On Puiseux roots of Jacobians

By Tzee-Char Kuo*) and Adam Parusiński**)

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Abstract: Take holomorphic $f(x, y)$, $g(x, y)$. A *polar arc* is a Puiseux root, $x = \gamma(y)$, of the Jacobian $J = f_y g_x - f_x g_y$, but not one of $f \cdot g$. We define the tree, $T(f, g)$, using the contact orders of the roots of $f \cdot g$, describe how polar arcs climb, and leave, the tree, and how to factor J in $\mathbf{C}\{x, y\}$. When collinear points/bars exist, the way the γ's leave the tree is not an invariant.

Key words: Puiseux roots; polar arcs; Jacobian.

Take holomorphic germs $f, g : (\mathbf{C}^2, O) \to (\mathbf{C}, O)$, and a coordinate system (x, y). The Puiseux factorizations ([7]) are of the form

$$(0.1)\qquad \begin{aligned} f(x, y) &= u(x, y) \cdot y^{E_1} \cdot \prod_{i=1}^{p} [x - \alpha_i(y)], \\ g(x, y) &= u'(x, y) \cdot y^{E_2} \cdot \prod_{j=1}^{q} [x - \beta_j(y)], \end{aligned}$$

where u, u' are units; $E_1 \geq 0$, $E_2 \geq 0$; α_i, β_j are fractional power series, $O_y(\alpha_i) > 0$, $O_y(\beta_j) > 0$.

We write $\alpha_1, \dots, \beta_q$ as $\lambda_1, \dots, \lambda_N$, $N := p + q$; and assume $\lambda_i \neq \lambda_j$ if $i \neq j$.

Definition 0.1. *A polar arc* of the pair (f, g) is a Puiseux root, $x = \gamma(y)$, with $O_y(\gamma) > 0$, of the Jacobian determinant

$$J(x, y) := J_{(f,g)}(x, y) := \begin{vmatrix} f_y & f_x \\ g_y & g_x \end{vmatrix},$$

which is not one of the λ_k's, that is: $J(\gamma(y), y) = 0$, $f(\gamma(y), y) g(\gamma(y), y) \neq 0$.

We use the contact orders $O(\lambda_s, \lambda_t) := O_y(\lambda_s(y) - \lambda_t(y))$, $1 \leq s, t \leq N$, to define the tree $T(f, g)$. Our Theorems T, N, and C, describe how the γ_j's climb, and leave, the tree (like vines); Theorems F and I describe how $J(x, y)$ can be factored in $\mathbf{C}\{x, y\}$, and how to compute the intersection multiplicities of some factors (possibly reducible) with the germs $C_f := f^{-1}(0)$, $C_g := g^{-1}(0)$, and $C := (f \cdot g)^{-1}(0)$. The detailed proofs of the theorems, and additional results, will appear elsewhere.

Our results generalize that in the one function case. Taking $g(x, y) = y$, $J(x, y)$ reduces to f_x, and $T(f, y) = T(f)$, the tree defined in [3]. The curve $f_x = 0$ is called a *polar curve*, whose irreducible components are called *polar branches*. In [6], Pham showed that the Zariski equisingularity types ([8]) of the polar branches need not be determined by that of $f = 0$. However, for polar arcs (Puiseux roots of f_x) the story is different. The contact orders, $C(f, f_x) := \{O(\alpha_i, \gamma_j)\}$, between the roots α_i of f and γ_j of f_x can be calculated using $T(f)$ *alone*. This result (now a corollary of Theorem T) was first proved in [3, 5] (see also [4]). *Thus, $C(f, f_x)$ is an equisingular invariant of f.* Theorems F and I generalize the theorems of Merle and Garcia-Barroso ([1, 5]).

In the general case, $T(f, g)$ may have what we call collinear points and bars (no such things exist in the one function case), and then we encounter a completely new phenomenon. Namely, it may not be possible anymore to know precisely where some of the polar arcs leave the tree.

Conventions. A fractional power series $\lambda(y)$ is called an "arc". If $O(\lambda, \mu) > q$, we write $\lambda \equiv \mu \bmod q^+$. We use $O(y^+)$ to represent a quantity which, as $y \to 0$, has the same order as y^e, $e > 0$.

