On metaplectic representations of unitary groups: II. Character formula

By Atsushi Murase

Department of Mathematics, Faculty of Science, Kyoto Sangyo University,
Kamigamo-Motoyama, Kita-ku, Kyoto 603-8555

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Abstract: Character formulas of metaplectic representations of unitary groups are given.

Key words: Metaplectic representation; Weil representation; unitary group; lattice model; character formula.

0. Introduction. Let G be a unitary group of degree n over a non-archimedean local field F of characteristic different from 2 and M a metaplectic representation of G. The object of the paper is to give a simple proof of character formula for M (Theorem 1.4). The main ingredient of the proof is a trace formula for M realized on a lattice model.

The character formula for M is first studied by Howe [H] when F is a finite field and $G = Sp_n$. D. Prasad [P] studied in the case where $G = U(1)$ and the residual characteristic of F is odd. For the archimedean case, we refer to Adams [A].

1. Main result. 1.1. Let K be a quadratic extension of a non-archimedean local field F of characteristic different from 2 and σ the nontrivial automorphism of K/F. Let $G = U(Q)$ be the unitary group of an anti-Hermitian matrix $Q \in GL_n(K)$. Define a nondegenerate alternating form $\langle\ ,\ \rangle$ on $W = K^n$ by $\langle w, w'\rangle = \mathrm{Tr}_{K/F}(w^*Qw')$ $(w, w' \in W)$, where $w^* = {}^t w^\sigma$. Denote by $H = W \times F$ the Heisenberg group attached to $(W, \langle\ ,\ \rangle)$ ([MVW, Ch. 2]; see also [M,§1.2]). Hereafter we fix a nontrivial additive character ψ of F. Let (ρ, V) be a smooth irreducible representation of H with central character $(0, x) \mapsto \psi(x)$. To simplify the notation, we write $\rho(w, x)$ for $\rho((w, x))$.

1.2. We now recall the definition of the metaplectic representation M of G attached to (ρ, V) given in [MVW, Ch.2]. If $g = 1$, we set $M(g) = \mathrm{Id}_V$. If $g \neq 1$, we put

$$M(g)v = \int_L \psi\Big(\frac{1}{2}\langle w, gw\rangle\Big)\rho((1-g)w, 0)v d_g w$$

for $v \in V$. Here L is a sufficiently large lattice of $W_g = W/\mathrm{Ker}(g-1)$ and $d_g w$ is the Haar measure on W_g self-dual with respect to the pairing $(w, w') \mapsto \psi(\langle w, (g-1)w'\rangle)$.

1.3. Let $\mathcal{X}$ be the set of unitary characters χ of $K^\times$ with $\chi|_{F^\times} = \omega$, where ω denotes the quadratic character of $F^\times$ corresponding to K/F. In [M], we have constructed a family $\{\mathcal{M}_\chi\}$ of splittings of M parametrized by $\chi \in \mathcal{X}$ given as follows. For $g \in G, g \neq 1$, we put $\nu_g = \dim_K(W_g)$ and

$$\xi_g = \det\big((w_i^* Q(g-1)w_j)_{1\le i,j\le \nu_g}\big) \in K^\times/N_{K/F}(K^\times),$$

where $\{w_i\}_{1\le i\le \nu_g}$ is a K-basis of W_g. For $g = 1$, we put $\nu_g = 0$ and $\xi_g = 1$. For $\chi \in \mathcal{X}$ and $g \in G$, we set

$$\gamma_\chi(g) = \lambda_K(\psi)^{-\nu_g}\chi(\xi_g),$$

where $\lambda_K(\psi)$ is the Weil constant ([W]; see also [M,§1.5]). Then

$$g \mapsto \mathcal{M}_\chi(g) = \gamma_\chi(g)M(g)$$

defines a smooth representation of G on V ([M, Theorem 1.8]). The main result of this paper is stated as follows:

1.4. Theorem. *The character of $\mathcal{M}_\chi(g)$ at $g \in G' = \{g \in G \mid \det(g-1) \neq 0\}$ is equal to*

$$\frac{\gamma_\chi(g)}{|\det(g-1)|_K^{1/2}},$$

where $|\cdot|_K$ is the normalized valuation of K.

