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ALTERNATIVE DESCRIPTION OF RIGID BODY KINEMATICS AND QUANTUM MECHANICAL ANGULAR MOMENTA

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Abstract. In the present paper we investigate an alternative two-axes decomposition method for rotations that has been proposed in our earlier research. It is shown to provide a convenient parametrization for many important physical systems. As an example, the kinematics of a rotating rigid body is considered and a specific class of solutions to the Euler dynamical equations are obtained in the case of symmetric inertial ellipsoid. They turn out to be related to the Rabi oscillator in spin systems well known in quantum computation. The corresponding quantum mechanical angular momentum and Laplace operator are derived as well with the aid of infinitesimal variations. Curiously, the coefficients in this new representation happen to depend only on one of the angles, which simplifies the corresponding system of ODE's emerging from separation of variables. Some applications of the hyperbolic and complex analogues of this construction in quantum mechanics and relativity are considered in a different paper cited below.
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Introduction

Finding new representations of an old problem may seem like a triviality, but quite often proves to be extremely fruitful both from a practical and theoretical point of view. Moreover, the technique in many cases comes from a most unexpected field and through an utterly vague analogy, e.g. the Lax representation in integrable systems that is now a standard in the theory. The approach proposed here is certainly nothing like the famous Lax representation, but it illustrates how an engineering idea may have consequences for the theory. Namely, we consider a decomposition of finite rotations alternative to the classical one proposed by Euler, in which the first axis is fixed while the second one depends dynamically on the compound transformation's parameters. A similar idea has been used in [11, 15] in a kinematical context, but not fully developed until [3]. Although our method is naturally related to the classical Euler decomposition, the advantages of working with a factorization into just two, rather than three rotations, are numerous. To begin with, the expressions are simplified greatly and unlike any Euler type scheme, it allows for a restriction to the rational case, which has some practical implementations as well. The interesting part begins, however, when we consider variations (Lie derivatives) of the previously obtained solutions. This allows us for instance to obtain a peculiar representation of the quantum mechanical angular momentum operator and the associated Laplacian (whose coefficients depend only on one of the decomposition angles) that projects well on homogeneous spaces of $\mathrm{SO}(3)$ such as the two-sphere. As can be expected, the spherical harmonics in this case can be obtained via separation of variables in terms of Legendre polynomials. On the other hand, considering rigid body kinematics in the dynamical reference frame described above is also quite advantageous - the system one obtains for the case of rotational inertial ellipsoid looks much simpler than the one resorting on Euler angles. Moreover, going one step farther we derive a specific class of solutions to the dynamical equations of a rotating rigid body that mimics the Rabi oscillator [14] well known in the quantum mechanics of spin systems and quantum computation. For a similar treatment of the hyperbolic case (both three and four-dimensional) we refer to a recent paper of ours [8] focused on the possible applications to relativity and quantum mechanical scattering on the line. It also provides various examples.

1. Vectorial Parametrization of Rotations

The vector-parameter for $\mathrm{SO}(3)$ was initially introduced as a vector $\mathbf{c}$ along the axis of rotation $\mathbf{n}$ with magnitude, equal to $\tau = \tan\dfrac{\phi}{2}\cdot$ However, as we argue in [4], this quantity is not a vector at all and should be treated as a point in projective space $\mathbb{RP}^3$, equipped with an additional operation. These properties are naturally obtained via projection from the spin covering group. Such parametrization appears to be very convenient for several reasons. On the one hand, it gives a topologically adequate description of the orthogonal group $\mathrm{SO}(3) \cong \mathbb{RP}^3$ unlike other known alternatives, such as the Euler angles, which use parameters on the torus

$\mathbb{T}^3$ and hence, involve singularities such as gimbal lock. On the other, quaternion multiplication projects to an efficient composition law for two vector-parameters in the form

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)} \tag{1}$$

which is associative and constitutes a (nonlinear) representation[1] of the group in its automorphisms, corresponding to left-deck transformations, i.e., $\mathcal{R}(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle) = \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$. Moreover, this representation follows naturally from a central projection of the three-sphere (division by the real part of the quaternion) and yields rational expressions for the matrix entries of the corresponding rotation via the Cayley map (or using Euler's substitution in Rodrigues' formula)

$$\mathcal{R}(\mathbf{c}) = \mathrm{Cay}(\mathbf{c}^{\times}) = \frac{1 - \mathbf{c}^2 + 2\,\mathbf{c}\,\mathbf{c}^t + 2\,\mathbf{c}^{\times}}{1 + \mathbf{c}^2} \tag{2}$$

where $\mathbf{c}\,\mathbf{c}^t$ denotes the tensor (dyadic) product and $\mathbf{c}^{\times}$ - the Hodge conjugate of $\mathbf{c}$ defined as $\mathbf{c}^{\times}\mathbf{a} = \mathbf{c} \times \mathbf{a}$.

