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RECURSION OPERATORS FOR RATIONAL BUNDLE ON $\mathfrak{sl}(3,\mathbb{C})$ WITH $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ REDUCTION OF MIKHAILOV TYPE

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Abstract. We consider the recursion operator related to a system introduced recently that could be considered as a generalization to a pole gauge generalized Zakharov-Shabat system on $\mathfrak{sl}(3,\mathbb{C})$ but involving rational dependence on the spectral parameter and subject to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ reduction of Mikhailov type. We calculate the hierarchies of nonlinear evolution equations related to this system through the recursion operators we introduce.

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1. Introduction. Systems on $\mathfrak{sl}(3)$ and the GMV System

The generalized Zakharov-Shabat system (GZS) and Caudrey-Beals-Coifman system (CBC) in pole gauge on the algebra $\mathfrak{sl}(3)$ initially has been studied as an application of the general results about GZS and CBC system in pole gauge, see [1] and references in [2]. As a result, the generating operator has been calculated and some systems of Heisenberg Ferromagnet (HF) type with possible physical applications, [9]. The interest in the pole gauge systems was renewed after the system that we refer as GMV (Gerdjikov-Mikhailov-Valchev) has been introduced [3–5]. At the beginning the GMV system study started independently, spectral properties were studied and generating operators were calculated. Later it was pointed out that GMV could be treated as $\mathfrak{sl}(3)$ GZS system in pole gauge with additional reductions of Mikhailov type, so that the generating operators found for the GMV system could be obtained from the generating operator for the general $\mathfrak{sl}(3)$ system and geometric interpretation has been clarified [12]. Let us introduce the GMV system. By this name we shall call the auxiliary linear problem

$$L_{S_1}\psi = (\mathrm{i}\partial_x + \lambda S_1)\psi = 0, \qquad S_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}. \tag{1}$$

In the above u, v (the potentials) are smooth complex valued functions on x belonging to the real line and by * is denoted the complex conjugation. In addition, the functions u and v satisfy the relation $u|^2 + |v|^2 = 1$. As described in [3, 4] the GMV system arises naturally when one looks for integrable system having a Lax representation $[L, A] = 0$ with L of the form $\mathrm{i}\partial_x + \lambda S$, where $S \in \mathfrak{sl}(3, \mathbb{C})$ and L, A subject to Mikhailov-type reduction requirements, see for example [7, 8]. In this particular case the Mikhailov reduction group G_0 is generated by the two elements g_0 and g_1 acting on the fundamental solutions of the system (1) as

$$\begin{aligned} g_0(\psi)(x,\lambda) &= \left[\psi(x,\lambda^*)^\dagger\right]^{-1} \\ g_1(\psi)(x,\lambda) &= H_1\psi(x,-\lambda)H_1, \qquad H_1 = \mathrm{diag}(-1,1,1) \end{aligned}$$

where $\dagger$ denotes Hermitian conjugation. Since $g_0 g_1 = g_1 g_0$ and $g_0^2 = g_1^2 = \mathrm{Id}$ we see that $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Denote $\mathcal{H}_1 : X \mapsto H_1 X H_1 = H_1 X H_1^{-1}$. Then it will be an involutive automorphism of $\mathfrak{sl}(3, \mathbb{C})$ which commutes with the complex conjugation σ that defines the real form $\mathfrak{su}(3)$ of $\mathfrak{sl}(3, \mathbb{C})$, ($\sigma(X) = -X^\dagger$). Next we introduce the spaces

$$\mathfrak{g}^{[j]} = \{X\,;\, \mathcal{H}_1(X) = (-1)^j X\}, \qquad j = 0, 1$$

and we get the splittings

$$\begin{aligned} \mathfrak{sl}(3,\mathbb{C}) &= \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \\ \mathfrak{su}(3) &= (\mathfrak{g}^{[0]} \cap \mathfrak{su}(3)) \oplus (\mathfrak{g}^{[1]} \cap \mathfrak{su}(3)) \\ \mathrm{i}\mathfrak{su}(3) &= (\mathfrak{g}^{[0]} \cap \mathrm{i}\mathfrak{su}(3)) \oplus (\mathfrak{g}^{[1]} \cap \mathrm{i}\mathfrak{su}(3)). \end{aligned} \tag{2}$$