Remark. A similar formula holds for metaplectic representations of Sp_n.

2. Proof of the main result. 2.1. In this section, we prove Theorem 1.4 by taking a lattice model as (ρ, V) and using a trace formula for $M(g)$ on V. We keep the notation of §1. By a lattice of $W = K^n$ we mean an $\mathcal{O}_F$-lattice of W. A lattice L of W is said to be *self-dual* if L coincides with its dual lattice $L^* = \{z \in W \mid \psi(\langle z, w\rangle) = 1 \text{ for any } w \in L\}$.

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2.2. Lemma. *There exist a lattice $\mathcal{L}$ of W and a smooth function $\alpha\colon \mathcal{L} \to F$ satisfying the following three conditions*:

(2.1) *$\mathcal{L}$ is self-dual.*

(2.2) $\alpha(0) = 0$ *and* $\alpha(-l) = \alpha(l)$ $(l \in \mathcal{L})$.

(2.3) $\psi\Big(\alpha(l_1 + l_2) - \alpha(l_1) - \alpha(l_2) + \dfrac{1}{2}\langle l_1, l_2\rangle\Big) = 1$ $(l_1, l_2 \in \mathcal{L})$.

Proof. Take an $A \in GL_n(K)$ so that $Q' = A^*QA$ is diagonal. Set

$$\mathcal{L} = A(\mathcal{O}_F^n + (2Q')^{-1}(\mathcal{D}_\psi^{-1})^n),$$

where $\mathcal{D}_\psi^{-1} = \{x \in F \mid \psi(xy) = 1 \text{ for any } y \in \mathcal{O}_F\}$. Let $l = A(x + (2Q')^{-1}y) \in \mathcal{L}$ and $w = A(u + (2Q')^{-1}v) \in W$, where $x \in \mathcal{O}_F^n, y \in (\mathcal{D}_\psi^{-1})^n, u, v \in F^n$. Since $\langle l, w\rangle = {}^t xv - {}^t yu$, we see that $\mathcal{L}$ is self-dual. Set

$$\alpha(l) = \frac{1}{4}\operatorname{Tr}_{K/F}\left({}^t(A^{-1}l)Q'(A^{-1}l)\right) \quad (l \in \mathcal{L}).$$

Clearly α satisfies (2.2). Let $l_1, l_2 \in \mathcal{L}$ and put $l_i' = A^{-1}l_i = x_i + (2Q')^{-1}y_i$ $(x_i \in \mathcal{O}_F^n, y_i \in (\mathcal{D}_\psi^{-1})^n)$ for $i = 1, 2$. Since ${}^tQ' = Q'$, we have

$$\begin{aligned}&\alpha(l_1 + l_2) - \alpha(l_1) - \alpha(l_2) + \frac{1}{2}\langle l_1, l_2\rangle \\ &= \operatorname{Tr}_{K/F}\left\{\frac{1}{4}\left({}^tl_2'Q'l_1' + {}^tl_1'Q'l_2'\right) + \frac{1}{2}(l_1')^*Q'l_2'\right\} \\ &= \frac{1}{2}\operatorname{Tr}_{K/F}({}^t(l_1' + (l_1')^\sigma)Q'l_2') = {}^tx_1y_2 \in \mathcal{D}_\psi^{-1},\end{aligned}$$

which implies (2.3). □

2.3. From now on, we fix a lattice $\mathcal{L}$ of W and a function $\alpha\colon \mathcal{L} \to F$ satisfying the conditions of Lemma 2.2. Note that (2.2) and (2.3) imply $\psi(\alpha(l)) = \pm 1$ for $l \in \mathcal{L}$. We normalize a Haar measure dz on W by $\operatorname{vol}(\mathcal{L}) = 1$. Define a function $\psi_\mathcal{L}$ on $H_\mathcal{L} = \mathcal{L} \times F$ by $\psi_\mathcal{L}((l, x)) = \psi(\alpha(l) + x)$ for $(l, x) \in H_\mathcal{L}$. In view of (2.3), $\psi_\mathcal{L}$ is a character of $H_\mathcal{L}$. By general theory (cf. [MVW, Ch.2, I.3]), $\operatorname{Ind}_{H_\mathcal{L}}^H \psi_\mathcal{L}$ is a smooth irreducible representation of H with central character $(0, x) \mapsto \psi(x)$. It is straightforward to see that $\operatorname{Ind}_{H_\mathcal{L}}^H \psi_\mathcal{L}$ is equivalent to (ρ, V), where

