A note on the mean value of the zeta and L-functions. XIV

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(Communicated by Shigefumi MORI, M. J. A., April 12, 2004)

Abstract: The aim of the present note is to develop a study on the feasibility of a unified theory of mean values of automorphic L-functions, a desideratum in the field. This is an outcome of the investigation commenced with the part XII ([14]), where a framework was laid on the basis of the theory of automorphic representations, and a general approach to the mean values was envisaged. Specifically, it is shown here that the inner-product method, which was initiated by A. Good [7] and greatly enhanced by M. Jutila [9], ought to be brought to perfection so that the mean square of the L-function attached to any cusp form on the upper half-plane could be reached within the notion of automorphy. The Kirillov map is our key implement. Because of its geometric nature, our method appears to extend to bigger linear Lie groups. This note is essentially self-contained.

Key words: Mean values of automorphic L-functions; automorphic representations of linear Lie groups; Kirillov map.

1. Basic notion. We collect here basics from the theory of automorphic representations. In the next section our problem is made precise. An idea to deal with it is given in the third section.

Let $G = \mathrm{PSL}(2, \mathbf{R})$ and $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. Write

$$\mathrm{n}[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \quad \mathrm{a}[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix},$$

$$\mathrm{k}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and

$$N = \{\mathrm{n}[x] : x \in \mathbf{R}\}, \quad A = \{\mathrm{a}[y] : y > 0\},$$
$$K = \{\mathrm{k}[\theta] : \theta \in \mathbf{R}/\pi\mathbf{Z}\},$$

Thus $G = NAK$ is the Iwasawa decomposition of the Lie group G. We read it as

$$G \ni \mathrm{g} = \mathrm{nak} = \mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta].$$

The coordinate (x, y, θ) will retain this definition.

The center of the universal enveloping algebra of G is the polynomial ring on the Casimir element

$$\Omega = y^2\left(\partial_x^2 + \partial_y^2\right) - y\partial_x\partial_\theta.$$

The Haar measures on the groups N, A, K, G are normalized, respectively, by $d\mathrm{n} = dx$, $d\mathrm{a} = dy/y$, $d\mathrm{k} = d\theta/\pi$, $d\mathrm{g} = d\mathrm{n}d\mathrm{a}d\mathrm{k}/y$, with Lebesgue measures dx, dy, $d\theta$.

The space $L^2(\Gamma\backslash G)$ is composed of all left Γ-automorphic functions on G, vectors for short, which are square integrable over $\Gamma\backslash G$ against $d\mathrm{g}$. Elements of G act unitarily on vectors from the right. We have the orthogonal decomposition

$$L^2(\Gamma\backslash G) = \mathbf{C}\cdot 1 \bigoplus {}^0L^2(\Gamma\backslash G) \bigoplus {}^eL^2(\Gamma\backslash G)$$

into invariant subspaces. Here ${}^0L^2$ is the cuspidal subspace spanned by vectors whose Fourier expansions with respect to the left action of N have vanishing constant terms. The subspace ${}^eL^2$ is spanned by integrals of Eisenstein series. Invariant subspaces of $L^2(\Gamma\backslash G)$ and Γ-automorphic representations of G are interchangeable concepts, and we shall refer to them in a mixed way.

The cuspidal subspace decomposes into irreducible subspaces

$${}^0L^2(\Gamma\backslash G) = \overline{\bigoplus V}.$$

The operator Ω becomes a constant multiplication in each V:

$$\Omega|_{V^\infty} = \left(\nu^2 - \frac{1}{4}\right)\cdot 1,$$

where V^∞ is the set of all infinitely differentiable vectors in V. Under our present supposition, V belongs

2000 Mathematics Subject Classification. 11F70.

to either the unitary principal series or the discrete series; accordingly, we have $\nu \in i\mathbf{R}$ or $\nu = \ell - \frac{1}{2}$ $(1 \le \ell \in \mathbf{Z})$.

