Sixteenth International Conference on Geometry, Integrability and Quantization June 6–11, 2014, Varna, Bulgaria Ivaïlo M. Mladenov, Andrei Ludu and Akira Yoshioka, Editors **Avangard Prima**, Sofia 2015, pp 139–163 doi: 10.7546/giq-16-2015-139-163



CLASSICAL AND QUANTIZATION PROBLEMS IN DEGENERATE AFFINE MOTION

JAN J. SŁAWIANOWSKI and EVA E. ROŻKO

Institute of Fundamental Technological Research, Polish Academy of Sciences $5^{\rm B}$, Pawińskiego Str., 02-106 Warsaw, Poland

Abstract. Discussed are classical and quantized models of affinely rigid motion with degenerate dimension, i.e., such ones that the geometric dimensions of the material and physical spaces need not be equal to each other. More precisely, the material space may have dimension lower than the physical space. Physically interesting are special cases m = 2 or m = 1 and n = 3, first of all m = 2, n = 3, i.e., roughly speaking, the affinely deformable coin in three-dimensional Euclidean space. We introduce some special coordinate systems generalizing the polar and two-polar decompositions in the regular case. This enables us to reduce the dynamics to two degrees of freedom. In quantum case this is the reduction of the Schrödinger equation to multicomponent wave functions of two deformation invariants.

MSC: 37N15, 70E15, 70H33, 81R05

Keywords: affinely rigid motion, degenerate dimension, polar and two-polar decompositions, multicomponent wave function, Schrödinger equation, quantized models, deformation invariants

1. Introduction

The mechanics of affine motion was a subject of plenty of our papers [16–18, 22–24, 26–29]. There are also papers of many other people who developed the subject and discussed various applications and generalizations [3, 5–9, 11, 13–15, 25, 33]. Below we consider the classical and quantum generalizations to dimensionally degenerate models, i.e., such ones that the material space may be lower–dimensional than the physical space. Of course, we have in mind mainly the special case of two–dimensional material space and the three–dimensional physical Euclidean space. Roughly speaking, this is the mechanics of affinely–rigid body moving in the usual

space. The degenerate problem is mathematically much more complicated than the non-degenerate one. Nevertheless, there are some common methods, like generalized polar and two-polar decomposition of degrees of freedom. One must be careful nevertheless, because many concepts well-defined in the regular case become ill-defined or should be modified in a remarkable way. This concerns e.g., affine velocities, because the inverse mappings to configurations are not defined. But, surprisingly enough, the corresponding Hamiltonian concepts like the affine spin, spatial or co-moving, are well-defined. Because of this it seems that Hamiltonian methods are more reliable then. This is especially interesting in the quantum case which is mathematically based on the classical Hamiltonian description. Some introductory part is presented for the general values of n, m. It has to do with the Stiefel and Grassmann manifolds. Nevertheless we concentrate on the special case m = n - 1, both because of practical, computational simplicity and because of our natural interest in the special case of deformable coin, when m = 3, n = 2.

The model, including the general case of (n, m) is interesting in itself from the point of view of purely mathematical theory of dynamical systems on homogeneous spaces. Let us notice that the configuration space of affinely rigid body with degenerate dimension is a homogeneous space of the affine group in the physical space, but not of the material affine group. Such models are mathematically very interesting.

Nevertheless, they are also applicable in physical problems. First of all, one can use them in the theory of structured bodies, e.g., ones consisting of planar or almost planar molecules. Let us mention, e.g., the historical model of what Max Born called "Schwungrad" in early days of quantum theory, in particular in dynamics of molecules [4]. Two–dimensional structure elements are physically realized as three–atom molecules like CO_2 , S_3 , H_2O . Also more complicated molecules concentrated around some almost "flat" core may be described in this way. Molecular physics and nanophysics must be described in quantum terms. They are particularly interesting also from the point of view of foundation of quanta. Namely, their theory may shed some light onto paradoxes of quantum mechanics, like the border between classical and quantum physics, decoherence and measurement paradoxes.

We are to follow here the classical \Rightarrow quantum line of reasoning. Namely, we begin with formulating the classical theory of affine constraints based on the classical d'Alembert procedure. The motion is then classically confined to some Riemannian submanifold of the primary configuration space. And then one performs the usual quantization procedure of the classical problem on the manifold of constraints. This is what we are going to do. But to be honest, one should mention the another, not fully equivalent to the above one, procedure. We mean here something that might be called the classical \Rightarrow quantum line of reasoning. So, according to this line, one should consider the rigorous quantum theory of the non–constrained

system, writing its Hamiltonian in the form

$$\widehat{H} = \widehat{H}_0 + W$$

where \hat{H}_0 is a background expression, and W is a potential term describing the confinement to the constraints manifold. The characteristic feature of W is that it vanishes on the submanifold of constraints but grows very quickly when departing from it. This leads to the splitting of quantum motion into quantum vibrations "across" the constraints manifold and the constrained quantum dynamics "along" the classical constraints. Because of this there appear some energy levels of the dynamics across constraints and they split into some bands of energy levels of the dynamics of the along–constraints motion. It is reasonable to expect that the only across–constraints factor that is relevant here is some ground state of vibrations. Otherwise there is something like the Jahn–Teller effect that may obscure the picture. But here we do not get into those problems and simply consider everything on the level classical \Rightarrow quantum.

