Trigonal quotients of modular curves $X_0(N)$

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Abstract: Let $W(N)$ be the group of Atkin-Lehner involutions on the modular curve $X_0(N)$. The purpose of this article is to give complementary result to [7, 8, 9]; namely, we determine trigonal curves of the form $X_0(N)/W'$, where W' is a subgroup of $W(N)$ such that $2 < |W'| < |W(N)|$.

Key words: Modular curve; modular form; gonality; plane curve.

Let $X_0(N)$ be the modular curve over $\mathbf{Q}$ corresponding to the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

It is known [1] that the group $\mathrm{Aut}_{\mathbf{Q}}\, X_0(N)$ of automorphisms of $X_0(N)$ over $\mathbf{Q}$ contains the group $W(N) = \{w_d\}_d$ of Atkin-Lehner involutions, where d runs through the set of positive divisors of N such that $\gcd(d, N/d) = 1$. The group $W(N)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime divisors of N.

Let W' be a subgroup of $W(N)$, and consider the quotient curve $X_0(N)/W'$. When $W' = \langle w_d \rangle$ (resp. $W' = W(N)$), this curve is denoted by $X_0^{+d}(N)$ (resp. $X_0^*(N)$). Note that $X_0(N)/W'$ and the natural map $X_0(N) \to X_0(N)/W'$ are also defined over $\mathbf{Q}$, since every Atkin-Lehner involution is defined over $\mathbf{Q}$. Furthermore, rational points of $X_0(N)/W'$ has deep connection with $\mathbf{Q}$-curves. (Recall that an elliptic curve E over $\bar{\mathbf{Q}}$ is called a $\mathbf{Q}$-curve if every Galois conjugate of E is isogeneous to E.)

A curve X is said to be D-gonal if it admits a finite morphism of degree D to the projective line $\mathbf{P}^1$. We are interested in D-gonal curves $X_0(N)/W'$ with small D. It is not difficult to determine all the pairs (N, W') for which $X_0(N)/W'$ is rational ($D = 1$) or elliptic ($D = 2$ and genus one). Moreover, all the hyperelliptic $X_0(N)/W'$ ($D = 2$ and genus $\geqq 2$) are completely determined by [3, 5, 6, 13].

Now let us consider the case $D = 3$, namely, the case where $X_0(N)/W'$ is trigonal. We have already determined the trigonal curves of type $X_0(N)/W'$ whenever $|W'| \leqq 2$ or $W' = W(N)$ (see [7, 8, 9]). In this article, we determine the remaining case, i.e. $2 < |W'| < 2^{\omega(N)} = |W(N)|$. Note that the existence of such a W' forces N to have at least 3 distinct prime divisors. Furthermore, we have a sharp upper bound for N:

Lemma 1. *If $X_0(N)/W'$ is trigonal for some $W' \subset W(N)$, then $X_0^*(N)$ is D'-gonal with $D' \leqq 3$. In particular, we have $N \leqq 570$.*

Proof. Let X be a D-gonal curve and suppose there is a finite morphism $X \to Y$. Then Y is D'-gonal with $D' \leqq D$ ([11, Thm. VII.2], [12, Lem. 1.3]). Now take $X_0^*(N)$ as Y. By [5, 9] every rational, elliptic, hyperelliptic, or trigonal $X_0^*(N)$ has level $N \leqq 570$, so we obtain the desired result. □

From now on, we always assume that $N \leqq 570$. In view of [7, 8, 9], it remains to check the trigonality of $X_0(N)/W'$ for (N, W') such that

$$\begin{cases} \omega(N) = 3 \quad \text{and} \quad |W'| = 4; \\ \omega(N) = 4 \quad \text{and} \quad |W'| = 4, 8. \end{cases}$$

It is known that every nonhyperelliptic curve of genus 3 or 4 is necessarily trigonal ([2], [4, pp. 345–346]). On the other hand, it is easy to see that any hyperelliptic curve of genus $\geqq 3$ is *not* trigonal. We can explicitly determine all the $X_0(N)/W'$ with genus 3 or 4, so by using the result of [3] we find there are 93 trigonal curves $X_0(N)/W'$ of genus 3 or 4, as listed in Tables I and II.