1. The tree $T(f, g)$.

To construct $T(f, g)$ (compare [3]), we first draw a horizontal bar, B_*, called the *ground bar*. Then draw a vertical line segment on B_* as the *main trunk* of the tree. Mark $[p, q]$ alongside the trunk to indicate that p α_i's and q β_j's are bundled together. Let $h_0 := \min\{O(\lambda_i, \lambda_j) \mid 1 \leq i, j \leq N\}$. Then draw a bar, B_0, on top of the main trunk. Call $h(B_0) := h_0$ the *height* of B_0. We define $h(B_*) := 0$.

The roots λ_k, $1 \leq k \leq N$, are divided into equivalence classes modulo h_0^+. Represent each equivalence class by a vertical line segment on top of B_0,

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*) School of Mathematics, University of Sydney, Sydney, NSW, 2006, Australia.

**) Département de Mathématiques, Université d'Angers, 2, BD Lavoisier, 49045 Angers, France.

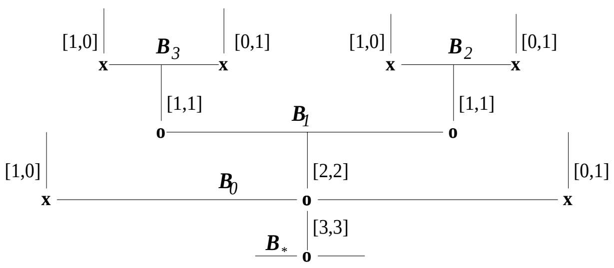

Fig. 1. (Example 1.1)

called a *trunk*. If a trunk consists of s α_i's and t β_j's ($s+t \geq 1$), it has *bi-multiplicity* $[s,t]$; we mark $[s,t]$ alongside and call $s+t$ the *total multiplicity*.

Repeat the same construction recursively on each trunk, getting more bars, then more trunks, etc. The construction finishes when all bars have infinite height; we omit drawing these bars.

Example 1.1. Take constants $A \neq 0 \neq B$, integers $0 < e < E$. Then consider

$$f(x,y) = (x+y)(x - y^{e+1} + Ay^{E+1}) \times (x + y^{e+1} + By^{E+1}),$$
$$g(x,y) = (x-y)(x - y^{e+1} - Ay^{E+1}) \times (x + y^{e+1} - By^{E+1}).$$

The tree $T(f,g)$ is shown in Fig. 1, with $h(B_0) = 1$, $h(B_1) = e+1$, $h(B_2) = h(B_3) = E+1$, where "$\circ$" and "$\times$" are defined in Convention 2.2.

Take a bar B, $h := h(B) < \infty$. Take λ_k whose modulo h^+ class is a trunk, T, on B. Let $\lambda_B(y)$ denote $\lambda_k(y)$ with all terms y^e, $e \geq h$, omitted. We write

$$\lambda_k(y) = \lambda_B(y) + cy^{h(B)} + \cdots, \quad c \in \mathbf{C},$$

where c is *uniquely* determined by T. We say T *grows* on B *at* c. Take a bar B^* on top of T. We say B^* is a *postbar* of B ([2]), *supported at* c; we write $B \perp_c B^*$, or simply $B \perp B^*$. We say B' *lies above* B (and also *above* c) if there is a postbar sequence:

$$B \perp B_1 \perp \cdots \perp B', \; B \perp_c B_1.$$

Definition 1.2. Take any arc ξ. If ξ has the form

$$\xi(y) = \lambda_B(y) + ay^{h(B)} + \cdots, \quad a \in \mathbf{C},$$

we say ξ *climbs* over B *at* a (like a vine). In this case, if no trunk grows at a we say ξ *leaves* the tree on B at a.

If $O(\xi, \lambda_B) < h(B)$, we say ξ is *bounded* by B.

2. Theorems T, N and C. Take a bar B, $h(B) < \infty$, a germ $F(x,y)$, a generic $z \in \mathbf{C}$, and $\eta(y)$. Let

$$\nu_F(B) := O_y(F(\lambda_B(y) + zy^{h(B)}, y));$$
$$\nu_F(\eta) := O_y(F(\eta(y), y)).$$

In particular, $\nu_f(B_*) = E_1$, $\nu_g(B_*) = E_2$, by (0.1).

