Estimates of automorphic L-functions in the discriminant-aspect

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Abstract: Let $L(s,\phi)$ be an automorphic L-function for a Bianchi group defined via an imaginary quadratic field with discriminant $d<0$. We give an upper bound for the absolute value of $L(s,\phi)$ in terms of $\Im(s)$, the Laplace eigenvalue and the discriminant d. The bound as $d\to\infty$ presents a new aspect in the study of L-functions.

Key words: Automorphic form; Maass form; convexity bound.

1. Introduction. Estimating the size of L-functions on the critical line $\Re(s)=1/2$ is a central subject in modern number theory.

The prototype was the estimate of the Riemann zeta function $\zeta(1/2+it)$ as $t\to\infty$. By the functional equation of $\zeta(s)$ and the Phragmén-Lindelöf convexity principle, it follows that

$$\zeta\left(\frac{1}{2}+it\right)=O\left(t^{\frac{1}{4}+\epsilon}\right). \tag{1}$$

The Lindelöf Hypothesis asserts that

$$\zeta\left(\frac{1}{2}+it\right)=O\left(t^{\epsilon}\right). \tag{2}$$

The bound (1) is the first result towards the Lindelöf Hypothesis (2).

The Lindelöf Hypothesis is generalized to more general L-functions and to more general aspects. Here an aspect means a parameter which goes to infinity with all other parameters being fixed. For example, let f be a nonholomorphic newform for $\Gamma_0(N)$. The automorphic L-function $L(1/2+it,f)$ satisfies the following estimate as $N\to\infty$:

$$L\left(\frac{1}{2}+it,f\right)=O_{\epsilon,t,\lambda}\left(N^{\frac{1}{4}+\epsilon}\right), \tag{3}$$

where the implied constant depends on ϵ, t, and the Laplace eigenvalue λ. It is natural to ask about the Lindelöf Hypothesis in this case as well:

$$L\left(\frac{1}{2}+it,f\right)=O_{\epsilon,t,\lambda}\left(N^{\epsilon}\right). \tag{4}$$

The crude bounds such as (1) and (3), which are proved by the functional equation and the Phragmeń-Lindelöf convexity principle, is called the convexity bound.

The aim of this paper is to propose a new aspect for the estimate of automorphic L-functions and to obtain the convexity bound. We consider the automorphic L-function $L(s,\phi)$ of a even Maass cusp form ϕ for a Bianchi group $\Gamma=PSL(2,\mathcal{O}_d)$ with $\mathcal{O}_d$ the integer ring of an imaginary quadratic field with discriminant $d<0$, $d\neq -3,-4$, and the class number $h(d)$. The main theorem gives the behavior of the size of $L(1/2+it,\phi)$ as $d\to\infty$. We prove

$$L\left(\frac{1}{2}+it,\phi\right)=O_{\epsilon,t,\lambda}\left(d^{\frac{1}{2}+\epsilon h(d)}\right),$$

where the implied constant depends on ϵ, t, and the Laplace eigenvalue λ. Indeed their dependency is explicitly written in Theorem 4.1.

The outline of the proof is as follows: We first prove the average estimate of eigenvalues of the Hecke operators, which is stated in Theorem 3.3. It is crucial to write down its dependency on d. This gives the estimate of $L(s,\phi)$ as $d\to\infty$ for $\Re(s)>1$. Next we write down the explicit form of the functional equation of $L(s,\phi)$:

$$\Lambda(1-s,\phi)=\Lambda(s,\phi), \tag{5}$$

where

$$\Lambda(s,\phi)$$
$$=(2\pi)^{-2s}|d|^{s}\Gamma\left(s+\frac{ir}{2}\right)\Gamma\left(s-\frac{ir}{2}\right)L(s,\phi).$$

Here again the key ingredient is the explicit form of the dependency on d. These results together with the standard technique of using the Phragmeń-Lindelöf convexity principle lead to Theorem 4.1.