2. Classical Ideas

Affine motion consists of spatial translations, rotations, and homogeneous deformations. The configuration space of an *n*-dimensional affinely–rigid body for some fixed Cartesian coordinates and reference coordinates is identified with

$$Q = \operatorname{GAf}(n, \mathbb{R}) \simeq \operatorname{GL}(n, \mathbb{R}) \otimes_s \mathbb{R}^n$$

where \mathbb{R}^n the center of mass motion (physical space identified with \mathbb{R}^n), $Q_{\text{int}} = \operatorname{GL}(n, \mathbb{R})$ is the general linear group which describes internal degrees of freedom. In continuum case one should use rather $Q_{\text{int}} = \operatorname{GL}^+(n, \mathbb{R})$ -the group of proper linear transformations. The sign \otimes_s denotes the semi-direct product of groups.

It is customary to use two logically different affine spaces (M, V)-the physical affine space of dimension n, and (N, U)-the material space of the same dimension n. Here V denotes the linear space of translations in M, and U respectively, the linear space of translations in N. The configuration space is identified with the manifold Q = AfI(N, M) of affine isomorphisms of N onto M. The co-moving mass distribution is described by a fixed, time-independent positive measure μ on N. When $\mu(N) < \infty$, the Lagrangian center of mass is well-defined as a point $v \in N$ such

$$\int \overrightarrow{va} \, \mathrm{d}\mu \left(a \right) = 0 \tag{1}$$

where $\overrightarrow{va} \in U$ denotes the translation vector carrying over v into a. If $\Phi \in AfI(N, M)$, then $v_{\Phi} := \Phi(v)$ denotes the current position of the center of mass in M.

The configuration space may be identified with the Cartesian product

 $Q = M \times \mathcal{L}\left(U, V\right)$

where $\mathcal{L}(U, V)$ denotes the manifold of linear isomorphisms of U onto V. The configuration $(x, \varphi) \in M \times \mathcal{L}(U, V)$ is understood in such a way that $x \in M$ is just the temporary position of the center of mass in M and $\varphi \in \mathcal{L}(U, V)$ describes the relative motion configuration. The material point labelled by $a \in N$ occupies such a spatial position $y \in M$ that $\overline{xy} = \varphi \cdot \overline{ya}$.

When some Cartesian coordinates are fixed in M and N, and the center of mass has Lagrangian (reference) coordinates $a^k = 0$, we write analytically that

$$\Phi \colon y^i = x^i + \varphi^i{}_K a^K.$$

The motion is described by the time dependence of x^i (translational motion) and $\varphi^i{}_K$ (internal, or relative, motion).

In usual mechanical models based on the d'Alembert principle the effective kinetic energy is obtained by a restriction of the multiparticle kinetic energy to the manifold of affine constraints. The resulting expression is controlled by two constant inertial quantities, i.e., the total mass \mathcal{M} and the second–order moment of the measure μ with respect to Lagrangian coordinates

$$\mathcal{M} = \int \mathrm{d}\mu\left(a\right), \qquad J^{KL} = \int a^{K} a^{L} \mathrm{d}\mu\left(a\right) = J^{LK}$$
(2)

where \mathcal{M} characterizes translational inertia and J characterize internal inertial properties of the body. It is interesting that the same quantity J describes rotational and deformative inertia. The usual tensor of inertia known from the rigid body mechanics is a linear function of J (and conversely). $J \in U \otimes U$ is a twice contravariant, symmetric, nonsingular, positively–definite tensor in U.

The multiparticle kinetic energy of affine motion is given by

$$T = T_{\rm tr} + T_{\rm int} = \frac{M}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{d}\varphi^i{}_A}{\mathrm{d}t} \frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t} J^{AB}$$

where $g \in V^* \otimes V^*$ denotes the metric tensor of the physical space. T_{tr} is the kinetic energy of the translational motion and T_{int} refers to the internal/relative motion.