The invariance under the reduction group G_0 means that if ψ is the common G_0-invariant fundamental solution of (1) and a linear problem of the type

$$\begin{aligned} &A\psi = \mathrm{i}\partial_t\psi + (\sum_{i=0}^{n} \lambda^k A_k)\psi = 0, \qquad A_k \in \mathfrak{sl}(3,\mathbb{C}) \\ &A_{2k+1} \in \mathfrak{g}^{[1]} \cap \mathrm{i}\mathfrak{su}(3), \qquad A_{2k} \in \mathfrak{g}^{[0]} \cap \mathrm{i}\mathfrak{su}(3), \qquad k = 0, 1, 2, \ldots. \end{aligned}$$

In the same way $S_1 \in \mathfrak{g}^{[1]} \cap \mathrm{i}\mathfrak{su}(3))$ which forces S_1 to be as in (1). In [3, 4] and in [10] have been considered the spectral theory aspects of the recursion operators related to (1) and their relation to the recursion operators in general position related to GZS system in pole gauge. The geometric aspects of the theory of those operators has been discussed in [12].

In the present article we shall consider a linear problem which is subject to the one more reduction, which has been also introduced in [3–5]. This linear problem is a sort of generalization of GMV problem but admits a bigger Mikhailov reduction group. It is generated by the three elements g_0, g_1 (as before) and g_2

$$g_2(\psi)(x,\lambda) = H_2\psi(x,\frac{1}{\lambda})H_2, \qquad H_2 = \mathrm{diag}(1,-1,1). \tag{3}$$

Since the elements g_i, $i = 0,1,2$ commute and $g_i^2 = \mathrm{Id}$ the Mikhailov reduction group is $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. As easily seen L_{S_1} cannot admit such such reduction group for which rational dependence on λ is needed. So in [3,4] has been considered the linear problem

$$L_{S_{\pm1}} = \mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1}S_{-1} \tag{4}$$

subject to reduction generated by g_0, g_1, g_2. (As it is clear the reduction group forces S_{-1} to be equal to $\mathcal{H}_2(S_1)$ where $\mathcal{H}_2(X) = H_2XH_2$. Later, in [6], has been considered the question of the recursion operators for the system (4) which we shall call rational GMV system. From (4) it is clear that the problem of the recursion operators for the rational GMV is much more complicated than that of GMV system. Here we address the algebraic aspects of the recursion operators related to (4), that is to see whether such operators arise when one resolves the relations equivalent to the Lax equation $[L_{S_{\pm1}}, A] = 0$.

2. Some Algebraic Preliminaries

We shall need some algebraic facts about the algebra $\mathfrak{sl}(3,\mathbb{C})$. It a is simple Lie algebra with Killing form $\langle X,Y\rangle = 6\,\mathrm{tr}\,XY$. If S is a regular element from $\mathfrak{sl}(3,\mathbb{C})$ it defines a Cartan subalgebra $\mathfrak{h}_S = \ker\mathrm{ad}_S = \{X\,;\,[S,X]=0\}$. In our case both S_1 and S_{-1} are regular,ant the corresponding subalgebras

$$\mathfrak{h}_{S_1} = \{X\,;\,[X,S_1]=0\}, \qquad \mathfrak{h}_{S_{-1}} = \{X\,;\,[X,S_{-1}]=0\}$$

are Cartan subalgebras. We shall denote the orthogonal complements (with respect to the Killing form) $\mathfrak{h}_{S_1}^{\perp}$ and $\mathfrak{h}_{S_{-1}}^{\perp}$ of the above spaces by by $\mathfrak{g}_{S_1}$ and $\mathfrak{g}_{S_{-1}}$ and the orthogonal projectors onto them by π_+, π_-. For $X\in\mathfrak{sl}(3,\mathbb{C})$ we shall put

$$\begin{aligned} \pi_+X &= X^{+a}, \qquad (\mathrm{Id}-\pi_+)X = X^{+d}\\ \pi_-X &= X^{-a}, \qquad (\mathrm{Id}-\pi_-)X = X^{-d}. \end{aligned}$$

We introduce now some facts about the matrices $S_{\pm1}$ that will be useful in our calculations. First, it is easy to see (for example using the fact that both $S_{\pm1}$ are simple matrices and have eigenvalues $0;\pm1$) that

1. $\mathfrak{h}_{S_1}$ is spanned by $\{S_1,\ S_2 = S_1^2 - (2/3)\mathbf{1}\}$
2. $\mathfrak{h}_{S_{-1}}$ is spanned by $\{S_{-1},\ S_{-2} = S_{-1}^2 - (2/3)\mathbf{1}\}$

3. $\operatorname{tr} S_1^2 = \operatorname{tr} S_{-1}^2 = 2$.