The right action of K induces the decomposition of V into K-irreducible subspaces

$$V = \overline{\bigoplus_{p=-\infty}^{\infty} V_p}\,, \quad \dim V_p \le 1.$$

If it is not trivial, V_p is spanned by a $\mathit{\Gamma}$-automorphic function on which the right translation by $\mathrm{k}[\theta]$ becomes the multiplication by the factor $\exp(2ip\theta)$. It is called a $\mathit{\Gamma}$-automorphic form of spectral parameter ν and weight $2p$.

Let V be in the unitary principal series. Then $\dim V_p = 1$ for all $p \in \mathbf{Z}$ and there exists a complete orthonormal system $\{\varphi_p \in V_p : p \in \mathbf{Z}\}$ of V such that

$$\varphi_p(\mathrm{g}) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\mathrm{sgn}(n)} \phi_p(\mathrm{a}[|n|]\mathrm{g}; \nu), \tag{1}$$

where $\phi_p(\mathrm{g}; \nu) = y^{\nu+\frac{1}{2}} \exp(2ip\theta)$, and

$$\mathcal{A}^{\delta} \phi_p(\mathrm{g}; \nu) = \int_{-\infty}^{\infty} \exp(-2\pi i\delta x) \phi_p(\mathrm{wn}[x]\mathrm{g}; \nu) dx,$$

with $\mathrm{w} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$, the Weyl element. It should be observed that the coefficients $\varrho_V(n)$ in (1) do not depend on the weight. In particular, we have the expansion

$$\varphi_0(\mathrm{g}) = \frac{2\pi^{\frac{1}{2}+\nu}}{\Gamma(\frac{1}{2}+\nu)} \cdot \sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \varrho_V(n) \exp(2\pi inx) K_\nu(2\pi|n|y), \tag{2}$$

where K_ν is the K-Bessel function of order ν. This is in fact a real analytic cusp form on the upper half-plane $\{x + iy : x \in \mathbf{R},\ y > 0\}$.

Next, let V be in the discrete series, with the spectral parameter $\ell - \frac{1}{2}$ as above. We have

$$\text{either} \quad V = \overline{\bigoplus_{p=\ell}^{\infty} V_p} \quad \text{or} \quad V = \overline{\bigoplus_{p=-\infty}^{-\ell} V_p}\,,$$

with $\dim V_p = 1$, corresponding to the holomorphic and the antiholomorphic discrete series. The involution $\mathrm{g} = \mathrm{nak} \mapsto \mathrm{n}^{-1}\mathrm{ak}^{-1}$ maps one to the other; thus we may restrict ourselves to the holomorphic case. Then we have a complete orthonormal system $\{\varphi_p \in V_p : p \ge \ell\}$ in V such that

$$\varphi_p(\mathrm{g}) = \pi^{\frac{1}{2}-\ell} \left(\frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \right)^{\frac{1}{2}} \cdot \sum_{n=1}^{\infty} \frac{\varrho_V(n)}{\sqrt{n}} \mathcal{A}^{+} \phi_p \left(\mathrm{a}[n]\mathrm{g}; \ell - \frac{1}{2} \right),$$

In particular, we have

$$\varphi_\ell(\mathrm{g}) = (-1)^\ell \frac{2^{2\ell}\pi^{\ell+\frac{1}{2}}}{\sqrt{\Gamma(2\ell)}} \exp(2i\ell\theta) \cdot y^\ell \sum_{n=1}^{\infty} \varrho_V(n) n^{\ell-\frac{1}{2}} \exp(2\pi in(x+iy)), \tag{3}$$

in which the sum is a holomorphic cusp form of weight 2ℓ on the upper half-plane.

It is convenient to put

$$\psi_V = \varphi_0 \quad \text{or} \quad \varphi_\ell\,,$$

according to the series to which V belongs, with the specification (2) and (3). The right Lie derivatives of ψ_V generate the space V.