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Remark 1.1. It is known that automorphic L-functions generally have analytic continuation to the entire plane, as shown by Hecke, and they satisfy the functional equations of the standard type. In this paper, we calculate the explicit form of it.

Remark 1.2. For $\mathbf{Q}(\sqrt{-1})$, i.e. $d = -4$, Petridis and Sarnak [6] improve the convexity bound. Koyama [5] obtains Theorem 3.3 in the case of $\mathbf{Q}(\sqrt{-1})$, i.e. $d = -4$.

2. Definitions. Let $\mathbf{H}^3 = \{\omega = (y, z) \mid y > 0,\ z = x_1 + ix_2 \in \mathbf{C}\}$ be the 3-dimensional hyperbolic space. The group $\mathrm{SL}(2, \mathbf{C})$ acts on $\mathbf{H}^3$ by

$$g \cdot \omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega = (y(g \cdot \omega), z(g \cdot \omega))$$

with

$$y(g \cdot \omega) = \frac{y}{|\gamma z + \delta|^2 + |\gamma|^2 y^2} > 0,$$

$$z(g \cdot \omega) = \frac{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\overline{\gamma} y^2}{|\gamma z + \delta|^2 + |\gamma|^2 y^2} \in \mathbf{C}.$$

Let K be an imaginary quadratic number field with discriminant $d < 0$, class number $h(d)$, and ring of integers $\mathcal{O}_d$. Note that $h(d) \ll |d|^{1/2}$. The group $\Gamma = PSL(2, \mathcal{O}_d)$ is a lattice in $\mathrm{SL}(2, \mathbf{C})$ and acts discontinuously on $\mathbf{H}^3$. The set of cusps of Γ is equal to $K \cup \{\infty\}$, and this set splits into $h(d)$ orbits with respect to Γ. The Laplacian Δ is given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial y}.$$

We define a Maass form for Γ to be a smooth function ϕ on $\mathbf{H}^3$ such that

(i) $\Delta\phi = (1 + r^2)\phi$,
(ii) $\phi(g \cdot \omega) = \phi(\omega),\ g \in \Gamma,\ \omega \in \mathbf{H}^3$,
(iii) $\phi(\omega) = O(y^k)$ as $y \to \infty$ for some $k > 0$.

Suppose that ϕ is an arbitrary Maass form for Γ. It follows from (ii) that ϕ has the Fourier expansion. If ϕ has no constant term in the Fourier expansion at any cusp, we call ϕ a Maass cusp form. From (1) in [8], the Fourier expansion at ∞ is given by

$$\phi(\omega) = \sum_{0 \neq \nu \in \mathcal{O}_d} c(\nu) y K_{ir} \left(2\pi \left| \frac{2\nu}{\sqrt{d}} \right| y \right) e\left(\Re\left(\frac{2\nu}{\sqrt{d}} z \right) \right),$$

where $K_{ir}(z)$ is the K-Bessel function, i.e. for $\Re(z) > 0$,

$$K_{ir}(z) = \frac{1}{2} \int_0^\infty \exp\left(-\frac{z}{2} \left(t + \frac{1}{t} \right) \right) t^{ir} \frac{dt}{t}.$$

Let $\iota : \mathbf{H}^3 \to \mathbf{H}^3$ be the antiholomorphic involution $\iota(y, z) = (y, -z)$. If ϕ is an eigenfunction of Δ, then $\phi \circ \iota$ is an eigenfunction with the same eigenvalue. Because $\iota^2 = 1$, its eigenvalues are ± 1. We may therefore diagonalize the Maass cusp forms with respect to ι. If $\phi \circ \iota = \phi$, we call ϕ even. In this case, $c(\nu) = c(-\nu)$. If $\phi \circ \iota = -\phi$, we call ϕ odd. In this case, $c(\nu) = -c(-\nu)$.