We note that

$$\begin{aligned} \mathcal{H}_1(S_2) &= S_2, & \mathcal{H}_1(S_{-2}) &= S_{-2} \\ \mathcal{H}_2(S_2) &= S_{-2}, & \mathcal{H}_2(S_{-2}) &= S_2 \\ \mathcal{H}_1([S_1, S_{-1}]) &= [S_1, S_{-1}], & \mathcal{H}_2([S_1, S_{-1}]) &= -[S_1, S_{-1}]. \end{aligned}$$

Since all of our matrices lie in $\mathrm{i}\mathfrak{su}(3)$, in the future if some vector space $\mathfrak{f}$ is defined in $\mathfrak{sl}(3,\mathbb{C})$ but we use $\mathfrak{f} \cap \mathrm{i}\mathfrak{su}(3)$ we shall continue to refer to it as $\mathfrak{f}$ 'forgetting' to write $\mathrm{i}\mathfrak{su}(3)$ in order to simplify the notation. We hope that this will not lead to confusion.

As mentioned already, see (2), the automorphism $\mathcal{H}_1$ splits the algebra $\mathfrak{sl}(3,\mathbb{C})$ into a direct sum. We shall denote the projectors defined by this splitting by $\pi^{[0,1]}$ and if $X \in \mathfrak{sl}(3,\mathbb{C})$ we shall put $\pi^{[0,1]}X = X^{[0,1]}$.

Remark 1. *Note that the projectors $\pi_\pm$ and $\pi^{[0,1]}$ commute.*

In the same way as we split the algebra $\mathfrak{sl}(3,\mathbb{C})$ we can obtain the splittings

a) For the Cartan subalgebras $\mathfrak{h}_{S_{\pm 1}}$

$$\mathfrak{h}_{S_{\pm 1}} = \mathfrak{h}_{S_{\pm 1}}^{[0]} \oplus \mathfrak{h}_{S_{\pm 1}}^{[1]}$$

because $\mathfrak{h}_{S_{\pm 1}}$ are invariant under $\mathcal{H}_1$: (Of course everything depends on x but we are slightly abusing the notation.)

b) For the orthogonal complements $\mathfrak{g}_{S_{\pm 1}} = \mathfrak{h}_{S_{\pm 1}}^{\perp}$ of $\mathfrak{h}_{S_{\pm 1}}$

$$\mathfrak{g}_{S_{\pm 1}} = \mathfrak{g}_{S_{\pm 1}}^{[0]} \oplus \mathfrak{g}_{S_{\pm 1}}^{[1]}$$

because the Killing form is invariant under automorphisms.

The matrices that are invariant under both automorphisms $\mathcal{H}_1$ and $\mathcal{H}_2$ are as easily seen diagonal. For example, in the above Lax pairs, $[S_1, A_{-1}] + [S_{-1}, A_1]$ is invariant under $\mathcal{H}_1, \mathcal{H}_2$ so it must be diagonal. The same is true for A_0.

3. The Recursion Relations Systems Related to GMV System and the Rational GMV System

Let us consider the following L, A pair on the algebra $\mathfrak{sl}(3,\mathbb{C})$ (first without imposing any other restrictions)

$$L = \mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1} S_{-1}, \qquad A = \mathrm{i}\partial_t + A_0 + \sum_{k=1}^{N} (\lambda^k A_k + \lambda^{-k} A_{-k}). \quad (5)$$