2. The problem. We define the automorphic L-function associated with the irreducible $\mathit{\Gamma}$-automorphic representation V by

$$L_V(s) = \sum_{n=1}^{\infty} \varrho_V(n) n^{-s},$$

which converges for $\mathrm{Re}\, s > 1$ and continues to an entire function of polynomial order in any fixed vertical strip.

The mean square of L_V is the integral

$$\mathcal{M}_2(L_V; g) = \int_{-\infty}^{\infty} \left| L_V\left(\frac{1}{2} + it\right) \right|^2 g(t) dt\,,$$

where the weight function g is assumed, for the sake of simplicity but without much loss of generality, to be even, entire, real on $\mathbf{R}$, and of fast decay in any fixed horizontal strip. This is an analogue of the fourth moment of the Riemann zeta-function

$$\mathcal{M}_4(\zeta; g) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 g(t) dt\,,$$

since the product of two values of ζ corresponds to the Eisenstein series in much the same way as L_V does to ψ_V. These quantities have been major subjects in Analytic Number Theory, as they are indispensable means to reveal the intriguing nature of the zeta- and L-functions.

A. Good [7] was the first to consider $\mathcal{M}_2(L_V; g)$ on the basis of automorphy. He dealt with the case where V is in the discrete series. Later the present author [12] devised an alternative argument for the same case. He started with the integral

$$\int_{-\infty}^{\infty} L_V(u+it)\overline{L_V(\bar{v}+it)}g(t)dt\,,$$

which is an entire function of u, v. The non-diagonal part of this is, in the region of absolute convergence,

$$(4) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varrho_V(n)\overline{\varrho_V(n+m)}}{n^u(n+m)^v}\hat{g}\left(\log\left(1+\frac{m}{n}\right)\right),$$

where $\hat{g}$ is the Fourier transform of g. Expressing the $\hat{g}$-factor in terms of a Mellin inversion, one sees that what is essential is to relate analytically the function

$$(5) \qquad \sum_{n=1}^{\infty} \frac{\varrho_V(n)\overline{\varrho_V(n+m)}}{(n+m)^s}$$

to the space V. For the discrete series this is by no means difficult. It suffices to consider the inner product

$$(6) \qquad \langle P_m(\cdot;\xi),\, |\psi_V|^2\rangle_{\Gamma\backslash G/K}\,,$$

where ξ is a complex parameter and P_m is the Poincaré series of the Selberg type, i.e., the one obtained by replacing the factor τ by the constant 1 that is implicit in (10) below. Thus the success of the arguments in [7] and [12] is due much to the fact that the exponential function $\exp(2\pi i(x+iy))$ appears in the Fourier expansion (3). Having such a relation between (5) and (6), the spectral decomposition of $\mathcal{M}_2(L_V; g)$ with V in the discrete series reduces to a matter of technicalities, though the procedure is never straightforward as can be seen from these two works.

On the other hand, when V is in the unitary principal series, (6) gives only an expression approximating (5) in an involved way. The difficulty stems from the expansion (2); that is, the presence of the K-Bessel function is the obstacle. Nevertheless, Jutila [9] pressed the matter and could develop a deep asymptotic study of $\mathcal{M}_2(L_V; g)$. A notable merit of his method is in that it is applicable to the Riemann zeta and any automorphic L-functions equally, even though it does not give complete spectral decompositions of the mean values.

As to $\mathcal{M}_4(\zeta; g)$, it is in a mixed status. An explicit spectral decomposition was established by the present author [13, Chapter 4]. Recently R. W. Bruggeman and the present author [4] gave a new proof of it. Both the arguments are sharply different from the inner-product approach mentioned above. In [13, Chapter 4], it is exploited that the expression corresponding to (4) has the sum of powers of divisors function σ_η in place of ϱ_V. Ramanujan's expansion of σ_η in terms of additive characters and a use of the functional equation for the Estermann zeta-function transform the expression into an instance of sums of Kloosterman sums; then the Kloosterman-spectral sum formula of N.V. Kuznetsov yields the spectral decomposition. In contrast, the work [4] dispenses with the use of the sum formula; instead, an approach via a particular Poincaré series on $\Gamma\backslash G$ is employed. Such a possibility was indicated already in [11] (see also [13, Section 4.2]), but its realization took a long time because of the necessity of a drastic change of means. It was required to employ the representational approach or to move from the upper half-plane to the group G. More precisely, the computation of the projection of the Poincaré series to an arbitrary irreducible subspace became the main issue, and it was accomplished, in [4], only after the authors had been inspired by the work [6] due to J. W. Cogdell and I. Piatetski-Shapiro.