If we use Cartesian orthonormal coordinates, $g_{ij} = \delta_{ij}$ and putting $M = N = U = V = \mathbb{R}^n$, $Q_{int} = GL(n, \mathbb{R})$ we can rewrite the kinetic energy in the following matrix description

$$T = T_{\rm tr} + T_{\rm int} = \frac{M}{2} \frac{\mathrm{d}x^T}{\mathrm{d}t} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{2} \operatorname{Tr} \left(J \frac{\mathrm{d}\varphi^T}{\mathrm{d}t} \frac{\mathrm{d}\varphi}{\mathrm{d}t} \right).$$
(3)

In non-dissipative models without external magnetic or gyroscopic forces one derives equations of motion from Lagrangians

$$L = T - V(x, \varphi)$$

as variational Euler–Lagrange equations. We concentrate just on such models and on the corresponding Hamiltonian description. The Legendre transformation

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i}, \qquad p^A{}_i = \frac{\partial L}{\partial \dot{\varphi}^i_A} = \frac{\partial T}{\partial \dot{\varphi}^i_A} \tag{4}$$

i.e., explicitly

$$p_i = M g_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}t}, \qquad p^A_{\ i} = g_{ij} \frac{\mathrm{d}\varphi^j{}_B}{\mathrm{d}t} J^{BA} \tag{5}$$

allows us to construct the Hamiltonian as a function of generalized coordinates x^i , $\varphi^i{}_A$ and their conjugate canonical momenta p_i , $p^A{}_i$

$$H = T + V = T_{tr} + T_{int} + V = \frac{1}{2M} g^{ij} p_i p_j + \frac{1}{2} \widetilde{J}_{AB} p^A{}_i p^B{}_j g^{ij} + V \quad (6)$$

where $\widetilde{J} \in U^* \otimes U^*$ is the reciprocal tensor of J, i.e., $\widetilde{J}_{AC} J^{CB} = \delta_A{}^B$. The linear groups $\operatorname{GL}(V)$, $\operatorname{GL}(U)$ act on the internal configuration space $Q_{\operatorname{int}} = \mathcal{L}(U, V)$ according to the following rule

$$A \in GL(V): \ \mathcal{L}(U,V) \ni \varphi \mapsto A\varphi \in \mathcal{L}(U,V)$$
(7)

$$B \in GL(U): \ \mathcal{L}(U,V) \ni \varphi \mapsto \varphi B \in \mathcal{L}(U,V).$$
(8)

They are, respectively, spatial and material transformations. Restricting A, B to orthogonal subgroups O(V,g), $O(U,\eta)$ ($\eta \in U^* \otimes U^*$ is the metric tensor of the material space), one can study isotropy properties and rotational invariance of dynamical models. For example, the kinetic energy is always spatially isotropic, but in general it is materially anisotropic, unless J is proportional to η .

 Q_{int} is a homogeneous space with respect to the actions of $\operatorname{GL}(V)$ and $\operatorname{GL}(U)$ (7). Its isotropy groups are trivial, i.e., the action of both groups is free. Let us stress, however, that the action of $\operatorname{GL}(V) \times \operatorname{GL}(U)$

$$\varphi \mapsto A\varphi B$$

is non-effective, because the dilatation subgroups of GL(V) and GL(U), i.e.,

$$\{(\ell \operatorname{Id}_V, \ell^{-1} \operatorname{Id}_U); \ell \in \mathbb{R}^+\}$$

act in the same way as transformation groups. Therefore, the non–effectiveness kernel consists of elements of the form $(\ell \operatorname{Id}_V, \ell^{-1} \operatorname{Id}_U)$, where ℓ runs over the set of non–vanishing reals, Id_V and Id_U are identity transformations, respectively, in V and U.

2.1. Degenerate Dimension, Deformable Coin

Now let us turn to the proper subject of our paper, i.e., to affinely-rigid bodies with degenerate dimension. The standard continuum theory deals with such objects, e.g., membranes, strings, etc. In fundamental physics strings and membranes are objects of intensive investigations within the framework of quantum field theory and the theory of elementary particles. Some classical and later on quantum toy models of such objects may be formulated as our affine bodies of degenerate dimension. We shall consider *m*-dimensional affinely-rigid bodies moving in *n*-dimensional spaces m < n. Obviously, in standard physical applications only the special cases n = 3, m = 1, 2 are of direct interest. And in fact, below we concentrate on n = 3, m = 2, i.e., two-dimensional affine plates moving in the usual three-dimensional space.

The material and physical spaces will be affine spaces (N, U), (M, V), respectively of dimensions m, n, where m < n. The configuration space of affinelyrigid body consists of affine injections, monomorphisms from N to M, Q = AfM(N, M). Just as previously, the Lagrangian (co-moving) mass distribution is described by some fixed, time-independent positive measure μ on N. The center of mass in N (cf (1)), the total mass \mathcal{M} and the inertial quadrupole $J \in U \otimes U$ given by (2) are defined just as in the regular case (when m = n). Lagrange coordinates a^K , $K = 1, \ldots, m$, are also chosen in such a way that $a^K = 0$ for the center of mass in N. The configuration space may be identified with

$$Q = \operatorname{AfM}(N, M) = M \times \operatorname{LM}(U, V)$$

where LM(U, V) is the set of linear monomorphisms (injections) from U to V. The formula

$$y^i = x^i + \varphi^i{}_K a^K$$

remains valid. The $n \times m$ matrix $[\varphi^i{}_K]$ has rank m.