The condition $[L, A] = 0$ is equivalent to a system of equations on the coefficients A_k which we do not write explicitely. We call it the L recursion system (where L is as in (5)). One can see that the L recursion system almost splits to two different systems – one for negative indexes k and another for positive k's. If one is able to resolve them recursively then one will obtain A_k $(k \geq 1)$ from A_s $(s > k)$ and A_{-k} $(k \geq 1)$ from A_{-s} $(s > k)$. The two recursion processes come together in the last stage and obtain the relations

$$\begin{aligned} \mathrm{i}A_{1;x} - \mathrm{i}S_{1;t} + [S_1, A_0] + [S_{-1}, A_2] &= 0 \\ \mathrm{i}A_{-1;x} - \mathrm{i}S_{-1;t} + [S_{-1}, A_0] + [S_1, A_{-2}] &= 0 \end{aligned}$$

which in fact give the system of NLEEs corresponding to $[L, A] = 0$ and

$$\mathrm{i}A_{0;x} + [S_1, A_{-1}] + [S_{-1}, A_1] = 0$$

which is a sort of a compatibility relation.

If we have Mikhailov group generated by g_0, g_1, g_2 it imposes the following requirements on the coefficients in (5)

i) $A_l^\dagger = A_l$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$ and $S_l^\dagger = S_l$ for $l = \pm 1$ where $\dagger$ denotes Hermitian conjugation.

ii) $\mathcal{H}_1(A_l) = (-1)^l A_l$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$ and $\mathcal{H}_1(S_l) = (-1)^l S_l$ for $l = \pm 1$ where $\mathcal{H}_1$ is the involution defined by $\mathcal{H}_1(X) = H_1 X H_1$, $H_1 = \mathrm{diag}(1, -1, -1)$.

iii) $\mathcal{H}_2(A_l) = A_{-l}$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$ and $\mathcal{H}_2(S_l) = S_{-l}$ for $l = \pm 1$ where $\mathcal{H}_2$ is the involution $\mathcal{H}_2(X) = H_2 X H_2$, $H_2 = \mathrm{diag}(1, -1, 1)$.

The L recursion system in which the coefficients A_k are subject to the above restrictions and S_1 is as in the GMV system we call rational GMV recursion system. As a result of iii) half of the equations equivalent to $[L, A] = 0$ become consequence of the other half. Since $[S_1, A_{-1}] + [S_{-1}, A_1] = (\mathrm{Id} + \mathcal{H}_2)[S_{-1}, A_1]$ we have the following independent equations which are equivalent to the rational GMV recursion system

$$\begin{aligned} &\mathrm{i}A_{0;x} + (\mathrm{Id} + \mathcal{H}_2)[S_{-1}, A_1] = 0 \\ &\mathrm{i}A_{1;x} - \mathrm{i}S_{1;t} + [S_1, A_0] + [S_{-1}, A_2] = 0 \\ &\mathrm{i}A_{k;x} + [S_1, A_{k-1}] + [S_{-1}, A_{k+1}] = 0, \qquad k = 2, 3, \ldots N - 1 \\ &\mathrm{i}A_{N;x} + [S_1, A_{N-1}] = 0 \\ &[S_1, A_N] = 0. \end{aligned} \tag{6}$$

The effect of i) and ii) on S_1, S_{-1}, A_1, A_{-1} must belong to $\mathfrak{g}^{[1]} \cap \mathrm{i}\mathfrak{su}(3)$. Now, before going to the rational GMV system, let us briefly analyse the GMV system recursion relations, that is the system that arises from the condition $[L_{S_1}, A] = 0$, where

$$L_{S_1} = \mathrm{i}\partial_x + \lambda S_1, \qquad A = \mathrm{i}\partial_t + \sum_{k=0}^{N} \lambda^k A_k. \tag{7}$$

Then the condition $[L_{S_1}, A] = 0$ is equivalent to the recursion system (L_{S_1} recursion system or GMV type recursion system) which is obtained from the system (6) putting in it formally $S_{-1} \equiv 0$. For it the things are relatively easy. Basically we need to find A_k if A_{k+1} is known. In dealing with systems of the type we have it is useful to use the following proposition which can be proved without difficulties.