$$\begin{aligned}V = \{\Phi \in \mathcal{S}(W) \mid &\Phi(z + l) \\ &= \psi\left(\frac{1}{2}\langle z, l\rangle + \alpha(l)\right)\Phi(z) \ (z \in W, l \in \mathcal{L})\}\end{aligned}$$

and

$$\rho(h)\Phi(z) = \psi\left(\frac{1}{2}\langle z, w\rangle + x\right)\Phi(z + w)$$
$$(h = (w, x) \in H, \Phi \in V, z \in W).$$

Here $\mathcal{S}(W)$ stands for the space of locally constant and compactly supported functions on W. We call the realization (ρ, V) the *lattice model*. Define an H-invariant inner product on V by

$$(\Phi, \Phi') = \int_W \Phi(z)\overline{\Phi'(z)}dz \qquad (\Phi, \Phi' \in V).$$

2.4. Set

$$\Phi_0(z) = \begin{cases}\psi(\alpha(z)) & \cdots\ z \in \mathcal{L} \\ 0 & \cdots\ z \in W - \mathcal{L}.\end{cases}$$

Then Φ_0 belongs to V and satisfies the following:

(2.4) We have $\rho(l, -\alpha(l))\Phi_0 = \Phi_0$ for $l \in \mathcal{L}$.

(2.5) For $\Phi \in V$, we have

$$\Phi(z) = (\rho(z, 0)\Phi, \Phi_0) \quad (z \in W).$$

2.5. We take a $\nu \in \mathcal{D}_{K/F}^{-1}$ satisfying $\nu + \nu^\sigma = 1$. For $w \in W$, put $x_w = 2^{-1}(\nu - \nu^\sigma)w^*Qw \in F$. For a lattice L of W, we put $H(L) = \{(w, x_w + x) \mid w \in L, x \in \mathfrak{a}_L\}$, where $\mathfrak{a}_L$ is the fractional ideal of F generated by $\operatorname{Tr}_{K/F}(\nu l^*Ql')$ $(l, l' \in L)$. Then $H(L)$ is an open compact subgroup of H. Let $V(L) = V^{H(L)}$ be the $H(L)$-invariant subspace of V. We have $V(L) = \{0\}$ unless

$$\psi|_{\mathfrak{a}_L} = 1. \tag{2.6}$$

For a lattice L satisfying (2.6), put

$$\mathcal{P}_L = \int_L \rho(w, x_w)d_Lw \in \operatorname{End}(V),$$

where d_Lw is the Haar measure on W normalized by $\operatorname{vol}(L) = 1$. Then we have $\mathcal{P}_L^2 = \mathcal{P}_L$ and $V(L) = \{\Phi \in V \mid \mathcal{P}_L\Phi = \Phi\}$.

2.6. Let M be the metaplectic representation of G attached to the lattice model (ρ, V) as in §1.2. It is easily seen that

(2.7)
$$(M(g)\Phi, \Phi') = (\Phi, M(g^{-1})\Phi') \quad (g \in G, \Phi, \Phi' \in V)$$

and that $\mathcal{P}_LM(g)$ is of trace class. To prove Theorem 1.4, it is sufficient to show the following fact:

2.7. Proposition. *For $g \in G'$, there exists a lattice L_g of W such that, for any lattice L with $L \subset L_g$, we have*

$$\operatorname{Tr}(\mathcal{P}_LM(g)) = |\det(g - 1)|_K^{-1/2}.$$

Proof. Take a sufficiently small lattice L_g of W such that $\psi|_{\mathfrak{a}_{L_g}} = 1$ and $x_w + (1/2)\langle(g - 1)^{-1}w, w\rangle \in \mathcal{D}_\psi^{-1}$ holds for any $w \in L_g$. By (2.5) and (2.7), we have

$$M(g)\Phi(z) = \int_W k_g(z,z')\Phi(z')dz' \quad (\Phi \in V, z \in W)$$

with a kernel function

$$k_g(z,z') = \overline{(M(g^{-1})\rho(-z,0)\Phi_0)(z')}.$$

Let L be any lattice of W contained in L_g. Then

$$\mathcal{P}_L M(g)\Phi(z) = \int_W k_{g,L}(z,z')\Phi(z')dz',$$

where

$$k_{g,L}(z,z') = \int_L \psi\left(\frac{1}{2}\langle z,w\rangle + x_w\right) k_g(z+w,z')d_Lw.$$

Observe that $z \mapsto k_{g,L}(z,z')$ is in $V(L)$ and $z' \mapsto \overline{k_{g,L}(z,z')}$ in V. Hence we have

$$\mathrm{Tr}(\mathcal{P}_L M(g)) = \int_{L_1} k_{g,L}(z,z)dz$$

with a sufficiently large lattice L_1 of W (depending only on g and L). Taking a sufficiently large lattice L_2 of W, we have

$$\begin{aligned} &k_g(z+w,z) \\ &\quad = \int_{g(L_2)} \psi\left(-\frac{1}{2}\langle w', g^{-1}w'\rangle\right) \\ &\quad \overline{\rho((1-g^{-1})w',0)\rho(-z-w,0)\Phi_0(z)}d_{g^{-1}}w' \\ &\quad = \int_{L_2} \psi\Big(\frac{1}{2}\langle w', gw'\rangle + \frac{1}{2}\langle (g-1)w', z+w\rangle \\ &\quad -\frac{1}{2}\langle z, (g-1)w'-w\rangle\Big)\overline{\Phi_0((g-1)w'-w)}d_gw' \end{aligned}$$

for $(z,w) \in L_1 \times L$, and hence

$$\begin{aligned} &\mathrm{Tr}(\mathcal{P}_L M(g)) \\ &= \int_{L_1} dz \int_L d_Lw\ \psi\left(\frac{1}{2}\langle z,w\rangle + x_w\right) k_g(z+w,z) \\ &= \int_{L_1} dz \int_L d_Lw \int_{L_2} d_gw' \\ &\quad \psi(\langle z, w-(g-1)w'\rangle + x_w + \frac{1}{2}\langle w', (g-1)w'\rangle \\ &\quad +\frac{1}{2}\langle (g-1)w', w\rangle)\overline{\Phi_0((g-1)w'-w)}. \end{aligned}$$

We may assume that $(g-1)^{-1}L \subset L_2$. Changing the variable w' into $w' + (g-1)^{-1}w$, we have

$$\begin{aligned} &\mathrm{Tr}(\mathcal{P}_L M(g)) \\ &= \int_L d_Lw \int_{L_2} d_gw' \int_{L_1} dz\ \psi(\langle z, -(g-1)w'\rangle) \\ &\quad \psi\left(x_w + \frac{1}{2}\langle (g-1)^{-1}w, w\rangle + \langle w', g^{-1}w\rangle\right) \\ &\quad \overline{\Phi_0((g-1)w')} \\ &= \mathrm{vol}(L_1)\int_L d_Lw \int_{L_2\cap(g-1)^{-1}L_1^*} d_gw'\psi(\langle w', g^{-1}w\rangle) \\ &\quad \overline{\Phi_0((g-1)w')}, \end{aligned}$$

since $\psi\left(x_w + (1/2)\langle (g-1)^{-1}w, w\rangle\right) = 1$ for $w \in L$. Taking L_1^* sufficiently small, we obtain

$$\begin{aligned} \mathrm{Tr}(\mathcal{P}_L M(g)) &= \mathrm{vol}(L_1)\,\mathrm{vol}((g-1)^{-1}L_1^*)\frac{d_gw'}{dw'} \\ &= |\det(g-1)|_K^{-1/2} \end{aligned}$$

as claimed. □

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