2. Two-Axes Decompositions

In [12] and [4] the generalized Euler decompositions of $\mathrm{SO}(3)$ and $\mathrm{SO}(2,1)$ have been thoroughly investigated. Here we focus on the case of two axes denoting $\mathbf{c}_k = \tau_k\,\hat{\mathbf{c}}_k$ $(k = 1, 2)$, where $\hat{\mathbf{c}}_k$ are the unit vectors along the axes of rotations in the decomposition and $\tau_k = \tan\dfrac{\phi_k}{2}$ - are the corresponding *scalar parameters*. Using the Euler invariant axis theorem, we easily obtain that the decomposition $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$ is possible if and only if $r_{21} = g_{21}$ with the notation

$$g_{ij} = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j), \qquad r_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j).$$

As for the solutions, they can be retrieved directly from the composition laws in the form

$$\tau_1 = \frac{\tilde{\rho}_3}{g_{1[2}\rho_{1]}}, \qquad \tau_2 = \frac{\tilde{\rho}_3}{g_{2[1}\rho_{2]}} \tag{3}$$

with the notation $\rho_k = (\mathbf{c}, \hat{\mathbf{c}}_k)$, $\tilde{\rho}_3 = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathbf{c})$ and $a_{[i}b_{j]} = a_i b_j - a_j b_i$.

In [2] we propose decomposition of rotations into two factors, one of which has a fixed axis and the other - a fixed angle. One slight variation of that approach is to fix, instead of the latter, the angle $\gamma = \arccos|g_{12}|$ between the two invariant axes

[1] The inverse element is represented by the opposite vector - $\mathbf{c}$, and the identity by the zero vector.

in the decomposition. Then, by formula (10) and (11) in [2] we have for the two angles

$$\phi_1 = 2\arctan(\frac{\rho_1 - g_{12}\tau_2}{1 + g_{12}\rho_1\tau_2}), \qquad \phi_2 = \arccos(\frac{r_{11} - g_{12}^2}{1 - g_{12}^2}) \tag{4}$$

where the second axis is determined from $\hat{\mathbf{c}}_2 = \tau_2^{-1}\langle \mathbf{c}, -\tau_1\hat{\mathbf{c}}_1\rangle$. Note that for $g_{12} \neq 0$ real solutions exist only if $|\arccos r_{11}| \leq 2\gamma$ (see [6] for details), so it is guaranteed only for $g_{12} = 0$, which is the case considered in [3, 11]. There, however, we take an arbitrary fixed axis $\hat{\mathbf{c}}_1$ and determine $\hat{\mathbf{c}}_2$ as

$$\hat{\mathbf{c}}_2 = \lambda\,\hat{\mathbf{c}}_1 \times \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \qquad \lambda = (1 - r_{11}^2)^{-1/2}. \tag{5}$$

This ensures that the necessary and sufficient condition $r_{21} = g_{21}$ is satisfied and since formula (1) yields

$$\rho_1\rho_2 + \tilde{\rho}_3 \tag{6}$$

the solutions for τ_1 and τ_2 are easily obtained in the form

$$\tau_1 = \rho_1, \qquad \tau_2 = \rho_2 = \sqrt{\frac{1 - r_{11}}{1 + r_{11}}} \tag{7}$$