All the above expressions for kinetic energy (3), Legendre transformation (4) (5), Hamiltonians (6), transformation groups (7) (8) are also formally correct. Some difference appears in the structure of group actions (7) (8). Namely, $Q_{\text{int}} = \text{LM}(U, V)$ is a homogeneous space for the left-hand side action of GL(V) but it fails to be homogeneous with respect to the material transformations, i.e., the righthand side action of GL(U). Indeed, the right-hand side action of GL(U) is nontransitive and its orbits consist of such φ -s that have the same images $\varphi(U) \subset V$. Only such φ_1, φ_2 for which $\varphi_1(U) = \varphi_2(U)$ may be joined by right action. When we put $V = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Q = \mathbb{R}^n \times \text{LM}(m, n)$, $Q_{\text{int}} = \text{LM}(m, n)$,

When we put $V = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Q = \mathbb{R}^n \times LM(m, n)$, $Q_{int} = LM(m, n)$, the above actions become matrix multiplications, just as in the regular case (7) (8) they were simply left and right regular translations on $GL(n, \mathbb{R})$ or $GL^+(n, \mathbb{R})$. Our afore-mentioned system has f = n + mn = n(m + 1) degrees of freedom, *n* translational and $m \cdot n$ internal (relative) ones. We are not particularly interested in the translational motion. We concentrate on φ -type degrees of freedom on Q_{int} . LM(U, V) is a homogeneous space with respect to spatial transformations, i.e., with respect to the left-hand side action of GL(V). Let us fix some standard linear monomorphism Ψ of U into V. So, we may say that if translational motion is neglected, N and M are identified with U and V respectively, then LM(U, V)may be obtained from Ψ by the left actions

$$LM(U, V) \ni \Psi \mapsto \varphi = A\Psi \in LM(U, V), \qquad A \in GL(V)$$

where A runs over $\operatorname{GL}(V)$. What is the stabilizer subgroup $H[\Psi] \subset \operatorname{GL}(V)$ of the reference configuration Ψ ? It consists of those elements of $\operatorname{GL}(V)$ which do not only preserve the linear subspace $\Psi(U) \subset V$, but also preserve separately any element of this subspace.

Let us put $U = \mathbb{R}^m$, $V = \mathbb{R}^n$. We identify \mathbb{R}^m with the subspace of \mathbb{R}^n and assume $\Psi(U)$ to have zeros at (n-m) places $[a^1, \ldots, a^m, 0, \ldots, 0]^T$

$$\Psi\left(a^{1},\ldots,a^{m}\right) = \begin{bmatrix} a^{1} \\ \vdots \\ a^{m} \\ o \end{bmatrix}$$

where o is an $(n - m) \times 1$ dimensional zero matrix. Then H is given by

$$\left[\begin{array}{cc} \mathbf{I}_m & A \\ o & B \end{array}\right]$$

where I_m is the $m \times m$ identity matrix, A is the $m \times (n - m)$ matrix, B is the $(n - m) \times (n - m)$ matrix, and o is the $(n - m) \times m$ zero matrix. The matrices A, B subject only to the restriction that the total matrix is nonsingular. When taken together, the matrices A, B involve $m(n - m) + (n - m)^2 = n(n - m)$ parameters and the quotient manifold $GL(n, \mathbb{R})/H : n^2 - n(n - m) = nm$ parameters, the dimension of L(m, n) on LM(m, n). H is indeed a subgroup

$$\begin{bmatrix} I & A_1 \\ o & B_1 \end{bmatrix} \begin{bmatrix} I & A_2 \\ o & B_2 \end{bmatrix} = \begin{bmatrix} I & A_1B_2 + A_2 \\ o & B_1B_2 \end{bmatrix}.$$

One defines affine velocity (Eringen's gyration) $\Omega \in L(V) \simeq GL(V)'$ in spatial representations and $\hat{\Omega} \in L(U) \simeq \mathfrak{gl}(U)'$ in material representations

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \,\varphi^{-1}, \qquad \hat{\Omega} = \varphi^{-1} \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \varphi^{-1} \Omega \varphi.$$

They are defined for m = n but do not exist when m < n. More precisely, the right inverse ρ such that $\varphi \rho = \operatorname{Id}_V$ does not exist at all. The left inverse λ , $\lambda \varphi = \operatorname{Id}_U$ does exist but is non–unique; various versions coincide only on $\varphi(U) \subset V$.