Proposition 2. *Suppose we need to solve with respect to X the equation*

$$\mathrm{i}\partial_x R + T = -[S_1, X] \tag{8}$$

(R, T, X are functions with values in $\mathfrak{sl}(3)$). Suppose the compatibility condition $(\mathrm{Id} - \pi_+)(\mathrm{i}\partial_x R + T) = (\mathrm{i}\partial_x R + T)^{+a} = 0$ holds. Then the general solution of (8) *is $X^{+a} + D^{+d}$ where D^{+d} is arbitrary function with values in $\mathfrak{h}_{S_1}$ and*

$$X^{+a} = \Lambda_{S_1} R^{+a} + \mathrm{ad}_{S_1}^{-1}(T^{+a} + \frac{\mathrm{i}}{12}\partial_x^{-1}(\langle T^{+d}, S_1\rangle)S_{1;x} + \frac{\mathrm{i}}{4}\partial_x^{-1}(\langle T^{+d}, S_2\rangle)S_{2;x}).$$

Here Λ_{S_1} is the operator

$$\Lambda_{S_1}(X) = -\,\mathrm{ad}_{S_1}^{-1}\pi_+\left(\mathrm{i}\partial_x X + \frac{\mathrm{i}}{12}S_{1;x}\partial_x^{-1}\langle X, S_{1;x}\rangle + \frac{\mathrm{i}}{4}S_{2;x}\partial_x^{-1}\langle X, S_{2;x}\rangle\right)$$

and X^{+a} could be written even more easily if we introduce the operator Θ_{S_1}

$$\Theta_{S_1}(T) = \mathrm{ad}_{S_1}^{-1}\left(\pi_+ T + \frac{\mathrm{i}}{12}\partial_x^{-1}\langle T^{+d}, S_1\rangle S_{1;x} + \frac{\mathrm{i}}{4}\partial_x^{-1}\langle T^{+d}, S_2\rangle S_{2;x}\right).$$

Then $X^{+a} = \Lambda_{S_1} R^{+a} + \Theta_{S_1}(T)$. Note that $\Theta_{S_1}(T^{+a}) = \mathrm{ad}_{S_1}^{-1}(T^{+a})$.

In the future we shall adopt the following notation. If $\mathfrak{f}_{S_1}$ is a field of spaces each defined for $S_1(x)$ (that is for each rel x we have the fixed linear subspace $\mathfrak{f}_{S_1}(x) \subset \mathfrak{sl}(3, \mathbb{C})$ we shall put $\mathfrak{F}(\mathfrak{f}_{S_1})$ for the vector space of smooth functions $x \mapsto \mathfrak{f}_{S_1}(x)$. The same logic should be applied understanding expressions of the type $\mathfrak{F}(\mathfrak{f}_{S_{-1}})$. Naturally, $\mathfrak{f}_{S_1(x)}$ are subalgebras then $\mathfrak{F}(\mathfrak{f}_{S_1})$ is a Lie algebra, a subalgebra of $\mathfrak{F}(\mathfrak{sl}(3, \mathbb{C}))$ – the set of rapidly decreasing functions with values in $\mathfrak{sl}(3, \mathbb{C})$. With the above notation we have

$$\Theta_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}), \qquad \Theta_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]})$$
$$\Lambda_{S_1}\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}), \qquad \Lambda_{S_1}\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}).$$

If we eliminate A_0 through a gauge transformation and apply the Proposition to the GMV recursion system we immediately get the soliton equations related to systems of GMV type (note that we did not use the fact that there is a reduction here, but simply that S_1 is regular) are

$$\mathrm{i}S_{1;t} = -\operatorname{ad}_{S_1}\Lambda_{S_1}^N A_N. \tag{9}$$

In systems that are not subject to $\mathbb{Z}_2$ reductions is interpreted as Λ_{S_1} being the recursion operator. For the GMV system in order that the soliton equations (9) is consistent with the involution related with $\mathcal{H}_1$, $\Lambda_{S_1}^N A_N$ must take values in $\mathfrak{g}^{[0]}$. So for even N, A_N is taken into the form αS_2 ($\alpha = \mathrm{const}$) and for odd N we have $A_N = \beta S_1$, $\beta = \mathrm{const}$. Thus effectively the hierarchy of the soliton equations related to GMV system is obtained by the action of $\Lambda_{S_1}^2$.