which can be expressed in terms of the corresponding angles as

$$\phi_1 = 2\arctan\rho_1, \qquad \phi_2 = \arccos r_{11}. \tag{8}$$

In particular, $\mathbf{n} \perp \hat{\mathbf{c}}_1$ yields $\phi_1 = 0$ and $\phi_2 = \phi$ as the second axis is, by construction, parallel to $\mathbf{n}$. One exception is the case of a half-turn, i.e., $\mathcal{R}(\mathbf{c}) = \mathcal{O}(\mathbf{n}) = 2\,\mathbf{n}\,\mathbf{n}^t - \mathcal{I}$, in which we end up with a one-parameter degenerate set of solutions, similar to the gimbal lock setting familiar from the classical Euler representation (cf. [1]). Moreover, for a specific choice of second axis $\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_1$ one has a decomposition in the form $\mathcal{O}(\mathbf{n}) = \mathcal{O}(\hat{\mathbf{c}}_2)\mathcal{R}(\mathbf{c}_1)$ with[2] $\phi_1 = 2\measuredangle(\mathbf{c}_2, \mathbf{n})$. In particular, on may safely choose $\mathbf{c}_2 = \mathbf{n} \times \mathbf{c}_1$ and factorize into a pair of half-turns, i.e., $\mathcal{O}(\mathbf{n}) = \mathcal{O}(\hat{\mathbf{c}}_2)\,\mathcal{O}(\hat{\mathbf{c}}_1)$.

Note that we may obtain the second vector-parameter directly from (1) as

$$\mathbf{c}_2 = \langle \mathbf{c}, -\mathbf{c}_1\rangle = \frac{\mathbf{c} - \rho_1\,(\mathcal{I} + \mathbf{c}^{\times})\,\hat{\mathbf{c}}_1}{1 + \rho_1^2}. \tag{9}$$

One easily completes the (orthonormal) basis with a third vector in the form

$$\hat{\mathbf{c}}_3 = \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 = \lambda\,[r_{11}\,\mathcal{I} - \mathcal{R}(\mathbf{c})]\,\hat{\mathbf{c}}_1.$$

[2]Here and bellow $\measuredangle(\mathbf{n}, \hat{\mathbf{c}}_k)$ denotes the minimal positive angle between the two axes.

In order to parameterize the group of rotations we choose the third parameter κ as the normal component of the rate, at which $\hat{\mathbf{c}}_2$ varies with respect to $\mathcal{R}$, i.e., if we let d_t denote the time derivative, we may write

$$\mathrm{d}_t\hat{\mathbf{c}}_2 = \mathrm{d}_t(\kappa)\,\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2. \tag{10}$$

2.1. Relation to the Classical Euler Angles

Note that if we choose $\hat{\mathbf{c}}_1$ to be the z-axis, the above factorization may be expressed in terms of the standard Euler angles as $\phi_1 = \varphi + \psi$ and $\phi_2 = \vartheta$. A direct way to see this is to write the Euler decomposition in the form (see [1])

$$\mathcal{R}(\mathbf{n},\phi) = \mathcal{R}(\hat{\mathbf{c}}_1,\psi)\mathcal{R}(\hat{\mathbf{c}}_2,\vartheta)\mathcal{R}(\hat{\mathbf{c}}_1,\varphi) = \mathcal{R}\left(\mathcal{R}(\hat{\mathbf{c}}_1,\psi)\,\hat{\mathbf{c}}_2,\vartheta\right)\mathcal{R}(\hat{\mathbf{c}}_1,\varphi+\psi)$$

where we use conjugation and the norm-preserving property of $\mathcal{R}(\hat{\mathbf{c}}_1,\psi)$. The next step is to show that there is a rotation about $\hat{\mathbf{c}}_1$ which sends the unit vector $\hat{\mathbf{c}}_2$ to the unit vector $(1-r_{11}^2)^{-1/2}\,\hat{\mathbf{c}}_1 \times \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1$. It is straightforward to see that it does and its scalar parameter is given by

$$\tau_3 = \frac{r_{21}}{\sqrt{1-r_{11}^2} - \mathring{\omega}_2}, \qquad \mathring{\omega}_2 = \left(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2\right). \tag{11}$$

Assuming $(\hat{\mathbf{c}}_1,\hat{\mathbf{c}}_2) = 0$ (otherwise not every rotation can be decomposed), we may label the axes $\hat{\mathbf{c}}_{1,2}$ with OZ and OX, respectively, and formula (11) coincides with one of the two solutions for the classical Euler setting, namely

$$\tau_3 = \frac{\mathcal{R}_{13}}{\sqrt{1-\mathcal{R}_{33}^2} - \mathcal{R}_{23}} = \frac{\sin\psi}{1+\cos\psi}. \tag{12}$$