To define the L-function, we introduce the Hecke operators $T(\nu) : L^2(\Gamma\backslash\mathbf{H}^3) \to L^2(\Gamma\backslash\mathbf{H}^3)$ as follows:

$$T(\nu)\phi(\omega) = \frac{1}{|\nu|} \sum_{\substack{\alpha\delta = \nu \\ \Re(\delta) > 0}} \sum_{\beta \bmod \delta} \phi\left(\frac{1}{\sqrt{\nu}} g \cdot \omega \right),$$

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in PSL(2, \mathcal{O}_d).$$

$T(\nu)$ satisfies the usual commutation relations with one another and commutates with the Laplacian Δ as well as the involution. Diagonalizing the $T(\nu)'s$ and Δ to get an orthogonormal basis ϕ of eigenfunctions of Δ, we have

$$\Delta\phi = (1 + r^2)\phi,$$

$$T(\nu)\phi = \lambda(\nu)\phi.$$

The eigenvalue $\lambda(\nu)$ has the following properties:

$$c(\nu) = c(1)\lambda(\bar{\nu}). \tag{6}$$

It follows from (6) that the coefficient $c(1)$ is nonzero.

Throughout this paper, we assume that ϕ is a Hecke eigen Maass cusp form for $\Gamma = PSL(2, \mathcal{O}_d)$, where $d \neq -3, -4$ and even.

To each ϕ, we associate its standard L-function, which takes the form

$$L(s, \phi) = \sum_{(\nu) \neq 0} \frac{\lambda(\nu)}{N(\nu)^s}.$$

This is well-defined when ϕ is even.

3. Average estimate of Hecke eigenvalues. Suppose $\mathfrak{a}_n$ $(1 \leq n \leq h(d),\ \mathfrak{a}_1 = \infty)$ are cusps for Γ. Let $\sigma_{\mathfrak{a}_n} \in \mathrm{SL}(2, K)$ be the scaling matrix, so $\sigma_{\mathfrak{a}_n} \cdot \infty = \mathfrak{a}_n$. Then ϕ has the Fourier expansion at $\mathfrak{a}_n$ of type

$$\phi(\sigma_{\mathfrak{a}_n} \cdot \omega) = \sum_{0 \neq \nu \in \mathcal{O}_d} c_{\mathfrak{a}_n}(\nu) y K_{ir} \left(2\pi \left| \frac{2\nu}{\sqrt{d}} \right| y \right) e\left(\Re\left(\frac{2\nu}{\sqrt{d}} z \right) \right).$$

Note that $c_\infty(\nu) = c(\nu)$.

Lemma 3.1. *For all $0 \neq \nu \in \mathcal{O}_d$,*

$$c_{\mathfrak{a}_n}(\nu) \ll_{d,r} |\nu|,$$

where $|X| \ll_{d,r} Y$ means that $|X| \leq CY$ for an implied positive constant C depending only on d and r.

Proof of Lemma 3.1. It follows from the definition of Maass form that ϕ is Γ invariant. Because ϕ is cuspidal, this function decays rapidly as ω approaches any cusp. Thus it is bounded on $\Gamma\backslash\mathbf{H}^3$ and there exists a constant C_1 such that $|\phi(\omega)| \ll_{d,r} C_1$ for all ω. Thus from (3.9) in [1], for any $y > 0$,

$$\begin{aligned} c_{\mathfrak{a}_n}(\nu) y K_{ir}\left(2\pi\left|\frac{2\nu}{\sqrt{d}}\right|y\right) &= \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}} \phi(\sigma_{\mathfrak{a}_n}\cdot\omega)\, e\left(-\Re\left(\frac{2\nu}{\sqrt{d}}z\right)\right) dz \\ &\ll_{d,r} C_1, \end{aligned}$$

where $|\mathcal{P}|$ denotes the Euclidean area of a fundamental parallelogram of $\mathcal{P}$ of $\mathcal{O}_d$. This estimate is independent of $0 \neq \nu \in \mathcal{O}_d$. We choose $y = c|2\nu/\sqrt{d}|^{-1}$, where $c > 0$ is a constant such that $K_{ir}(2\pi c) \neq 0$, and obtain $c_{\mathfrak{a}_n}(\nu) \ll_{d,r} |\nu|$ as required. □

