+1 & \\ (3\mu-2)\alpha^2+\alpha/3 & \\ \qquad\text{if } 2\alpha+1\le 3\alpha\mu\le 3\alpha+1 & \\ (4\mu-3)\alpha^2 & \text{if } 3\alpha+1\le 3\alpha\mu.\end{cases}$$

Noting the relation $\mathcal{S}_\alpha^*\subset\mathcal{C}(\varphi_\alpha,\varphi_\alpha)$, we would have an estimate for strongly starlike functions of order α. When $3\alpha\mu\le 2\alpha-1$ or $3\alpha\mu\ge 3\alpha+1$, that estimate incidentally coincides with the sharp estimate for strongly starlike functions of order α obtained previously by Ma and Minda [5].

Now, by using Lemma 2.3, we investigate convolution properties of functions in $\mathcal{C}(\varphi,\psi)$. First, we recall results due to Ma and Minda. The following form is slightly different from the original one, so we include its proof here.

Proposition 3.4 [6]**.**

(a) Let $\varphi\in\mathcal{N}(0)$. For $g\in\mathcal{K}$ and $h\in\mathcal{S}^*(\varphi)$, we have $g*h\in\mathcal{S}^*(\varphi)$.

(b) Let $\varphi\in\mathcal{N}(1/2)$. For $g\in\mathcal{S}^*(1/2)$ and $h\in\mathcal{S}^*(\varphi)$, we have $g*h\in\mathcal{S}^*(\varphi)$.

Proof. First, we prove (a). Set $G=zh'/h\prec\varphi$. Since $z(g*h)'=g*(zh')=g*(Gh)$, from Lemma 2.3, we see

$$\frac{z(g*h)'(z)}{(g*h)(z)}=\frac{(g*Gh)(z)}{(g*h)(z)}\in\overline{\mathrm{co}}G(\mathbf{D})\subset\overline{\varphi(\mathbf{D})}.$$

Hence, we have $z(g*h)'/g*h\prec\varphi$. Assertion (b) can be shown similarly. □

With the aid of the above result, we can now prove the following.

Theorem 3.5.

(a) Let $\varphi\in\mathcal{N}(0)$ and $\psi\in\mathcal{N}$. Then, for $g\in\mathcal{K}$ and

$f \in \mathcal{C}(\varphi, \psi)$, we have $g * f \in \mathcal{C}(\varphi, \psi)$.

(b) Let $\varphi \in \mathcal{N}(1/2)$ and $\psi \in \mathcal{N}$. Then, for $g \in \mathcal{S}^*(1/2)$ and $f \in \mathcal{C}(\varphi, \psi)$, we have $g * f \in \mathcal{C}(\varphi, \psi)$.

Proof. We show only (a). We can handle (b) in the same fashion. Let $\varphi \in \mathcal{N}(0)$ and $\psi \in \mathcal{N}$. If $f \in \mathcal{C}(\varphi, \psi)$, there is a function $h \in \mathcal{S}^*(\varphi)$ such that $zf'/h \prec \psi$. Set $G(z) = zf'(z)/h(z)$. Then $G(\mathbf{D}) \subset \psi(\mathbf{D})$ and $z(g*f)' = g*(zf') = g*Gh$. Since $\psi(\mathbf{D})$ is convex and since $z(g*f)'/(g*h)$ is analytic, Lemma 2.3 implies that

$$\frac{z(g*f)'(z)}{(g*h)(z)} = \frac{(g*Gh)(z)}{(g*h)(z)}$$

lies in $\psi(\mathbf{D})$, in other words, $z(g*f)'/g*h \prec \psi$. Now Proposition 3.4 ensures $g*h \in \mathcal{S}^*(\varphi)$. Hence we find from definition (1.2) that $g*f \in \mathcal{C}(\varphi, \psi)$, which completes the proof of Theorem 3.5. □

Remark. If we apply the above theorem to the case $\varphi = \psi_{0,0}$ and $\psi = \psi_{\beta,0}$ for $|\beta| < \pi/2$, then Theorem 3.5 would immediately yield that $f*g \in \mathcal{C}$ for $f \in \mathcal{C}$ and $g \in \mathcal{K}$ (see [1, Theorem 8.7]).

References

[1] Duren, P. L.: Univalent Functions. Springer, Berlin-Heidelberg-New York (1983).

[2] Goodman, A. W.: Univalent Functions, vol. 2. Mariner Pub. Co., Tampa (1983).

[3] Kim, Y. C., and Sugawa, T.: Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions (preprint).

[4] Koepf, W.: On the Fekete-Szegö problem for close-to-convex functions. Proc. Amer. Math. Soc., **101**, 89–95 (1987).

[5] Ma, W., and Minda, D.: An internal geometric characterization of strongly starlike functions. Ann. Univ. Mariae Curie-Skłodowska, Sect. A, **45**, 89–97 (1991).

[6] Ma, W., and Minda, D.: A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis (eds. Li, Z., Ren, F., Yang, L., and Zhang, S.). Internat. Press, Cambridge, MA, pp.157–169 (1992).

[7] Rogosinski, W. W.: On the coefficients of subordinate functions. Proc. London Math. Soc., **48**, 48–82 (1943).

[8] Ruscheweyh, St., and Sheil-Small, T.: Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture. Comment. Math. Helv., **48**, 119–135 (1973).