4. Rational GMV Recursion System

Let us consider now the recursion system for the rational GMV system (6). The calculations which for lack of space we cannot present here show that it indeed could be resolved, A_{k+1} could be found if A_k, A_{k-1} are known. However, this recursion process is not in a form suggesting that there exists recursion operator. So let us try another idea, namely to use some linear combinations of the coefficients A_k, $-N \leq k \leq N$. We shall put

$$\begin{aligned} P_k &= A_k + A_{-k} = A_k + \mathcal{H}_2(A_k) = (\operatorname{Id} + \mathcal{H}_2)(A_k) \\ Q_k &= A_{k-1} + A_{-(k+1)} = A_{k-1} + \mathcal{H}_2(A_{k+1}). \end{aligned}$$

We extend the definition of the matrices P_k and Q_k for arbitrary $k \in \mathbb{Z}$ assuming that $A_k = 0$ if $|k| > N$. Thus we have $P_k = 0$ for $|k| > N$ and $Q_k = 0$ for $|k| > N+1$, in particular, $Q_N = A_{N-1}$, $Q_{N+1} = A_N$ and $Q_0 = 2A_{-1}$, $P_0 = 2A_0$. Directly from the definition of P_k and Q_k we obtain that they have the properties

$$\begin{aligned} &\mathcal{H}_1 P_k = (-1)^k P_k, \qquad \mathcal{H}_1 Q_k = (-1)^{k+1} Q_k \\ &\mathcal{H}_2 P_k = P_k, \qquad P_k = P_{-k}, \qquad Q_k = Q_{-k}. \end{aligned}$$

Further, for $k \geq 1$

$$A_k = Q_{k+1} - \mathcal{H}_2(Q_{k+3}) + Q_{k+5} - \mathcal{H}_2(A_{k+7}) + \ldots$$

(since $Q_s = 0$ for $s > N+1$ the above series is finite).
In particular, since $Q_0 = 2\mathcal{H}_2(A_1)$ we have that

$$\frac{1}{2}Q_0 = \mathcal{H}_2(A_1) = \mathcal{H}_2(Q_2) - Q_4 + \mathcal{H}_2(Q_6) - \ldots \equiv \frac{1}{2}F(Q).$$

It follows that we have

Proposition 3. *The set of Q_k, $k = 0, 1, \ldots N+1$ determines uniquely the quantities A_k. Q_0 (A_1) is a linear combination $F(Q)$ of the $Q_{2s}, \mathcal{H}_2(Q_{2s})$ for $s \geq 1$.*

Using the rational GMV system recursion relations it is easy to check that for $|k| > 2$ we have

$$\mathrm{i}\partial_x P_k + (\mathrm{Id} + \mathcal{H}_2)[S_1, Q_k] = 0$$
$$\mathrm{i}\partial_x Q_k - [S_1 - S_{-1}, P_k] + [S_1, Q_{k-1} + Q_{k+1}] = 0.$$

We solve the first equation for P_k and introduce into the second one. We get

$$\mathrm{i}\partial_x Q_k - [S_1 - S_{-1}, \mathrm{i}\partial_x^{-1}(\mathrm{Id} + \mathcal{H}_2)[S_1, Q_k]] = -[S_1, Q_{k+1} + Q_{k-1}]. \tag{10}$$

This system is exactly of the type that is considered in Proposition 2 but in order to write the things in a concise way let us introduce some notation. First, we introduce the operator Ω_{S_1} by

$$\Omega_{S_1}(Z) = [S_1 - S_{-1}, \mathrm{i}\partial_x^{-1}(\mathrm{Id} + \mathcal{H}_2)[S_1, Z]]$$

where Z is a function on x with values in $\mathfrak{g}$. It is obvious that in fact $\Omega_{S_1}(Z)$ depends only on the projection of Z on the space $\mathfrak{g}_{S_1}$, that is $\Omega_{S_1}(Z) = \Omega_{S_1}(Z^{+a})$. We also have

$$\Omega_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}), \qquad \Omega_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}).$$

Now, using Proposition 2 we get

$$(\Lambda_{S_1} - \Theta_{S_1} \circ \Omega_{S_1})Q_k^{+a} = Q_{k+1}^{+a} + Q_{k-1}^{+a}.$$

It is obvious that if we put

$$\mathbf{\Lambda}_{S_1} = \Lambda_{S_1} - \Theta_{S_1} \circ \Omega_{S_1}. \tag{11}$$

The above equation will have even nicer form $\mathbf{\Lambda}_{S_1} Q_k^{+a} = Q_{k+1}^{+a} + Q_{k-1}^{+a}$. One immediately checks that $\mathbf{\Lambda}_{S_1}$ has the properties

$$\mathbf{\Lambda}_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}), \qquad \mathbf{\Lambda}_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}). \tag{12}$$