But affine spins Σ , $\hat{\Sigma}$ and affine momenta $p^A{}_j$ do exist. Affine spins are objects $\Sigma \in L(V)^* \simeq L(V)$, $\hat{\Sigma} \in L(U)^* \simeq L(U)$ (identification through the trace formula, $\langle A, B \rangle = \text{Tr}(AB)$) well-defined as follows

$$\Sigma^{i}{}_{j} = \varphi^{i}{}_{A} p^{A}{}_{j}, \qquad \hat{\Sigma}^{A}{}_{B} = p^{A}{}_{i} \varphi^{i}{}_{B}$$
⁽⁹⁾

where $p \in L(V, U) \simeq L(U, V)^*$ is the canonical momentum conjugate to φ . Just as in the regular case, the components $\Sigma^i{}_j$ are Hamiltonian generators [2] of GL(V)left-acting on LM(U, V), and $\hat{\Sigma}^A{}_B$ are Hamiltonian generators of the right action of GL(U) on LM(U, V). They are not spatial and material components of any object, because there is no isomorphism between U and V.

The canonical spin $S^{i}{}_{j}$ and vorticity $\mathcal{V}^{A}{}_{B}$ are also defined as the doubled skewsymmetric parts of Σ and $\hat{\Sigma}$ (9)

$$S^{i}{}_{j} := \Sigma^{i}{}_{j} - g^{ik}g_{jl}\Sigma^{l}{}_{k}, \qquad \mathcal{V}^{A}{}_{B} := \hat{\Sigma}^{A}{}_{B} - \eta^{AC}\eta_{BD}\hat{\Sigma}^{D}{}_{C} \qquad (10)$$

where $g \in V^* \otimes V^*$, $\eta \in U^* \otimes U^*$ denote the metric tensors in V and U. S and \mathcal{V} are Hamiltonian generators (momentum mapping) of the proper orthogonal subgroups $\mathrm{SO}(V,g) \subset \mathrm{GL}(V)$, $\mathrm{SO}(U,\eta) \subset \mathrm{GL}(U)$ acting, respectively, on the left and right on $\mathrm{LM}(U,V)$, i.e., Hamiltonian generators of spatial and material rotations.

Contravariant tensor objects may be transferred (push-forwarded) from the material space to the physical one, but not conversely, because φ is non-invertible. Unlike this, covariant tensors may be transferred from V to U (pull-back), but not from U to V. For example, the translational canonical momentum $p \in V^*$ conjugate to x gives rise to the object $\hat{p} \in U^*$

$$\hat{p}_A = p_i \varphi^i{}_A.$$

Affine spin and canonical momentum (9) are Hamiltonian generators of the spatial affine group GAf(M). Poisson brackets are given by structure constants of this group

$$\begin{split} \left\{ \Sigma^{i}{}_{j}, \Sigma^{k}{}_{l} \right\} &= \delta^{i}{}_{l}\Sigma^{k}{}_{j} - \delta^{k}{}_{j}\Sigma^{i}{}_{l} \\ \left\{ \Sigma^{i}{}_{j} + x^{i}p_{j}, p_{k} \right\} &= \delta^{i}{}_{k}p_{j} \\ \left\{ p_{i}, p_{j} \right\} &= 0 \\ \left\{ \Sigma^{i}{}_{j} + x^{i}p_{j}, \Sigma^{k}{}_{l} + x^{k}p_{l} \right\} &= \delta^{i}{}_{l} \left(\Sigma^{k}{}_{j} + x^{k}p_{j} \right) - \delta^{k}{}_{j} \left(\Sigma^{i}{}_{l} + x^{i}p_{l} \right) \\ \left\{ \Sigma^{i}{}_{j}, p_{k} \right\} &= 0. \end{split}$$

For the material affine spin the following holds

$$\left\{\widehat{\Sigma}^{A}{}_{B},\widehat{\Sigma}^{C}{}_{D}\right\} = \delta^{C}{}_{B}\widehat{\Sigma}^{A}{}_{D} - \delta^{A}{}_{D}\widehat{\Sigma}^{C}{}_{B}, \qquad \left\{\Sigma^{i}{}_{j},\widehat{\Sigma}^{A}{}_{B}\right\} = 0$$

because the left action of GL(V) commutes with the right action of GL(U). The following Poisson brackets hold

$$\left\{x^{i}, p_{j}\right\} = \delta^{i}_{j}, \qquad \left\{\varphi^{i}{}_{A}, p^{B}{}_{j}\right\} = \delta^{i}_{j}\,\delta^{B}_{A}.$$

We do not quote similar formulas for S_{j}^{i} , \mathcal{V}_{B}^{A} (10) in terms of structure constants of SO(V, g), $SO(U, \eta)$. If F depends only on the configuration (x, φ) , then

$$\left\{\Sigma^{i}{}_{j},F\right\} = -\varphi^{i}{}_{A}\frac{\partial F}{\partial \varphi^{j}{}_{A}}, \qquad \left\{\widehat{\Sigma}^{A}{}_{B},F\right\} = -\varphi^{k}{}_{B}\frac{\partial F}{\partial \varphi^{k}{}_{A}}.$$

Let us also quote the covariant Green tensor $G \in U^* \otimes U^*$ and contravariant deformation Cauchy tensor $\widetilde{C} \in V^* \otimes V^*$

$$G = \varphi^* \cdot g, \qquad \widetilde{C} = \varphi^* \cdot \widetilde{\eta} \tag{11}$$

analytically

$$G_{AB} = g_{ij} \varphi^i{}_A \varphi^j{}_B, \qquad \widetilde{C}^{ij} = \varphi^i{}_A \varphi^j{}_B \eta^{AB}$$
(12)

and in matrix terms

$$G = \varphi^T \varphi, \qquad \widetilde{C} = \varphi \varphi^T \tag{13}$$

where $U = \mathbb{R}^m$, $V = \mathbb{R}^n$, and $\varphi \in \mathrm{LM}(m, n)$.