On the other hand, as we argue in [6], the Euler decomposition of the rotation $\mathcal{R}(\mathbf{n},\phi)$ is possible with respect to non-orthogonal axes for an arbitrary angle ϕ as long as $\measuredangle(\mathbf{n},\hat{\mathbf{c}}_1) \leq \arccos|g_{12}| = \measuredangle(\hat{\mathbf{c}}_1,\hat{\mathbf{c}}_2)$ and for an arbitrary axis $\mathbf{n}$ as long as the angle satisfies the condition $|\phi| \leq 2\arccos|g_{12}|$, in which case the analogy with (8) is lost and one needs to use the more general formula (4) instead.

2.2. Rational Decompositions

This new type of decomposition provides one more interesting opportunity: namely, it yields a setting that guarantees rational decomposition for an arbitrary rotation with rational matrix entries $\mathcal{R} \in \mathrm{SO}(3,\mathbb{Q})$. It is easy to show that this is not the case with the standard Euler-type representations (see [5]), in which one always ends up with some non-trivial Pythagorean relation[3]. Here, on the other hand, one only needs to start with a unit rational vector $\hat{\mathbf{c}}_1$ and the above construction yields

$$\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_1 \times \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1$$

[3] In the hyperbolic case such a universal settings exist, e.g. the Iwasawa decomposition.

which is not unit this time, since normalization over the rational numbers is non-trivial. This effectively multiplies τ_2 by the normalizing factor λ, i.e.,

$$\tau_1 = \rho_1, \qquad \tau_2 = \frac{1}{1 + r_{11}} \tag{13}$$

and the solutions are rational too. Note that the only non-trivial part in this construction is the initial choice of a unit vector with rational components. Fortunately, we have generators of Pythagorean quadruples at our disposal, which may produce infinitely many vectors with this property. Thus, we effectively introduce rational parametrization of $\mathrm{SO}(3, \mathbb{Q}) \cong \mathbb{PQ}^3$, which appears to be very convenient for various kinematical and dynamical considerations, as seen in the continuous analogue. Here, however, the time parameter should also be discrete and differentials - either introduced purely algebraically or replaced by finite differences.

3. The Angular Momentum and the Laplacian

Next, we consider variations with respect to left shifts of the type

$$u^L_{\hat{\boldsymbol{\varsigma}}}(t)\mathbf{c} = \langle t\,\hat{\boldsymbol{\varsigma}}, \mathbf{c}\rangle, \qquad \hat{\boldsymbol{\varsigma}} \in \mathbb{S}^2. \tag{14}$$

Direct differentiation yields in this case

$$\begin{aligned} \partial_{\hat{\boldsymbol{\varsigma}}}\phi_1 &= (1+\rho_1^2)^{-1}\left[(\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1) + \rho_1(\hat{\varsigma}, \mathbf{c}) + (\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\varsigma})\right] \\ \partial_{\hat{\boldsymbol{\varsigma}}}\phi_2 &= \lambda\,\delta\left[(\hat{\boldsymbol{\varsigma}}, \mathbf{c}) - \rho_1(\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1) - \rho_1(\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\boldsymbol{\varsigma}})\right] \end{aligned} \tag{15}$$

and without much trouble we also find that $\partial_{\hat{\varsigma}}\kappa = (\partial_{\hat{\varsigma}}\hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$, or

$$\partial_{\hat{\boldsymbol{\varsigma}}}\kappa = \lambda^2\delta\big\{(1+\rho_1^2)(\hat{\varsigma}, [\mathcal{I} - \mathcal{R}(\mathbf{c})]\,\hat{\mathbf{c}}_1) + (1-r_{11})[(1+\rho_1)(\mathbf{c}, \hat{\varsigma}) + (\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\varsigma})]\big\}\,.$$