We shall normalize the Fourier coefficients by putting

$$\hat{c}_{\mathfrak{a}_n}(\nu) = \left(\frac{\sqrt{|d|}}{||\phi|| e^{\pi r}}\right)^{\frac{1}{2}} c_{\mathfrak{a}_n}(\nu).$$

Applying the method of Iwaniec [3], it follows that

$$\sum_{N(\nu)\leq X} |\hat{c}_{\mathfrak{a}_n}(\nu)|^2 \ll |r| + \frac{X}{|r|}. \tag{7}$$

We evaluate the sum by means of the Rankin-Selberg L-function

$$L_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}) = \sum_{0\neq\nu\in\mathcal{O}_d} \frac{|\hat{c}_{\mathfrak{a}_n}(\nu)|^2}{N(\nu)^s}.$$

By virtue of Lemma 3.1 and (7), the above series converges absolutely on $\Re(s) > 1$, and satisfies the following integral representation with the Eisenstein series $E_{\mathfrak{a}_n}(\omega, s)$,

$$\Lambda_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}) = \int_{\Gamma\backslash\mathbf{H}^3} E_{\mathfrak{a}_n}(\omega, 2s-1)\, |\phi(\omega)|^2 \frac{dz dy}{y^3}, \tag{8}$$

where

$$\Lambda_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}) = \frac{|d|^{s+\frac{1}{2}}\Gamma(s)^2\Gamma(s+ir)\Gamma(s-ir)}{16(2\pi)^{2s}\Gamma(2s)} L_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}),$$

$$E_{\mathfrak{a}_n}(\omega, s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}_n}\backslash\Gamma} \left(y\left(\sigma_{\mathfrak{a}_n}^{-1}\gamma\cdot\omega\right)\right)^{s-1}.$$

From (8) and the properties of the Eisenstein series (see Theorem 5.8 in [2]) we deduce that the Rankin-Selberg L-function has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\begin{aligned} \Lambda_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}) &= \frac{2\pi}{\sqrt{|d|}(2s-1)} \\ &\times \sum_{j=1}^{h(d)} \left[\Omega\left(\frac{L(2s-1,\chi_l)}{L(2s,\chi_l)}\delta_{lk}\right)\Omega^{-1}T\right]_{n,j} \\ &\times \Lambda_{\mathfrak{a}_n}(1-s, \phi\otimes\bar{\phi}), \end{aligned} \tag{9}$$

where $L(s,\chi_l)$ is defined by

$$L(s,\chi_l) = \sum_{0\neq\mathfrak{n}\subset\mathcal{O}_d} \frac{\chi_l(\mathfrak{n})}{N(\mathfrak{n})^s},$$

χ_l ($1 \leq l \leq h(d)$) are the characters on the group $\mathfrak{M}/K^\times = \{\mathfrak{m}_1, \cdots, \mathfrak{m}_{h(d)}\}$, $\mathfrak{M}$ is the set of all fractional ideals $\neq \{0\}$ of K,

$$\Omega = \left(\frac{1}{\sqrt{h(d)}}\chi_k(\mathfrak{m}_{\mathfrak{l}})\right)_{l,k=1,\ldots,h(d)},$$

and $T \in GL(n, \mathbf{Z})$ is the permutation matrix such that

$$\exists\lambda_l \in \mathcal{O}_d \quad \text{s.t.} \quad {\mathfrak{m}_{\mathfrak{l}}}^{-1} = \lambda_l \mathfrak{m}_{s(l)},$$

$$\begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(h(d)) \end{pmatrix} = T \begin{pmatrix} 1 \\ 2 \\ \vdots \\ h(d) \end{pmatrix}.$$

It is known from class field theory that the product of $L(s,\chi_l)$ is the Dedekind zeta function $\zeta_F(s)$ where F is the Hilbert class field of K:

$$\zeta_F(s) = \prod_{l=1}^{h(d)} L(s,\chi_l).$$

Every $L_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi})$ is holomorphic for $\Re(s) > 1/2$ except for a simple pole at $s = 1$ with residue

$$(10)\qquad R := \operatorname*{res}_{s=1} L_{\mathfrak{a}_n}(s, \phi\otimes\bar{\phi}) = \frac{512\pi^4}{|d|^2\Gamma(1+ir)\Gamma(1-ir)\zeta_K(2)e^{\pi|r|}} \gg \frac{1}{|d^2 r|},$$

where $\zeta_K(s)$ is the Dedekind zeta function of K.