At any event, both [13, Chapter 4] and [4] depend much on the arithmetic peculiarity of the function σ_η, and as such it does not seem to extend to $\mathcal{M}_2(L_V; g)$ with an arbitrary V.

Thus, there exists a difference among methods for mean values of automorphic L-functions. It has long been desired to find a unified way to treat them, indeed since Good's pioneering works. This is the problem we are dealing with.

3. An idea. From the above, one might surmise that Jutila's inner-product argument [9] should be more on the right track than other approaches, because of its generality. We are going to show that this might be the case. Namely, albeit a certain restriction is imposed on the variables u, v, we shall prove that (4) can be reached via an extension of (6), regardless to which series the representation V belongs. Our idea is to employ the Kirillov map $\mathcal{K}$ to prove a quasi-analogue of (3) for any V in the unitary principal series.

To this end we invoke

Lemma 1. *Let $\nu \in i\mathbf{R}$, and introduce the Hilbert space*

$$U_\nu = \overline{\bigoplus_{p=-\infty}^{\infty} \mathbf{C}\phi_p}, \quad \phi_p(\mathrm{g}) = \phi_p(\mathrm{g};\nu),$$

equipped with the norm

$$\|\phi\|_{U_\nu} = \sqrt{\sum_{p=-\infty}^{\infty} |c_p|^2}, \quad \phi = \sum_{p=-\infty}^{\infty} c_p\phi_p.$$

Then

$$\mathcal{K}\phi(u) = \mathcal{A}^{\mathrm{sgn}(u)}\phi(\mathrm{a}[|u|])$$

is a unitary map from U_ν *onto* $L^2(\mathbf{R}^\times, \pi^{-1}d^\times)$.

Proof. Here $\mathbf{R}^\times = \mathbf{R}\backslash\{0\}$ and $d^\times u = du/|u|$, as usual. This seems due originally to A. A. Kirillov [10] (see also [6, Section 4.2]). A proof of the unitaricity is given in [14, Theorem 1], though disguised in the context of automorphy. The surjectivity is proved in [4, Lemma 4]. As to analogues for other series of representations, see [4, Section 4]. □

The following assertion is a consequence:

Lemma 2. *Let* V *be in the unitary principal series. Let* α *be an arbitrary complex number with* $\mathrm{Re}\,\alpha$ *being positive and sufficiently large. Then there exists an element* $\Phi(\cdot,\alpha)$ *in* V *such that*

$$(7) \quad \Phi(\mathrm{n}[x]\mathrm{a}[y],\alpha) = y^\alpha \sum_{n=1}^{\infty} \varrho_V(n) n^{\alpha-\frac{1}{2}} \exp(2\pi i n(x+iy)).$$

Remark. This brings us to a situation similar to (3). In fact, we notice a correspondence between α and ℓ. Our proof gives a lower bound for $\mathrm{Re}\,\alpha$, which is, however, not uniform in V.