In particular, one may choose to work with the invariant axis of the compound rotation $\hat{\boldsymbol{\varsigma}} = \mathbf{n}$ and derive

$$\partial_\phi\phi_1 = \frac{\sin\phi_1}{\sin\phi}, \qquad \partial_\phi\phi_2 = \frac{\sin\phi_2}{\sin\phi}\Big(\frac{\cos\phi_1 - \cos\phi}{1 - \cos\phi_2}\Big), \qquad \partial_\phi\kappa = \frac{\sin\phi_1\sin\phi_2}{\sin\phi}$$

thus obtaining for example the differential of ϕ in the form

$$\mathrm{d}\phi = \sin\phi(\csc\phi_1\,\mathrm{d}\phi_1 + \frac{1-\cos\phi_2}{\cos\phi_1 - \cos\phi}\,\csc\phi_2\,\mathrm{d}\phi_2 + \csc\phi_1\csc\phi_2\,\mathrm{d}\kappa).$$

Note that the direct differentiation yields

$$\mathrm{d}\phi = \cot\frac{\phi}{2}(\tan\frac{\phi_1}{2}\,\mathrm{d}\phi_1 + \tan\frac{\phi_2}{2}\,\mathrm{d}\phi_2)$$

instead, but it is not correct for arbitrary rotations, since it neglects the fact that ϕ actually affects the direction of the second axis. For the derivation of the above formulae we use the compound vector-parameter representation

$$\mathbf{c} = (\tau_1,\ \tau_2,\ -\tau_1\tau_2)^t \quad \Rightarrow \quad 1 + \mathbf{c}^2 = (1+\tau_1^2)(1+\tau_2^2) \tag{16}$$

and thus $\delta = \dfrac{2}{1+\mathbf{c}^2} = \dfrac{1}{2}\,(1+\cos\phi_1)(1+\cos\phi_2)$.

Without much effort, we obtain the variations of the parameters ϕ_1, ϕ_2 and κ with respect to right deck transformations in the form

$$\begin{aligned} \partial_{\hat{\boldsymbol{\varsigma}}}\phi_1 &= (1+\rho_1^2)^{-1}\left[(\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1) + \rho_1(\hat{\boldsymbol{\varsigma}}, \mathbf{c}) - (\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\varsigma})\right] \\ \partial_{\hat{\boldsymbol{\varsigma}}}\phi_2 &= \lambda\,\delta\left[(\hat{\boldsymbol{\varsigma}}, \mathbf{c}) - \rho_1(\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1) + \rho_1(\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\boldsymbol{\varsigma}})\right] \end{aligned} \tag{17}$$

and finally

$$\begin{aligned} \partial_{\hat{\boldsymbol{\varsigma}}}\kappa = \lambda^2\delta\big\{(1-\rho_1^2)(\hat{\boldsymbol{\varsigma}}, [\mathcal{I}-\mathcal{R}]\,\hat{\mathbf{c}}_1) + (1-r_{11})[\rho_1(\mathbf{c}, \hat{\boldsymbol{\varsigma}}) - (\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\boldsymbol{\varsigma}}) - (\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1)] \\ +\ 2\rho_1(\hat{\boldsymbol{\varsigma}}, \hat{\mathbf{c}}_1, \mathcal{R}\hat{\mathbf{c}}_1)\big\} \end{aligned}$$

which allows for a straightforward representation of the quantum mechanical angular momentum operator (the technique is thoroughly explained in [7, 16])

$$\begin{aligned} L_1 &= \frac{\partial}{\partial\phi_1} \\ L_2 &= \sin\phi_1\tan\frac{\phi_2}{2}\frac{\partial}{\partial\phi_1} + \cos\phi_1\frac{\partial}{\partial\phi_2} + \sin\phi_1\csc\phi_2\frac{\partial}{\partial\kappa} \\ L_3 &= \cos\phi_1\tan\frac{\phi_2}{2}\frac{\partial}{\partial\phi_1} - \sin\phi_1\frac{\partial}{\partial\phi_2} + \cos\phi_1\csc\phi_2\frac{\partial}{\partial\kappa}\cdot \end{aligned} \tag{18}$$

The associated Laplace operator (quantum hamiltonian), namely

$$\Delta = \sec^2\frac{\phi_2}{2}\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2} - \tan\frac{\phi_2}{2}\frac{\partial}{\partial\phi_2} + \csc^2\phi_2\frac{\partial^2}{\partial\kappa^2} \tag{19}$$

has one advantage - its coefficients depend solely on the second angle ϕ_2.