Let T_k be the trunks on B, $1 \leq k \leq l$; T_k grows at z_k with bi-multiplicity $[p_k, q_k]$. We write

$$\Delta_B(z_k) := \begin{vmatrix} \nu_f(B) & p_k \\ \nu_g(B) & q_k \end{vmatrix}, \quad 1 \leq k \leq l.$$

Define the *rational function associated to* B by

$$\mathcal{M}_B(z) := \sum_{k=1}^{l} \frac{\Delta_B(z_k)}{z - z_k}, \quad z \in \mathbf{C}.$$

Definition 2.1. We say z_k, $1 \leq k \leq l$, is a *collinear* point on B if $\Delta_B(z_k) = 0$; otherwise, *non-collinear*.

Let $C(B)$ and $N(B)$ denote respectively the sets of collinear and non-collinear points:

$$C(B) \cup N(B) = \{z_1, \ldots, z_l\}.$$

Their (finite) cardinal numbers are denoted by $c(B)$ and $n(B)$ respectively.

Convention 2.2. A collinear point is indicated by $\circ$; a non-collinear one by $\times$.

Definition 2.3. We call B *collinear* if $\Delta_B(z_k) = 0$ for all k, $1 \leq k \leq l$; otherwise, call it *non-collinear*. Call B *purely non-collinear* if $C(B) = \emptyset$ ($\neq N(B)$).

In Example 1.1, B_1 and B_* are collinear, B_2, B_3 are purely non-collinear.

If $\mathcal{M}_B(z) = 0$, z is called a *mero-zero* on B. Let $m_B(z)$ denote its multiplicity. Let $M(B)$ denote the set of mero-zeros. We write: $m(B) := \sum_{z \in M(B)} m_B(z)$.

Suppose $N(B) \neq \emptyset$. A non-collinear z_k is a pole, hence not a mero-zero:

$$N(B) \cap M(B) = \emptyset, \quad n(B) \geq m(B) + 1.$$

On the other hand it may happen that $C(B) \cap M(B) \neq \emptyset$. If $z \in M(B) \setminus C(B)$, we say z is a *pure mero-zero*. It can happen that $\Delta(z_k) = 0$ for all k; in this case, $\mathcal{M}_B \equiv 0$, $N(B) = \emptyset$, and $M(B) = \mathbf{C}$. It can also happen that $M(B) = \emptyset$. (Take $f(x,y) = x$ and $g(x,y) = x^2 - y^2$. Then $\mathcal{M}_B(z) = 2z^{-1}[z^2 - 1]^{-1}$.)

Take a non-collinear bar B. Define the *total multiplicity function* by

$$\tau_B(z) = \begin{cases} p_k + q_k, & \text{if } z = z_k; \\ 0, & \text{otherwise,} \end{cases}$$

and the *mero-multiplicity function* by

$$\mu_B(z) = \begin{cases} m_B(z), & \text{if } z \in M(B); \\ -1, & \text{if } z \in N(B); \\ 0, & \text{otherwise.} \end{cases}$$

We also write

$$\tau(B) := \sum_{z \in \mathbf{C}} \tau_B(z), \ \mu(B) := \sum_{z \in \mathbf{C}} \mu_B(z).$$

Note that $\mu(B) = m(B) - n(B)$, a negative integer.

Let $\mathcal{T}_B(z)$ denote the total number of polar arcs (counting multiplicities) which climb over B at z, and $\mathcal{T}(B)$ denote that of those which climb over B.

Theorem T. *Let B be a non-collinear bar. Then*

$$\mathcal{T}_B(z) = \tau_B(z) + \mu_B(z), \quad z \in \mathbf{C}, \tag{2.1}$$

and, consequently,

$$\mathcal{T}(B) = \tau(B) + \mu(B). \tag{2.2}$$

In particular, if a polar arc climbs over B at z, then

$$z \in N(B) \cup C(B) \cup M(B).$$

Corollary 2.4. *Let z be a pure mero-zero. There are exactly $m_B(z)$ polar arcs (counting multiplicities) climbing over B at z. (They all leave $T(f, g)$ at z.)*

Corollary 2.5. *If $\sum_{z_k \in N(B)} \Delta_B(z_k) \neq 0$, then*

$$m(B) + 1 = n(B); \ \mathcal{T}(B) = \sum_{k=1}^{l} (p_k + q_k) - 1.$$

In particular, if $pE_2 - qE_1 \neq 0$, the total number of polar arcs is $p + q - 1$.