Lemma 3.2. *For $\Re(s) = 1 - 1/(h(d)+2)$,*

$$(s-1)L_\infty(s, \phi\otimes\bar{\phi}) \ll |d(1+|t|+|r|)|^{C_3},$$

where C_3 is a positive constant.

Proof of Lemma 3.2. Put

$$G(s) = (s-1)L_\infty(s, \phi\otimes\bar{\phi})\zeta_F(2s).$$

By (9) we deduce that $G(s)$ is holomorphic in the whole s-plane. Hence by a standard application of the Phragmén-Lindelöf convexity principle [7], the functional equations of $L(s, \chi_l)$ $(1 \le l \le h(d))$, and (9), we obtain

$$G(s) = |d|^{2h(d)(1-\sigma)}(1+|t|)^{h(d)(1-\sigma)}(1+|r|)^{\sigma}\,\zeta_F(2s),$$

for $-1/(h(d)+2) \le \Re(s) = \sigma \le 1 + 1/(h(d)+2)$. Put $\Re(s) = 1 - 1/(h(d)+2)$, there exists a positive constant C_3 such that

$$G(s) \ll |d(1+|t|+|r|)|^{C_3}\zeta_F\left(2(1+\frac{1}{h(d)+2}+it)\right)$$

Hence we get a bound of the form

$$(s-1)L_\infty(s, \phi\otimes\bar{\phi}) \ll |d(1+|t|+|r|)|^{C_3}\frac{\zeta_F\left(2(1+\frac{1}{h(d)+2}+it)\right)}{\zeta_F\left(2(1-\frac{1}{h(d)+2}+it)\right)} \ll |d(1+|t|+|r|)|^{C_3}$$

for $\Re(s) = 1 - 1/(h(d)+2)$. □

Theorem 3.3. *For all $X > 0$,*

$$\sum_{N(\nu)\le X} |\lambda(\nu)|^2 \ll_\epsilon |d(1+|r|)|^{\epsilon h(d)}X, \quad \forall \epsilon > 0.$$

Proof of Theorem 3.3. Applying Perron's formula to $L_\infty(s, \phi\otimes\bar{\phi})$ and Lemma 3.2, we infer that

$$(11)\qquad \sum_{N(\nu)\le X} |\hat{c}(\nu)|^2 = RX + O\left(X^{1-\frac{1}{h(d)+2}}|d(1+|r|)|^{C_3}\right)$$

for all $X > 0$, where $\hat{c}(\nu) = \hat{c}_\infty(\nu)$.

From (10) and (11), if X is large such that

$$X \gg |d(1+|r|)|^{C_4 h(d)} \ge \left(\frac{|d(1+|r|)|^{C_3}}{R}\right)^{h(d)+2},$$

where C_4 is a positive constant, we have

$$RX \ll \sum_{N(\nu)\le X} |\hat{c}(\nu)|^2 \ll RX.$$

Hence, we shall evaluate the bound for $\sum_{N(\nu)\le X} |\hat{c}(\nu)|^2 X^{-1/2}$ using the partial summation and get

$$(12)\qquad RX^{\frac12} \ll \sum_{N(\nu)\le X} \frac{|\hat{c}(\nu)|^2}{X^{\frac12}} \ll \sum_{N(\nu)\le X} \frac{|\hat{c}(\nu)|^2}{N(\nu)^{\frac12}} \ll RX^{\frac12}$$

for $X \ge |d(1+|r|)|^{C_4 h(d)}$.