Skipping the technical details we obtain

Proposition 4. *The system of equations equivalent to the rational* GMV *recursion system could be resolved in the following way*: *At the first stage we resolve*

$$\mathbf{\Lambda}_{S_1} Q_k = Q_{k+1}^{a} + Q_{k-1}^{+a}, \qquad P_k = \Omega_{S_1}(Q_k^{+a}), \qquad N + 1 \leq k \leq 2$$

to find Q_2 as a function of S_1, S_2 and their x-derivatives. Then using the value of Q_2 and knowing that $Q_0 = F(Q)$ we resolve

$$\mathrm{i}\partial_x Q_1 + \frac{1}{2}(\mathrm{Id} + \mathcal{H}_2)[S_1, Q_0] + \frac{1}{2}(\mathrm{Id} - \mathcal{H}_2)[S_{-1}, Q_0] + [S_1, Q_2] = 0$$

to find Q_1 and obtain the corresponding soliton equation either in the form

$$\mathrm{i}\,\mathrm{ad}_{S_1}^{-1} S_{1;t} = Q_3^{+a} - \mathbf{\Lambda}_{S_1} Q_2^{+a} + Q_1^{+a} \tag{13}$$

or in the form

$$\mathrm{i}\,\mathrm{ad}_{S_1}^{-1} S_{-1;t} = Q_1^{+a} - \frac{1}{2}\mathbf{\Lambda}_{S_1} Q_0^{+a}. \tag{14}$$

Finally $P_0 = \Omega_{S_1}(Q_0^{+a})$ *and* $P_1 = \frac{1}{2}(\mathrm{Id} + \mathcal{H}_2)Q_0$ *are also determined thus finding all the functions* A_k *needed for the Lax representation of* (13) (or (14)).

The above suggests that $\mathbf{\Lambda}_{S_1}$ defined in (11) could be the recursion operator we are looking for since we can obtain recursively all Q_k^{+a} though the situation is not quite as it is usally. In support of this opinion we remark that the operator $\mathbf{\Lambda}_{S_1}$ (or rather its square) has been proposed in [6] to be the recursion operator for the rational GMV system using some other type of argument. In order to introduce it we sketch the construction from that work the more reason that the form of the relations in [6] does not permit to verify immediately our claim.

5. The Recursion Operator Defined Through Adjoint Solutions

According to [6] the recursion operator could be constructed from the requirement that for it some combinations of the adjoint solutions for $L_{S_{\pm 1}}$ (see (4)) are eigenfunctions. Let χ be a fundamental solution to $L_{S_{\pm 1}}\chi = 0$. Then if A is a fixed constant matrix, the function $\Phi_A = \chi A \hat{\chi}$ will satisfy the equation

$$\mathrm{i}\partial_x \Phi_A + [\lambda S_1 + \lambda^{-1} S_{-1}, \Phi_A] = 0.$$

Let us introduce the functions

$$\Phi_{A;k} = \lambda^k \Phi_A + \lambda^{-k}\mathcal{H}_2(\Phi_A), \qquad k = 0, \pm 1, \pm 2, \dots .$$

The interest in these functions is motivated by the fact that they enter in some Wronskian identities (in that case A must be diagonal) which are essential in the theory (see [3] for their derivation and more explanations). In fact one can notice that in the relations enter only the projections $\pi_+ \Phi_{A;1}(x,\lambda) = \Phi_{A;1}^{+a}(x,\lambda)$ on the orthogonal complement $\mathfrak{g}_{S_1}$ to the Cartan subalgebra $\mathfrak{h}_{S_1}$. Even more, because all diagonal matrices in $\mathfrak{g}$ belong to $\mathfrak{g}^{[0]}$ in fact in the Wronskian relations enters only the projection of $\pi_+ \Phi_{A;1}^{[1]}(x,\lambda)$ on $\mathfrak{g}^{[0]}$ which we denote by $\Phi_{A;1}^{[1]+a}(x,\lambda)$ (recall that π_+ commutes with $\pi^{[0]}$ and $\pi^{[1]}$) on the orthogonal complement $\mathfrak{g}_{S_1}$ to the Cartan subalgebra $\mathfrak{h}_{S_1}$. More generally, we introduce the splittings