Theorem N. *Take $z \in N(B)$. Let B^* be the postbar of B supported at z. Then $m(B^*) + 1 = n(B^*)$; in particular, B^* is non-collinear. Moreover, every polar arc which climbs over B at z must also climb over B^*. That is, there is no polar arc, γ, such that*

$$h(B) < O(\gamma, \lambda_{B^*}) < h(B^*).$$

Take $c \in C(B)$. A set $\{\bar{B}_1, \ldots, \bar{B}_r\}$ of non-collinear bars is called a (non-collinear) *cover* of c if the following holds. Each $\bar{B}_s$ lies over c and is *minimal* in the sense that there is a postbar sequence

$$B \perp B_1^* \perp \cdots \perp B_{r(s)}^* \perp \bar{B}_s, \ B \perp_c B_1^*,$$

where either $r(s) = 0$ (i.e. $B \perp_c \bar{B}_s$), or else all B_i^*, $1 \le i \le r(s)$, are collinear. Moreover, each root λ_k climbing over B at c also climbs over a (unique) $\bar{B}_s$. (In Fig. 1, $\{B_2, B_3\}$ is a cover of $0 \in C(B_0)$.)

Theorem C. *Let B be a non-collinear bar. Take $c \in C(B)$ with cover $\{\bar{B}_1, \ldots, \bar{B}_r\}$. Then there are exactly $m_B(c) + \sum_{s=1}^{r} [n(\bar{B}_s) - m(\bar{B}_s)]$ polar arcs which climb over B at c, bounded by every $\bar{B}_s$, $1 \le s \le r$.*

To prove Theorem T, take B, non-collinear. Let

$$\mathcal{M}_B(z, y) := \begin{vmatrix} \nu_f(B) \displaystyle\sum_{i=1}^{p} \frac{y^{h(B)}}{x - \alpha_i(y)} \\ \nu_g(B) \displaystyle\sum_{j=1}^{q} \frac{y^{h(B)}}{x - \beta_j(y)} \end{vmatrix},$$

where $x := \lambda_B(y) + z y^{h(B)}$ is a substitution.

Note that $\mathcal{M}_B(z) = \mathcal{M}_B(z, 0)$ $(\not\equiv 0)$, whence, by Rouché's Theorem, for $|y|$ small,

$$\oint_{\mathcal{C}} \frac{d}{dz} \log \mathcal{M}_B(z, y)\, dz = \oint_{\mathcal{C}} \frac{d}{dz} \log \mathcal{M}_B(z)\, dz.$$

We can write $J(x, y)$ as

$$\begin{aligned} J(x, y) &= y^{-1} f g \begin{vmatrix} \dfrac{y f_y}{f} & \dfrac{f_x}{f} \\ \dfrac{y g_y}{g} & \dfrac{g_x}{g} \end{vmatrix} \\ &= y^{-h(B)-1} f g [\mathcal{M}_B(z, y) + \mathcal{P}_B(z, y)], \end{aligned}$$

$$\begin{aligned} \mathcal{P}_B(z, y) := &\begin{vmatrix} \dfrac{y f_y}{f} - \nu_f(B) & y^{h(B)} \dfrac{f_x}{f} \\ \dfrac{y g_y}{g} - \nu_g(B) & y^{h(B)} \dfrac{g_x}{g} \end{vmatrix} \\ &+ y^{h(B)} \begin{vmatrix} \nu_f(B) & \dfrac{u_x}{u} \\ \nu_g(B) & \dfrac{u'_x}{u'} \end{vmatrix}. \end{aligned}$$

Let $a \in \mathbf{C}$, $|y| \ll \varepsilon$, and $\mathcal{C}$ the contour $|z - a| = \varepsilon$. Then

$$\begin{aligned} &\oint_{\mathcal{C}} \frac{d}{dz} \log[\mathcal{M}_B(z, y) + \mathcal{P}_B(z, y)]\, dz \\ &= \oint_{\mathcal{C}} \frac{d}{dz} \log \mathcal{M}_B(z)\, dz = 2\pi i \mu_B(a). \end{aligned}$$

Take $a = z_k$ on B. There are $\tau_B(z_k)$ roots of $f(x,y)g(x,y)$ within $|z - z_k| = \varepsilon$. Hence $J(x,y)$ has $\tau_B(z_k) + \mu_B(z_k)$ roots therein. Since ε is arbitrarily small, these roots all climb over B at z_k.