Next we will follow the method of Iwaniec [4] (19). Putting

$$L(X) = \sum_{N(\nu)\le X} \frac{|\lambda(\nu)|^2}{N(\nu)^{\frac12}},$$

we have

$$(13)\qquad L(X)^2 \ll_\epsilon X^\epsilon L(X^2), \quad \forall \epsilon > 0.$$

Inserting (6) and (12) into (13) we obtain

$$\frac{R^2}{|\hat{c}(1)|^4}X \ll L(X)^2 \ll_\epsilon X^\epsilon L(X^2) \ll X^\epsilon \frac{R}{|\hat{c}(1)|^2}X.$$

Therefore,

$$\frac{R}{|\hat{c}(1)|^2} \ll_\epsilon X^\epsilon, \quad \forall \epsilon > 0,$$

for all $X > 0$ such that $X \gg |d(1+|r|)|^{C_4 h(d)}$. Hence

$$\frac{R}{|\hat{c}(1)|^2} \ll_\epsilon |d(1+|r|)|^{\epsilon h(d)}, \quad \forall \epsilon > 0.$$

From (6) and (11), we conclude that

$$\sum_{N(\nu)\le X} |\lambda(\nu)|^2 \ll \frac{R}{|\hat{c}(1)|^2}X \ll_\epsilon |d(1+|r|)|^{\,\epsilon h(d)}X$$

for any $\epsilon > 0$, with the implied constant depending on ϵ.

This completes the proof of Theorem 3.3 in the case of $X \gg |d(1+|r|)|^{C_4 h(d)}$. In the short range of $X \ll |d(1+|r|)|^{C_4 h(d)}$, Theorem 3.3 follows by the method of Iwaniec [3] Theorem 8.3. □

4. Main theorem.

Theorem 4.1. *For any $\epsilon > 0$ we have*

$$L\left(\frac{1}{2}+it,\phi\right) \ll_\epsilon |\sqrt{d}(1+|r|)|^{1+\epsilon h(d)}(1+|t|)^{1+\epsilon},$$

where the implied constant depends only on ϵ.

Remark 4.2. If we restrict the growth of the class number to $h(d) < C_2$, where C_2 is any fixed constant, it follows that

$$L\left(\frac{1}{2}+it,\phi\right) \ll_\epsilon |\sqrt{d}(1+|t|+|r|)|^{1+\epsilon},$$

where the implied constant depends only on ϵ.

Proof of Theorem 4.1. By Theorem 3.3, $L(s,\phi)$ converges absolutely on $\Re(s) > 1$ and it satisfies that

$$L(1+\epsilon+it) \ll_\epsilon |d(1+|r|)|^{\epsilon h(d)}, \quad \forall\epsilon > 0. \tag{14}$$

On the other hand, from the functional equation (5),

$$\begin{aligned}|L(s,\phi)| &= \left(\frac{\sqrt{|d|}}{2\pi}\right)^{2-4\sigma} \\ &\times \left|\frac{\Gamma\left(1-s+\frac{ir}{2}\right)\Gamma\left(1-s-\frac{ir}{2}\right)}{\Gamma\left(s+\frac{ir}{2}\right)\Gamma\left(s-\frac{ir}{2}\right)}\right| |L(1-s,\phi)|.\end{aligned}$$

Hence by (14) and Lemma 3 in [7]

$$\begin{aligned}&L(-\epsilon+it,\phi) \\ &\ll_\epsilon \left(\frac{\sqrt{|d|}}{2\pi}\right)^{2+4\epsilon} (1+|t|+|r|)^{2+4\epsilon}|d(1+|r|)|^{\epsilon h(d)}\end{aligned}$$

for any $\epsilon > 0$, the implied constant depending on ϵ. It follows from the Phragmén-Lindelöf convexity principle that

$$L\left(\frac{1}{2}+it,\phi\right) \ll_\epsilon |\sqrt{d}(1+|r|)|^{1+\epsilon h(d)}(1+|t|)^{1+\epsilon},$$

for any $\epsilon > 0$, the implied constant depends on ϵ. This completes the proof of Theorem 4.1 □

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