Proof. Let $\nu \in i\mathbf{R}$ be the spectral parameter of V, and let $\phi \in U_\nu$ be such that

$$(8) \qquad \mathcal{K}\phi(u) = \begin{cases} u^\alpha \exp(-2\pi u) & \text{for } u \geq 0, \\ 0 & \text{for } \ u < 0. \end{cases}$$

This is possible, for $\mathcal{K}$ is surjective and the member on right side is obviously in $L^2(\mathbf{R}^\times, \pi^{-1}d^\times)$. Let

$$\phi(\mathrm{g}) = \sum_{p=-\infty}^{\infty} a_p\phi_p(\mathrm{g}), \quad \phi_p(\mathrm{g}) = \phi_p(\mathrm{g};\nu),$$

where $a_p = a_p(\nu,\alpha)$. We put

$$\Phi(\mathrm{g},\alpha) = \sum_{p=-\infty}^{\infty} a_p\varphi_p(\mathrm{g}),$$

with φ_p as in (1). We shall later prove briefly that

$$(9) \qquad \Phi(\mathrm{g},\alpha) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\mathrm{sgn}(n)}\phi(\mathrm{a}[|n|]\mathrm{g}),$$

provided $\mathrm{Re}\,\alpha$ is sufficiently large. This gives (7), since

$$\mathcal{A}^{\mathrm{sgn}(n)}\phi(\mathrm{a}[|n|]\mathrm{n}[x]\mathrm{a}[y]) = \exp(2\pi i n x)\mathcal{K}\phi(ny).$$

We shall indicate how to prove (9). This is via an explicit computation of the coefficients a_p. The unitaricity of $\mathcal{K}$ gives

$$a_p = \frac{1}{\pi}\int_0^\infty u^{\alpha-1}\exp(-2\pi u)\overline{\mathcal{A}^+\phi_p(\mathrm{a}[u])}du\,.$$

As is well known, the $\mathcal{A}$-factor can be related to the confluent hypergeometric function (see, e.g., [4, (2.16)]). Then we use the formula 7.621 (3) of [8]. Or one may rather proceed directly with the present definition. We find that

$$a_p = (-1)^p 2^{-2\alpha}\pi^{-\nu-\alpha-\frac{1}{2}} \cdot \frac{\Gamma\left(\alpha+\nu+\frac{1}{2}\right)\Gamma\left(\alpha-\nu+\frac{1}{2}\right)}{\Gamma(\frac{1}{2}-\nu+p)\Gamma(\alpha+1-p)}.$$

In particular,

$$a_p \ll (|p|+1)^{-\mathrm{Re}\,\alpha-\frac{1}{2}},$$

as $|p|$ tends to infinity, and $\nu \in i\mathbf{R}$ is bounded. Thus, indeed $\phi \in U_\nu$ if $\mathrm{Re}\,\alpha > 0$, and ϕ becomes smoother if we take $\mathrm{Re}\,\alpha$ larger. The confirmation of (9) follows from this and the uniform bound

$$\mathcal{A}^\delta\phi_p(\mathrm{a}[y]) \ll (|p|+|\nu|+1)y^{-\frac{1}{2}} \cdot \exp\left(-\frac{y}{|\nu|+|p|+1}\right)$$

(see [4, (4.5)]). □

Next, we move to an inner-product argument, corresponding to (6): Let $\tau(\theta)$ be an infinitely differentiable function supported on a small neighbourhood of $\theta = 0$, and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tau(\theta)d\theta = 1.$$

Let m be a positive integer, and $\mathrm{Re}\,\xi > 1$. Put

$$h(\mathrm{g}) = y^\xi \exp(2\pi m i(x+iy))\tau(\theta).$$

Further, put

$$(10) \qquad \mathcal{P}h(\mathrm{g}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(\gamma\mathrm{g}), \quad \Gamma_\infty = N \cap \Gamma.$$