We are particularly interested in doubly–isotropic dynamical models, when the potential V depends on φ through deformation invariants only. Let us mention that deformation invariants λ , ℓ are assigned to the pairs of tensors (G_{AB}, η_{AB}) , (C^{ij}, g^{ij})

$$\det \left[G_{AB} - \lambda \eta_{AB}\right] = 0, \qquad \det \left[C^{ij} - \frac{1}{\ell}g^{ij}\right] = 0.$$

If m = n, then $\lambda = \ell$. If m < n, then λ -s depends on ℓ -s.

In the case of non–degenerate affine bodies, when m = n, we based on the polar and two–polar decompositions of $\varphi \in \mathrm{GL}^+(n,\mathbb{R})$

$$\varphi = RL = \Lambda R, \qquad \varphi = VDU^{-1}$$

where $R, V, U \in SO(n, \mathbb{R})$ are proper rotations (orthogonal matrices of determinant +1), L and $\Lambda = RLR^{-1}$ are symmetric and positively definite, and D is diagonal and positive.

There are counterparts of the above decompositions in the mechanics of degenerate affine bodies, when m < n. So, we write

$$\varphi = R \left[\begin{array}{c} L \\ o \end{array} \right]$$

where $R \in SO(n, \mathbb{R})$ is a special orthogonal matrix, $L \in Symm(m, \mathbb{R})$ is a symmetric $m \times m$ matrix, o is the $(n - m) \times m$ zero matrix. In the above formula and in almost all forthcoming ones we simply put $U = \mathbb{R}^m$, $V = \mathbb{R}^n$, $LM(U, V) = LM(m, n) \subset L(m, n)$.

It is obvious that dim SO $(n, \mathbb{R}) = n(n-1)/2$ and that one of dim Symm $(m, \mathbb{R}) = m(m+1)/2$. Then it is seen that for general values of m, n the total number of these parameters

$$\frac{n(n-1)}{2}+\frac{m(m+1)}{2}$$

does not equal the number of internal degrees of freedom, i.e., to (nm). Because of some redundant variables the configuration space cannot be identified with the Cartesian product $SO(n, \mathbb{R}) \times Symm(m, \mathbb{R})$. Because the subgroup $SO(n-m, \mathbb{R})$ of rotations acting on the (n-m)-tuple of the last variables in \mathbb{R}^n does not affect $\begin{bmatrix} L \end{bmatrix}$ when multiplying it on the left

 $\begin{bmatrix} L \\ o \end{bmatrix}$ when multiplying it on the left.

Let us take the subgroup $K \subset SO(n, \mathbb{R})$ composed of

$$R = \left[\begin{array}{cc} \mathbf{I}_m & \underline{o}^T \\ o & u \end{array} \right]$$

where I_m is an $m \times m$ identity matrix, while $o(n-m) \times m$ is the zero matrix and $u \in SO(n-m, \mathbb{R})$ is an arbitrary $(n-m) \times (n-m)$ rotation matrix. The subgroup K (isomorphic with $SO(n-m, \mathbb{R})$) is (n-m)(n-m-1)/2-dimensional. The quotient manifold of left cosets, $SO(n, \mathbb{R})/K$, has the dimension n(n-1)/2 - (n-m)(n-m-1)/2 = mn - m(m+1)/2. The configuration space of internal (relative) degrees of freedom Q_{int} is diffeomorphic with $(SO(n, \mathbb{R})/K) \times Symm(m, \mathbb{R})$. And the Cartesian product is an *mn*-dimensional manifold, just as $Q_{int} = LM(m, n, \mathbb{R})$ itself.

Let $\Psi \in \text{LM}(U, V)$ be a reference configuration. It is a linear mapping from U to V. $\Psi(U) \subset V$ is a linear subspace and $K(\Psi) \subset \text{SO}(V,g)$ denote a subgroup preserving every point of $\Psi(U)$, the more $\Psi(U)$ itself. It acts trivially on $\Psi(U)$ and is the group of rotations on $\Psi(U)^{\perp}$. The quotient manifold $\text{SO}(V,g)/K(\Psi)$ describes rotational degrees of freedom. Without using Ψ : this manifold is F(V,g;m), Stiefel manifold. When $V = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(V,g;m) = \text{SO}(n,\mathbb{R})/\text{SO}(n-m,\mathbb{R})$.