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Table I. List of W' ($g'=3$)

N	$p\|N$	W'
84	$2,3,7$	$\langle 2,7\rangle$, $\langle 3,7\rangle$, $\langle 12,28\rangle$
90	$2,3,5$	$\langle 2,9\rangle$, $\langle 2,5\rangle$, $\langle 18,10\rangle$
102	$2,3,17$	$\langle 2,17\rangle$, $\langle 6,17\rangle$, $\langle 6,34\rangle$
114	$2,3,19$	$\langle 2,57\rangle$, $\langle 6,38\rangle$
120	$2,3,5$	$\langle 3,5\rangle$
130	$2,5,13$	$\langle 2,65\rangle$, $\langle 26,5\rangle$
132	$2,3,11$	$\langle 44,3\rangle$, $\langle 12,11\rangle$
138	$2,3,23$	$\langle 2,23\rangle$
140	$2,5,7$	$\langle 20,7\rangle$, $\langle 20,28\rangle$
150	$2,3,5$	$\langle 2,75\rangle$, $\langle 50,3\rangle$
156	$2,3,13$	$\langle 3,13\rangle$, $\langle 12,52\rangle$
174	$2,3,29$	$\langle 3,29\rangle$, $\langle 2,87\rangle$
182	$2,7,13$	$\langle 14,26\rangle$
190	$2,5,19$	$\langle 2,95\rangle$, $\langle 10,38\rangle$
195	$3,5,13$	$\langle 3,65\rangle$, $\langle 15,39\rangle$
210	$2,3,5,7$	$\langle 2,5,7\rangle$, $\langle 3,5,7\rangle$, $\langle 2,3,35\rangle$, $\langle 6,5,7\rangle$, $\langle 10,14,3\rangle$, $\langle 6,14,5\rangle$
222	$2,3,37$	$\langle 6,74\rangle$
231	$3,7,11$	$\langle 3,77\rangle$
238	$2,7,17$	$\langle 7,17\rangle$, $\langle 2,119\rangle$, $\langle 14,34\rangle$

Table II. List of W' ($g'=4$)

N	$p\|N$	W'
102	$2,3,17$	$\langle 2,3\rangle$, $\langle 34,3\rangle$
114	$2,3,19$	$\langle 2,3\rangle$, $\langle 3,19\rangle$, $\langle 6,19\rangle$
120	$2,3,5$	$\langle 8,3\rangle$, $\langle 40,3\rangle$
126	$2,3,7$	$\langle 2,7\rangle$, $\langle 18,7\rangle$
130	$2,5,13$	$\langle 2,5\rangle$, $\langle 5,13\rangle$, $\langle 10,13\rangle$
132	$2,3,11$	$\langle 3,11\rangle$, $\langle 4,33\rangle$, $\langle 12,44\rangle$
138	$2,3,23$	$\langle 2,69\rangle$
140	$2,5,7$	$\langle 4,5\rangle$, $\langle 5,7\rangle$, $\langle 28,5\rangle$
150	$2,3,5$	$\langle 2,25\rangle$, $\langle 3,25\rangle$, $\langle 6,25\rangle$
154	$2,7,11$	$\langle 7,11\rangle$, $\langle 2,77\rangle$, $\langle 22,7\rangle$
165	$3,5,11$	$\langle 3,11\rangle$, $\langle 5,11\rangle$
168	$2,3,7$	$\langle 56,3\rangle$
170	$2,5,17$	$\langle 2,85\rangle$, $\langle 34,5\rangle$
174	$2,3,29$	$\langle 6,29\rangle$, $\langle 6,58\rangle$
182	$2,7,13$	$\langle 2,91\rangle$, $\langle 26,7\rangle$, $\langle 14,13\rangle$
186	$2,3,31$	$\langle 62,3\rangle$
210	$2,3,5,7$	$\langle 2,3,7\rangle$, $\langle 2,5,21\rangle$, $\langle 2,15,7\rangle$, $\langle 14,3,5\rangle$, $\langle 2,15,21\rangle$
220	$2,5,11$	$\langle 5,11\rangle$, $\langle 4,55\rangle$, $\langle 20,44\rangle$
222	$2,3,37$	$\langle 2,111\rangle$
231	$3,7,11$	$\langle 33,7\rangle$, $\langle 21,11\rangle$, $\langle 21,33\rangle$
255	$3,5,17$	$\langle 51,5\rangle$

Table II. (cont.)

N	$p\|N$	W'
285	$3,5,19$	$\langle 3,95\rangle$
286	$2,11,13$	$\langle 2,143\rangle$
330	$2,3,5,11$	$\langle 6,10,11\rangle$

Here we abbreviate $\langle w_{d_1}, w_{d_2}, \dots\rangle$ to $\langle d_1, d_2, \dots\rangle$.

In the following, we assume that $X_0(N)/W'$ is of genus $g' \geqq 5$. Some cases are concluded to be nontrigonal, by the following two lemmas.

Lemma 2 (See [7, §4] and the references given there). *Let X be a curve of genus g, and let w be an involution on X. Let $\bar{g}$ be the genus of $X/\langle w\rangle$. If $g > 2(\bar{g}+1)$, then X is not trigonal.*

Lemma 3 (cf. [13]). *Let $\widetilde{X}$ be the reduction of $X_0(N)/W'$ at a prime p not dividing N. If*

$$\left|\widetilde{X}(\mathbf{F}_{p^n})\right| > 3(1+p^n)$$

for some n, then $X_0(N)/W'$ is not trigonal.