$$\Phi_{A;0} = \Phi_{A;0}^{[0]} + \Phi_{A;0}^{[1]}, \qquad \Phi_{A;1} = \Phi_{A;1}^{[0]} + \Phi_{A;1}^{[1]}$$

corresponding to the splitting $\mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]}$ and let us put

$$\Phi_{A;1}^{[0]+a} = (\Phi_{A;1}^{[0]})^{+a} = \pi_+ \Phi_{A;1}^{[0]}, \qquad \Phi_{A;1}^{[1]+a} = (\Phi_{A;1}^{[1]})^{+a} = \pi_+ \Phi_{A;1}^{[1]}.$$

(recall that π_+ and the projectors $\pi^{[0]}$ and $\pi^{[1]}$ commute).

The experience one has from the study of GZS system and the CBC system is that the functions $\Phi_{A;1}^{[1]+a}(x,\lambda)$ involved into these relations are eigenfunctions for the generating operators. Developing that idea, in [6] the equation for $\Phi_{A;1}$ into the following equivalent form

$$\mathrm{i}\partial_x\Phi_{A;1} - [S_1 - S_{-1}, \Phi_{A;0}] = -(\lambda + \lambda^{-1})[S_1, \Phi_{A;1}].$$

Our further calculations are in fact the same as in [6] but give them another from that suits better our purposes. We notices that the above equation is similar to the one we had in (10). This permits us to write immediately

$$\mathbf{\Lambda}_{S_1}\Phi^{+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{+a}_{A;0}. \tag{15}$$

Using (12) from (15) we get

$$\mathbf{\Lambda}_{S_1}\Phi^{[0]+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{[1]+a}_{A;1}, \qquad \mathbf{\Lambda}_{S_1}\Phi^{[1]+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{[0]+a}_{A;1}.$$

As a consequence

$$\mathbf{\Lambda}^2_{S_1}\Phi^{[0]+a}_{A;1} = (\lambda + \lambda^{-1})^2\Phi^{[0]+a}_{A;1}, \qquad \mathbf{\Lambda}^2_{S_1}\Phi^{[1]+a}_{A;1} = (\lambda + \lambda^{-1})^2\Phi^{[1]+a}_{A;1}. \tag{16}$$

There are number of terms that cancel when one calculates explicitly the action of $\mathbf{\Lambda}^2_{S_{\pm 1}}$ on $(\Phi^{[0]}_{A;1})^{+a}$ and $(\Phi^{[0]}_{A;1})^{+a}$ so in [6] were introduced two operators Λ_1, Λ_2 such that

$$\mathbf{\Lambda}^2_{S_1}\Phi^{[0]+a}_{A;1} = \Lambda_2\Lambda_1\Phi^{[0]+a}_{A;1}, \qquad \mathbf{\Lambda}^2_{S_1}\Phi^{[1]+a}_{A;1} = \Lambda_1\Lambda_2\Phi^{[1]+a}_{A;1}. \tag{17}$$

The situation as in (16) and (17) is rather typical when one has $\mathbb{Z}_2$ reductions, see [11] and the comments at the end of section 3. What one has is that the operator $\mathbf{\Lambda}_{S_1}$ restricted to functions taking values in $\mathfrak{g}^{[0]}$ is equal to $-\Lambda_1$ and restricted to functions taking values in $\mathfrak{g}^{[1]}$ is equal to $-\Lambda_2$.

6. Conclusions

As we mentioned the recursion operators associated with an auxiliary linear problem L appear in several different roles:

1. Resolve the recursion systems for the soliton equations associated with L.
2. For them some projections of the adjoint solutions of L are eigenfunctions.
3. Their adjoint relate two Hamiltonian structures for the NLEEs associated with L.

We have shown that he operators $\mathbf{\Lambda}_{S_1}$ resolve the recursion relations and we have seen for them the projections of the adjoint solutions entering in the Wronskian relations are eigenfunctions. It remains of course to establish completeness relations for these eigenfunctions in order to develop the theory. As to the geometric properties of $\mathbf{\Lambda}_{S_1}$ which will involve the Hamiltonian structures for the NLEEs associated with the rational GMV until now it has not been treated and of course this will be an interesting task for the future research.

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