Remark 1. Stiefel manifold differs from Grassmann one. The latter are sets of *m*dimensional linear subspaces in V, these subspaces are simply linear shells of the mentioned *m*-frames, thus, they contain less information. Grassmann manifolds may be identified with

 $\mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-m,\mathbb{R})\times\mathrm{SO}(m,\mathbb{R}).$

Their dimension equals to m(n-m)*.*

The "polar" decomposition identifies the internal configuration space with

 $(\mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-m,\mathbb{R})) \times \mathrm{Symm}(m,\mathbb{R}).$

In general, it is not very easy to parametrize Stiefel manifolds. Fortunately when m = n - 1 (physically two), then SO $(1, \mathbb{R}) = \{1\}$ and SO $(n, \mathbb{R})/SO(1, \mathbb{R}) = SO(n, \mathbb{R})$. In this case Q_{int} is just

$$Q_{\text{int}} = \text{SO}(n, \mathbb{R}) \times \text{Symm}(n-1, \mathbb{R}).$$

In particular, when physically n = 3 and m = 2 we have

$$Q_{\text{int}} = \text{SO}(3, \mathbb{R}) \times \text{Symm}(2, \mathbb{R}).$$

Our practical interest is concentrated on this special case where n = 3 and m = 2, i.e., homogeneously deformable flat membrane.

The "two-polar" decomposition has the form

$$\varphi = V \begin{bmatrix} D \\ o \end{bmatrix} U^{-1}$$

where $V \in SO(n, \mathbb{R})$ with the same provisos concerning fictitious degrees of freedom described by $SO(n - m, \mathbb{R})$, $D = \operatorname{diag}(D_1, \ldots, D_m)$ is an $m \times m$ positive diagonal matrix, and $U \in SO(m, \mathbb{R})$ is an arbitrary $m \times m$ proper orthogonal matrix.

Physically, when n = 3 and m = 2 the configurations are correctly represented by the triples (V, D, U). The "polar" and "two–polar" decompositions have the forms

$$\varphi = R \begin{bmatrix} \xi & \alpha \\ \alpha & \zeta \\ 0 & 0 \end{bmatrix}, \qquad \varphi = V \begin{bmatrix} \lambda & 0 \\ 0 & \mu \\ 0 & 0 \end{bmatrix} U^{-1}(\theta)$$

where $R, V \in SO(3, \mathbb{R}), U[\theta] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \in SO(2, \mathbb{R})$. As every angular variable, θ is taken modulo 2π and (ξ, α, ζ) . (), μ) are systems of real coordinates

variable, θ is taken modulo 2π , and (ξ, α, ζ) , (λ, μ) are systems of real coordinates subject only to the positive-definiteness conditions

 $\xi>0,\qquad \xi\zeta-\alpha^2>0,\qquad \lambda>0,\qquad \mu>0$

where (ξ, α, ζ) , (λ, μ) are global variables (although subject to unilateral constraints). If our planar affine system of material points is discrete, or if affine degrees of freedom are essentially internal, the mentioned unilateral constraints may be weakened or even omitted.

The group $SO(3, \mathbb{R})$ is parametrized by local coordinates well–known from the rigid body mechanics and from geometry of the corresponding group manifold like e.g., Euler angles, rotation vector etc. The particular choice depends on the structure of dynamical model one uses.

We introduce angular velocities in the co-moving representation [2]

$$\omega = R^{-1} \frac{\mathrm{d}R}{\mathrm{d}t} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$
$$\chi = V^{-1} \frac{\mathrm{d}V}{\mathrm{d}t} = \begin{bmatrix} 0 & \chi_3 & -\chi_2 \\ -\chi_3 & 0 & \chi_1 \\ \chi_2 & -\chi_1 & 0 \end{bmatrix}$$
$$\vartheta = U^{-1} \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t} U^{-1} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The internal kinetic energy in the "polar" representation has the form

$$T = -\frac{1}{2} \operatorname{Tr} \left(\begin{bmatrix} LJL & o^T \\ o & O_{n-m} \end{bmatrix} \omega^2 \right) + \operatorname{Tr} \left(\begin{bmatrix} LJ\dot{L} & o^T \\ o & O_{n-m} \end{bmatrix} \omega \right) + \frac{1}{2} \operatorname{Tr} \left(J\dot{L}^2 \right)$$

where O_{n-m} is a $(n-m) \times (n-m)$ -dimensional zero matrix. The three terms of above energy are interpreted as

- 1. T_{rot} it has the form of the kinetic energy of the top with degrees of freedom described by $SO(n, \mathbb{R})$. This term describes the coupling between angular velocity and the deformation matrix L.
- 2. $T_{\rm rot-def}$ this term describes the coupling between angular velocity and the deformation velocity, i.e., Coriolis term.
- 3. T_{def} this term describes the kinetic energy of deformation.