Theorems N and C are derived from Theorem T.

3. Theorems F and I. We say B is *conjugate* to $\bar{B}$, written as $B \sim \bar{B}$, if $h(B) = h(\bar{B})$ and there exists an irreducible $p(x,y) \in \mathbf{C}\{x,y\}$, of which one (Puiseux) root climbs over B and one climbs over $\bar{B}$. In this case, *any* irreducible $q(x,y)$ which has a root climbing over B must have one climbing over $\bar{B}$. For a conjugate class $\mathbf{B}$, either all $B \in \mathbf{B}$ are collinear, or else all are non-collinear.

Let $\mathbf{B}_i$, $1 \le i \le s$, denote the set of all conjugate classes of non-collinear bars with positive height. Take i, define $P_i(x,y) := \prod_j [x - \gamma_j(y)]$, taking over all j such that γ_j *leaves* the tree on some $B \in \mathbf{B}_i$. Define $Q_i(x,y) := \prod [x - \gamma_j(y)]$, taking over all j such that γ_j climbs over some $B \in \mathbf{B}_i$ at a collinear point c, bounded by every bar of the cover of c. If $\mathbf{B}_i$ consists of purely non-collinear bars, define $Q_i(x,y) := 1$. For the ground bar B_*, let $Q_{B_*}(x,y) := \prod [x - \gamma_j(y)]$, taking over all j such that γ_j is bounded by all non-collinear bars of minimal height.

Theorem F. *The Jacobian admits a factorization*

$$J(x,y) = unit \cdot y^E \cdot Q_{B_*}(x,y) \cdot \prod_i^s P_i(x,y) \cdot Q_i(x,y),$$

in $\mathbf{C}\{x,y\}$, *where* $E \ge 0$.

Let P_i denote the germ $P_i(x,y) = 0$, and $m^*(B)$ the number of pure mero-zeros on B. Let $m^*(\mathbf{B}) := \sum_{B \in \mathbf{B}} m^*(B)$, $\nu_f(\mathbf{B}) := \nu_f(B)$, $\nu_g(\mathbf{B}) := \nu_g(B)$.

Theorem I. *The intersection multiplicities with* P_i *are as follows*:

$$I(C_f, P_i) = \nu_f(\mathbf{B}) m^*(\mathbf{B}); \; I(C_g, P_i) = \nu_g(\mathbf{B}) m^*(\mathbf{B});$$
$$I(C, P_i) = [\nu_f(\mathbf{B}) + \nu_g(\mathbf{B})] m^*(\mathbf{B}), \quad 1 \le i \le s.$$

Note that we have no formulae for $I(Q_i, C_f)$, etc.

4. What theorem C does not say. Theorem C does not say precisely where the polar arcs leave the tree. We use examples to show that the coefficients of the λ_i's may also play a rôle. First, take $e < E < 2e$ in Example 1.1, where $\nu_f(B_2) = \nu_g(B_2) = \nu_f(B_3) = \nu_g(B_3) = E + e + 3$, B_1 being collinear. By Theorem T, there are four polar arcs climbing over B_0, all at 0. Put $x = zy^{e+1}$. Then

$$\frac{yg_y}{g} - \frac{yf_y}{f} = 2y^{E-e} \left[\frac{ezy^{2e-E}}{z^2 y^{2e} - 1} - \frac{(E-e)A(z-1)}{(z-1)^2 - A^2 y^{2(E-e)}} - \frac{(E-e)B(z+1)}{(z+1)^2 - B^2 y^{2(E-e)}} \right];$$

$$\frac{g_z}{g} - \frac{f_z}{f} = 2y^{E-e} \left[\frac{y^{2e-E}}{z^2 y^{2e} - 1} + \frac{A}{(z-1)^2 - A^2 y^{2(E-e)}} + \frac{B}{(z+1)^2 - B^2 y^{2(E-e)}} \right];$$