Remark 1.

(a) For $X = X_0(N)/W'$ and $w = w_d \pmod{W'} \in W(N)/W'$, it is not difficult to compute the genus of $X/\langle w\rangle$.

(b) If $X/\mathbf{Q}$ is a trigonal curve of genus $g \geqq 5$, then X has a $\mathbf{Q}$-rational finite morphism of degree 3 to a rational curve over $\mathbf{Q}$ ([12, Thm. 2.1]). If in addition X has good reduction at a prime p, then the reduced curve $\widetilde{X}/\mathbf{F}_p$ has a finite morphism of degree $d' \leqq 3$ to a rational curve over $\mathbf{F}_p$ ([12, Lem. 5.1]), so we must have an inequality $\left|\widetilde{X}(\mathbf{F}_{p^n})\right| \leqq 3(1+p^n)$. From this observation, together with the fact that $X_0(N)/W'$ has good reduction at any prime p not dividing N, we obtain Lemma 3. One can compute the number of rational points of $X_0(N)/W'$ over finite fields by using trace formulas of Hecke operators ([10, 15]).

Thus we reduce the set of (N, W') for which $X_0(N)/W'$ is possibly trigonal. Finally, to each $X_0(N)/W'$ such that (N, W') belongs to this set, we apply Proposition 2 in [8], which gives a criterion for trigonality. (Alternatively, one may use Petri's theorem ([2]).) See [7, 8] for details.

Our main result is as follows:

Theorem. *Let N be a positive integer, and let W' be a subgroup of $W(N)$. Assume that $2 < |W'| < 2^{\omega(N)}$. Then $X_0(N)/W'$ is trigonal of genus $g' \geqq 5$, if and only if W' is in the following list:*

N	$p\|N$	W'	g'
154	2, 7, 11	$\langle 14, 11\rangle$	5
170	2, 5, 17	$\langle 10, 17\rangle$	5
204	2, 3, 13	$\langle 68, 3\rangle$	5
270	2, 3, 5	$\langle 10, 27\rangle$	7
330	2, 3, 5, 11	$\langle 10, 3, 11\rangle$	5

(*Notation for* W' *is the same as in Tables* I *and* II.)

The following list gives the plane models of $X_0(N)/W'$.

Plane model of $X_0(N)/W'$
$X_0(154)/\langle 14, 11\rangle$: $(t^2+1)s^3 + t(t-1)s^2 + t(2t-1)(t-2)s - t(t-1)(t^2-3t+1) = 0$
$X_0(170)/\langle 10, 17\rangle$: $(t^2-t+1)s^3 + (t^2-t+3)s^2 + (2t^3+3t+3)s - (t^4-2t^3-4t-1) = 0$
$X_0(204)/\langle 68, 3\rangle$: $(t^2-2t-2)s^3 + (3t^3-8t^2-2t+1)s^2 + 2(t^4-3t^3-t-1)s - 4t^2 = 0$
$X_0(270)/\langle 10, 27\rangle$: $(t+1)(t^2-t+1)s^3 + 3t(t^3-t+1)s^2 + 3t^2(t-1)(t^2+t-1)s - (3t^4+6t^3+1) = 0$
$X_0(330)/\langle 10, 3, 11\rangle$: $(t^2+t+1)s^3 - (t^3+2t^2+4t+2)s^2 + (t^4+2t^2+3t+2)s - (t^3-t^2+t+1) = 0$

We refer to [8, §3] for the method of computing plane models (cf. [14]).

Remark 2. Let N be a positive integer, and let W' be a subgroup of $W(N)$. We see from the theorem above and the results of [7, 8, 9] that there are eighteen pairs of (N, W') for which $X_0(N)/W'$ is a trigonal curve of genus $\geqq 5$. Here we summarize the results for $|W'| \leqq 2$ and $W' = W(N)$.

If $|W'| = 1$ then $X_0(N)/W'$ is just the curve $X_0(N)$, and every trigonal curve $X_0(N)$ has genus < 5 ([7]).

If $|W'| = 2$, then $X_0(N)/W' = X_0^{+d}(N)$ for some $1 < d|N$ such that $(d, N/d) = 1$. In this case there are 8 pairs of (N, d) for which $X_0(N)/\langle w_d\rangle$ is trigonal of genus $\geqq 5$ ([8]); i.e. $(N, d) = (122, 122)$, $(146, 146), (147, 3), (181, 181), (227, 227)$ for genus 5, $(N, d) = (117, 13), (164, 164)$ for genus 6, and $(N, d) = (162, 162)$ for genus 7.

Finally, there are 5 values of N for which $X_0^*(N) = X_0(N)/W(N)$ is trigonal of genus $\geqq 5$ ([9]); i.e. $N = 253, 302, 323, 555$ for genus 5 and $N = 351$ for genus 6.

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