This is in $L^2(\Gamma\backslash G)$. Then, consider the inner product

$$\langle \mathcal{P}h, |\Phi|^2\rangle_{\Gamma\backslash G}.$$

Let us assume that α is positive and sufficiently large. The unfolding argument gives

$$\begin{aligned}&\langle \mathcal{P}h, |\Phi|^2\rangle_{\Gamma\backslash G}\\ &= \frac{1}{\pi}\int_0^\infty \int_0^1 y^{\xi-2}\exp(2\pi mi(x+iy))\\ &\cdot\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\tau(\theta)|\Phi(\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta],\alpha)|^2 d\theta dx dy.\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{\tau}\langle \mathcal{P}h, |\Phi|^2\rangle_{\Gamma\backslash G}\\ &= \frac{1}{\pi}\int_0^\infty \int_0^1 y^{\xi-2}\exp(2\pi mi(x+iy))\\ &\cdot|\Phi(\mathrm{n}[x]\mathrm{a}[y],\alpha)|^2 dx dy,\end{aligned}$$

where the support of τ shrinks to the point 0. The expression (7) implies that

$$\begin{aligned}&\sum_{n=1}^\infty \frac{\varrho_V(n)\overline{\varrho_V(n+m)}}{(n+m)^\xi(1+m/n)^{\alpha-\frac{1}{2}}}\\ &= \frac{\pi(4\pi)^{\xi+2\alpha-1}}{\Gamma(\xi+2\alpha-1)}\lim_{\tau}\langle \mathcal{P}h, |\Phi|^2\rangle_{\Gamma\backslash G}.\end{aligned}$$

With this, we may use the argument of [12, Section 1] and attain the inner sum of (4). In fact, it suffices for us to multiply both sides by the factor

$$\begin{aligned}&m^{-u-v+\xi}\Gamma(u+v-\xi)\\ &\cdot\frac{1}{2\pi i}\int_{\mathrm{Im}\,t=-c}\frac{\Gamma(\frac{3}{2}-u-\alpha+it)}{\Gamma(v+\frac{3}{2}-\alpha-\xi+it)}g(t)dt\end{aligned}$$

with $c>0$ sufficiently large, and integrate with respect to ξ along an appropriate vertical line. Provided α is sufficiently large and $\mathrm{Re}(u+v) > \mathrm{Re}\,\xi > 1$, the necessary absolute convergence holds throughout our procedure. Inserting the thus obtained expression into (4), we find that (4) admits an expression in terms of $\langle \mathcal{P}h, |\Phi|^2\rangle_{\Gamma\backslash G}$, provided $\mathrm{Re}(u+v) > 2$.

This ends the treatment of the unitary principal series. The case of the Eisenstein series or that pertaining to $\mathcal{M}_4(\zeta; g)$ is obviously analogous.

Therefore we have proved that (4) with V in any series of representations can be reached within the notion of automorphy, provided u, v are to be restricted appropriately. Admittedly, it remains for us to discuss the spectral decomposition that should follow, especially its analytic continuation to the central point $(u,v) = \left(\frac{1}{2},\frac{1}{2}\right)$. Nevertheless, we may claim that the above supports the view that there ought to exist a unified theory of mean values of automorphic L-functions.

Concluding remark. Probably our choice (7), i.e., (8), will turn out too drastic in practice. We shall then need to take into account the smoothness of Φ when the variable g approaches to the Bruhat cell NA from inside the big cell. Namely, we expect that we shall have to use instead a sequence whose limit is the present ϕ, in a way similar to the situation that is experienced in [4] with the seed function of the Poincaré series used there. In this context, the above should be regarded as a precursor of a more rigorous discussion to come.

Our argument can readily be extended to the setting $G = \mathrm{PSL}(2,\mathbf{C})$ and $\Gamma = \mathrm{PSL}(2,\mathbf{Z}[\sqrt{-1}])$. All necessary facts are given in [2] and [3] (see also [1]). Bigger groups could also be taken into consideration. For instance, we presume that a certain *double* mean value of the 12th power of the Riemann zeta-function could be grasped within the setting $G = \mathrm{PSL}(3,\mathbf{R})$ and $\Gamma = \mathrm{PSL}(3,\mathbf{Z})$. Here the term double transpires from the real rank of G. D. Bump's work [5] is relevant to this motivation of ours.

Acknowledgement. We are much indebted to R.W. Bruggeman for his expert comments to an earlier version of the present note. We thank to M. Jutila and A. Ivić for their encouraging comments.

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