On an elliptic curve over $\mathbf{Q}(t)$ of rank ≥ 14

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Abstract: We show an elliptic curve over $\mathbf{Q}(t)$ of rank ≥ 14.

Key words: Elliptic curve; rank.

In [2] and [3] Mestre showed an elliptic curve over $\mathbf{Q}(t)$ of rank ≥ 11 and ≥ 12 respectively. Also Nagao showed an example of rank ≥ 13 in [4]. We showed by using Nagao's example that there are infinitely many elliptic curves over $\mathbf{Q}$ of rank ≥ 14 in [1]. In this note we show an elliptic curve over $\mathbf{Q}(t)$ of rank at least 14.

Let $a_1 = 0$, $a_2 = (2p^2 + pq + 2q^2)^2$, $a_3 = 2(p + q)^2(2p^2+pq+q^2)$, $a_4 = q^2(4p^2-pq+4q^2)$, $a_5 = p(2p-q)(2p^2+4pq+5q^2)$, $a_6 = 4p^4+8p^3q+9p^2q^2-2pq^3+2q^4$ and $b_i = u + a_i$, $b_{i+6} = -u + a_i$ $(1 \leq i \leq 6)$.

We consider the polynomial $F(x) = \prod_{i=1}^{12}(x - b_i) \in K[x]$ where $K = \mathbf{Q}(u, p, q)$. There are unique polynomials $G(x)$, $r(x) \in K[x]$ with $\deg G(x) = 6$, $\deg r(x) \leq 5$ and $F(x) = G(x)^2 - r(x)$.

By the direct calculation, we see the x^5-coefficient of $r(x)$ is 0, and $\deg r(x) = 4$. Then the curve

$$\mathcal{E} \qquad\qquad y^2 = r(x)$$

is an elliptic curve over K with 12 K-rational points $\mathrm{P}_i = (x_i, y_i)$ $(1 \leq i \leq 12)$ where $x_i = b_i$, $y_i = G(b_i)$. From now on we express the x-coordinate of K-rational point P on $\mathcal{E}$ as $x(\mathrm{P})$.

Then there is one more K-rational point P_{13} on $\mathcal{E}$ where

$$x(\mathrm{P}_{13}) = \left(\frac{2p^2 + 4pq + 5q^2}{2p^2 + 2pq + 3q^2}\right) u + \frac{1}{2p^2 + 2pq + 3q^2} \times (8p^6 + 28p^5q + 58p^4q^2 + 69p^3q^3 + 76p^2q^4 + 40pq^5 + 22q^6).$$

Now by specializing $p = t^2(8 + 3t^2)$, $q = -6(2 + t^2) \times(4 + t^2)$ and $u = 4(2 + t^2)(2304 + 2400t^2 + 928t^4 + 150t^6+9t^8)(1152+1632t^2+860t^4+201t^6+18t^8)/t$, we have 2 $\mathbf{Q}(t)$-rational points P_{14} and P_{15} on $\mathcal{E}$ where

$$\begin{aligned} x(\mathrm{P}_{14}) &= -4(1152 + 1632t^2 + 860t^4 + 201t^6 + 18t^8) \\ &(10616832 - 18579456t + 33619968t^2 - 51535872t^3 \\ &+45895680t^4 - 61848576t^5 + 35397888t^6 \\ &-41945856t^7 + 16968640t^8 - 17591104t^9 \\ &+5232272t^{10} - 4675248t^{11} + 1035180t^{12} \\ &-769824t^{13} + 126252t^{14} - 71874t^{15} \\ &+8559t^{16} - 2916t^{17} + 243t^{18})/(t(2304 + 3168t^2 \\ &+1580t^4 + 339t^6 + 27t^8)) \end{aligned}$$

$$\begin{aligned} x(\mathrm{P}_{15}) &= 4(-48 + 24t - 34t^2 + 16t^3 - 6t^4 + 3t^5) \\ &\times(96 + 80t^2 + 4t^3 + 18t^4 + 3t^5) \\ &\times(1152 + 1632t^2 + 860t^4 + 201t^6 + 18t^8)/t. \end{aligned}$$

Now we regard $\mathcal{E}$ as an elliptic curve over $\mathbf{Q}(t)$ whose group structure is given by P_{15} as origin.

Then we have the following,

Theorem. *$\mathbf{Q}(t)$-rank of $\mathcal{E}$ is at least* 14.

Proof. This is shown by specializing $t = 2$. Let $\mathrm{R}_1, \ldots, \mathrm{R}_{14}$ be the rational points obtained from $\mathrm{P}_1, \ldots, \mathrm{P}_{14}$ by the above specialization, by using calculation system PARI, we see that the determinant of the matrix $(\langle \mathrm{R}_i, \mathrm{R}_j\rangle)$ $(1 \leq i, j \leq 14)$ associated to the canonical height is 221792776617402574.10. Since this determinant is non-zero, we see that $\mathrm{R}_1, \ldots, \mathrm{R}_{14}$ are independent points. So we see that $\mathrm{P}_1, \ldots, \mathrm{P}_{14}$ are independent. $\square$

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References

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