$$J(x,y) = y^{-e-2} \cdot f \cdot g \cdot \begin{vmatrix} \dfrac{yf_y}{f} & \dfrac{f_z}{f} \\ \dfrac{yg_y}{g} & \dfrac{g_z}{g} \end{vmatrix} = 2y^{E-2e-2} \cdot f \cdot g \cdot \Delta(z,y),$$

where, for $y = 0$, the shorthand $\Delta(z,y)$ reduces to

$$\Delta(z,0) = (z^2-1)^{-2}[(A+B)(2E+3)z^2 + 2(A-B)(E+e+3)z + (2e+3)(A+B)].$$

Observe that if $A + B \ne 0$, there are two zeros. This means that two polar arcs climb over B_1, the remaining two are bounded by B_1. If, however, $A + B = 0$, then there is only one zero. This means that one polar arc climbs over B_1, three are bounded by B_1. *Thus, in general, one cannot tell the positions of polar arcs relative to collinear bars.*

Example 4.1. Take $N > 0$, $2e > E > e > 0$. Let

$$f(x,y) := [x^2 - y^{2(e+1)}][(x-y)^2 - y^{2(e+1+N)}],$$
$$g(x,y) := [x + y^{E+1}][x + y].$$

There are four bars, B_*, B_1, B_2, B_3, with

$$h(B_1) = 1, \; h(B_2) = e+1, \; h(B_3) = e+1+N;$$
$$\mathcal{M}_{B_1}(z) = \frac{8}{z^2-1}, \quad \mathcal{M}_{B_2}(z) = \frac{-2(e+2)}{z(z^2-1)}.$$

By Theorem C, three polar arcs, γ_i, $1 \le i \le 3$, climb over B_1 at 0, bounded by B_2. Let us write $x = Xy$. The arcs $\eta_i(y) := y^{-1}\gamma_i(y)$ are Puiseux roots of

$$X^3(8 + \cdots) - Xy^E[2(E+2) + \cdots] - y^{2e}[2(e+2) + \cdots] = 0.$$

If $3E < 4e$, then the Newton Polygon of this equation has vertices $(3,0)$, $(1,E)$, and $(0,2e)$. Two η_i's have order $(E/2)$, one has $2e-E$. Thus, two polar arcs have order $(E/2)+1$, one has order $2e-E+1$. Let us take $e=7$, $E_1 := 8$, $E_2 := 9$ and

$$g_k(x,y) := (x + y^{E_k+1})(x+y), \quad k = 1, 2.$$

Then $T(f,g_1) = T(f,g_2)$, but, as $E_1 \neq E_2$, the polar arcs split away from the trees at different heights between B_1 and B_2.

In summary, the number of polar arcs in Theorem C is determined by $T(f,g)$, but their contact orders with $T(f,g)$ need *not* be.

References

[1] Garcia-Barroso, E.: Sur les courbes polaires d'une courbe plane réduite. Proc. London Math. Soc. (3), **81**, 1–28 (2000).

[2] Izumi, S., Koike, S., and Kuo, T.-C.: Computations and stability of the Fukui invariants. Compositio Math., **130**, 49–73 (2002).

[3] Kuo, T.-C., and Lu, Y.C.: On analytic function germs of two complex variables. Topology, **16**, 299–310 (1977).

[4] Kuo, T.-C., and Parusiński, A.: Newton polygon relative to an arc. Real and Complex Singularities (São Carlos, 1998) (eds. Bruce, J.W., and Tari, F.). Chapman and Hall Res. Notes Math. no. 412, Chapman and Hall, Boca Raton-London-New York-Washington D.C., pp. 76–93 (2000).

[5] Merle, M.: Invariants polaires des courbes planes. Invent. Math., **41**, 299–310 (1977).

[6] Pham, F.: Deformations equisingularitiés des ideaux jacobiens de courbes planes. Proc. Liverpool Singularities Symposium, II (ed. Wall C.T.C.). Lecture Notes in Math. vol. 209, Springer-Verlag, Berlin-Heidelberg-New York, pp. 218–233 (1971).

[7] Walker, R. J.: Algebraic Curves. Springer-Verlag, Berlin-Heidelberg-New York (1972).

[8] Zariski, O.: Studies in equisingularity I. Amer J. Math. 2, **87**, 507–536 (1965).