With the aid of the standard notation $(\phi_1, \phi_2) \to (\varphi, \vartheta)$ one may rewrite the above as

$$\Delta = \sec^2\frac{\vartheta}{2}\left[\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial\vartheta}\left(\cos^2\frac{\vartheta}{2}\frac{\partial}{\partial\vartheta}\right)\right] + \csc^2\vartheta\frac{\partial^2}{\partial\kappa^2}\cdot \tag{20}$$

Clearly, for orbits, on which the second axis $\hat{\mathbf{c}}_2$ is stationary, we obtain the restriction of Δ on the projective quadric (6), which turns out to be locally isomorphic to a two-dimensional sphere, in the form

$$\Delta_{\mathbb{S}^2} = \sec^2\frac{\vartheta}{2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\vartheta^2} - \tan\frac{\vartheta}{2}\frac{\partial}{\partial\vartheta}\cdot \tag{21}$$

With the aid of separation of variables the above can easily be decomposed into a system of ODE's in the form

$$\ddot{\Phi} + \nu^2\Phi = 0, \qquad (1+\tau^2)\,\ddot{\Theta} - 2\tau\dot{\Theta} - 4\nu^2\Theta = 0 \tag{22}$$

for the spherical harmonic

$$\Psi(\varphi, \vartheta) = \Phi(\varphi)\Theta(\tau), \qquad \tau = \tan\frac{\vartheta}{2}\cdot$$

The explicit solutions involve Legendre polynomials just as in the generic case

$$\Psi(\varphi, \vartheta, \kappa) = \Phi(\varphi)\Theta(\tau)\chi(\kappa).$$

Note that with the change of variables $\vartheta \to \tau$ the nonconstant coefficients in the ODE determining the second angle are all rational (both in the two and three-dimensional case), which also simplifies our work.

4. Applications to Rigid Body Mechanics

The $\{\phi_1, \phi_2, \kappa\}$ coordinates have been successfully exploited in [11] for the derivation of the system of differential equations governing the kinematics of rigid bodies in the form[4]

$$\begin{aligned} \dot{\phi}_1 &= \Omega_1 - \Omega_3 \tan\frac{\phi_2}{2} \\ \dot{\phi}_2 &= \Omega_2 \\ \dot{\kappa} &= \Omega_1 + \Omega_3 \cot\phi_2 \end{aligned} \tag{23}$$

where Ω_k denote the components of the angular velocity in the so chosen basis. The derivation is based on the explicit relation between the time derivative of the vector-parameter and the angular velocity in the rotating frame [9, 10, 13]

$$\dot{\mathbf{c}} = \frac{1}{2}\left(\boldsymbol{\Omega} + (\mathbf{c}, \boldsymbol{\Omega})\,\mathbf{c} - \mathbf{c}\times\boldsymbol{\Omega}\right) \tag{24}$$

and take into account formula (10) and (16). Inverting the matrix of the above system, one easily obtains

$$\begin{aligned} \Omega_1 &= \dot{w} - \cos v\,\dot{u} \\ \Omega_2 &= \dot{v} \\ \Omega_3 &= \sin v\,\dot{u} \end{aligned} \tag{25}$$

where we make use of the notation $u = \kappa - \phi_1$, $v = \phi_2$ and $w = \kappa$.

As far as dynamics is concerned, the above construction provides a suitable set of parameters for the description of various systems. Consider the Euler equations of motion

$$\dot{\Omega}_1 = \mu_1\,\Omega_2\,\Omega_3, \qquad \dot{\Omega}_2 = \mu_2\,\Omega_1\,\Omega_3, \qquad \dot{\Omega}_3 = \mu_3\,\Omega_1\,\Omega_2 \tag{26}$$

[4] To our knowledge, a particular case of this construction was first considered in [15].

where the constants μ_k are expressed in terms of the eigenvalues I_k of the inertial tensor as

$$\mu_1 = \frac{I_2 - I_3}{I_1}, \qquad \mu_2 = \frac{I_3 - I_1}{I_2}, \qquad \mu_3 = \frac{I_1 - I_2}{I_3}.$$

If we assume the so chosen coordinate system is canonical for the inertial tensor, direct differentiation of (25) yields

$$\begin{aligned} \ddot{u} &= \mu_3 \, \dot{v} \, \dot{w} - (\mu_3 + \csc v) \cos v \, \dot{u} \, \dot{v} \\ \ddot{v} &= \mu_2 (\dot{w} - \cos v \, \dot{u}) \sin v \, \dot{u} \\ \ddot{w} &= \left(\mu_1 \sin v - (1 + \mu_3 \cos^2 v) \csc v\right) \, \dot{u} \, \dot{v} + \mu_3 \cot v \, \dot{v} \, \dot{w}. \end{aligned} \tag{27}$$

In particular, for the case of symmetric inertial ellipsoid, i.e., $\mu_1 = -\mu_2 = \mu$, $\mu_3 = 0$, the above system is reduced to

$$\begin{aligned} \ddot{u} &= -\cot v \, \dot{u} \, \dot{v} \\ \ddot{v} &= \mu (\cos v \, \dot{u} - \dot{w}) \sin v \, \dot{u} \\ \ddot{w} &= (\mu \sin v - \csc v) \, \dot{u} \, \dot{v} \end{aligned} \tag{28}$$

which is significantly simpler compared to the standard representation based on Euler angles for example. In particular, we have an obvious conservation law: from the third equation in (25) and the first one in (28) it follows that

$$\dot{u} = \Omega_3 \csc v, \qquad \Omega_3 = \text{const}. \tag{29}$$

Combining the second and the third equation of (28), one obtains for the middle parameter in the form (a, b and $\varphi_\circ$ being constants of integration)

$$\dddot{v} + \omega^2 \, \dot{v} = 0 \qquad \Rightarrow \qquad v = a \cos(\omega t + \varphi_\circ) + b \tag{30}$$

with the notation $\omega = \mu \, \Omega_3$. The latter yields

$$\begin{aligned} \dot{u} &= \Omega_3 \csc \left(a \cos(\omega t + \varphi_\circ) + b\right) \\ \dot{w} &= \Omega_3 \cot \left(a \cos(\omega t + \varphi_\circ) + b\right) + a\omega \cos(\omega t + \varphi_\circ). \end{aligned} \tag{31}$$

Although the integrals for u and w cannot be resolved in terms of elementary functions, one may easily obtain the solutions for the angular velocity components Ω_k by direct substitution in (25). The evolution of the system turns out to be described in these coordinates by the circle

$$\Omega_1(t) = a\omega \cos(\omega t + \varphi_\circ), \qquad \Omega_2(t) = -a\omega \sin(\omega t + \varphi_\circ). \tag{32}$$

However, expressing $u(t)$ and $w(t)$ from the system (31) in far non-trivial even if we parameterize with the solution for $v(t)$, which yields the integrals

$$u = \mp\frac{1}{\mu}\int\frac{\csc v\,\mathrm{d}v}{\sqrt{a^2-(v-b)^2}}\,, \qquad w = \mp\frac{1}{\mu}\int\frac{\cot v+\mu(v-b)}{\sqrt{a^2-(v-b)^2}}\,\mathrm{d}v \tag{33}$$

where the sign equals $\operatorname{sgn}(\omega t+\varphi_\circ)$ and we have a singularity at the origin.

Solving the above integrals, however, is unnecessary since we have already reduced the dynamical problem to a kinematical one. More precisely, in the so chosen coordinate system we have a precessing angular velocity, for which case there are well-known solutions due to Rabi [14], for which he won a Nobel Prize. Nowadays Rabi cycles are famous mostly due to their implementation in quantum computers, but the solutions may be directly related to the kinematics of a rigid body with precessing angular velocity. We discuss this analogy in much more detail elsewhere.

Final Remarks

The suggested alternative parametrization for the kinematics of a rigid body is based on a specific decomposition method proposed earlier in [3]. A particular class of solutions to the Euler dynamical equations is obtained as an example, which is related to the well-known Rabi oscillator, and the quantum mechanical angular momentum is given an alternative representation. One may ask the question of possible extensions to both the hyperbolic case and higher dimensions. These topics are thoroughly discussed in our recent paper [8] with an emphasis on the Lorentz groups $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3,1)$ as well as the possible applications in special relativity and quantum mechanics. Moreover, the construction is shown to work in general for $\mathrm{SO}(4)$, $\mathrm{SO}(2,2)$ and $\mathrm{SO}^*(4)$ as real forms of $\mathrm{SO}(4,\mathbb{C})$, which seems to be the ultimate extension to our approach due to the invertibility of the quaternion algebra.

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