We obtain the complete form of the kinetic energy for such a case

$$T = T_{\rm rot} + T_{\rm rot-def} + T_{\rm def}$$

where

$$T_{\rm rot} = \frac{J_1 \alpha^2 + J_2 \zeta^2}{2} \omega_1^2 + \frac{J_1 \xi^2 + J_2 \alpha^2}{2} \omega_2^2 + \frac{J_1 \xi^2 + J_2 \zeta^2 + (J_1 + J_2) \alpha^2}{2} \omega_3^2 - (J_1 \xi + J_2 \zeta) \alpha \omega_1 \omega_2 T_{\rm rot-def} = \left(J_1 \alpha \frac{d\xi}{dt} + (J_2 \zeta - J_1 \xi) \frac{d\alpha}{dt} - J_2 \alpha \frac{d\zeta}{dt}\right) \omega_3 T_{\rm def} = \frac{J_1}{2} \left(\frac{d\xi}{dt}\right)^2 + \frac{J_2}{2} \left(\frac{d\zeta}{dt}\right)^2 + \frac{J_1 + J_2}{2} \left(\frac{d\alpha}{dt}\right)^2.$$

In the "two–polar" case, when $J^{AB} = J\delta^{AB}$ is isotropic, the kinetic energy has the following form

$$T = \frac{J}{2} \left(\mu^2 \chi_1^2 + \lambda^2 \chi_2^2 + \left(\lambda^2 + \mu^2 \right) \chi_3^2 \right) + 2J\lambda \mu \chi_3 \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{J \left(\lambda^2 + \mu^2 \right)}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 + \frac{J}{2} \left(\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}\mu}{\mathrm{d}t} \right)^2 \right).$$

This is the high symmetry case when the system is doubly isotropic, i.e., invariant under both physical and material rigid rotations.

We assume that the potential V depends only on the deformation, i.e., depends on φ through the Green deformation tensor; in other words, we assume that the problem is isotropic in the physical space. The Green deformation tensor G(11)(12)(13) is given by the following formula $G_{AB} = \delta_{ij}\varphi^i{}_A \varphi^j{}_B$. Obviously, $G^T = G$. The Green matrix G is non-sensitive to the left orthogonal mappings. Of course, the deformation invariants, i.e., eigenvalues of the symmetric matrix G, are invariant under both space and material rigid rotations.

The potential V adapted to the "two-polar" decomposition is a function of quantities λ , μ . In the "polar" description it is convenient to use V as a function of α , ξ , ζ . Our Lagrangian has the form $L = T - V(\varphi)$ with velocity-independent potential.

In the Hamiltonian formalism all physical quantities, in particular the energy, are expressed in terms of canonical momenta and generalized coordinates. Starting with Lagrangian description based on velocities one has to invert the Legendre transformation (4) (5) and express generalized velocities through canonical momenta and generalized coordinates. We describe the Legendre transformation in the following form $p_i = \partial L/\partial \dot{q}^i = \partial T/\partial \dot{q}^i$, $s_i = \partial L/\partial \omega^i = \partial T/\partial \omega^i$, where p_i are canonical momenta conjugate to the all generalized coordinates q^i except those contained in \mathbb{R} , and s_i are canonical spins [2].

Legendre transformation in the "polar" case

$$p_{\alpha} = (J_1 + J_2) \frac{\mathrm{d}\alpha}{\mathrm{d}t} - (J_1\xi - J_2\zeta) \omega_3$$

$$p_{\xi} = J_1 \left(\frac{\mathrm{d}\xi}{\mathrm{d}t} + \alpha\omega_3\right), \ p_{\zeta} = J_2 \left(\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \alpha\omega_3\right)$$

$$s_1 = (J_1\alpha^2 + J_2\zeta^2) \omega_1 - \alpha (J_1\xi + J_2\zeta) \omega_2$$

$$s_2 = -\alpha (J_1\xi + J_2\zeta) \omega_1 + (J_1\xi^2 + J_2\alpha^2) \omega_2$$

$$s_3 = \alpha \left(J_1\frac{\mathrm{d}\xi}{\mathrm{d}t} - J_2\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right) - \frac{\mathrm{d}\alpha}{\mathrm{d}t} (J_1\xi - J_2\zeta)$$

$$+ (J_1\xi^2 + (J_1 + J_2) \alpha^2 + J_2\zeta^2) \omega_3.$$

10

Legendre transformation in the "two-polar" case

$$p_{\lambda} = J \frac{d\lambda}{dt}, \qquad p_{\mu} = J \frac{d\mu}{dt}, \qquad p_{\theta} = J \left(\lambda^2 + \mu^2\right) \frac{d\theta}{dt} + 2J\lambda\mu\chi_3$$
$$s_1 = J\mu^2\chi_1, \qquad s_2 = J\lambda^2\chi_2, \qquad s_3 = 2J\lambda\mu\frac{d\theta}{dt} + J \left(\lambda^2 + \mu^2\right)\chi_3.$$

After inverting these expressions, i.e., expressing \dot{q}^i , ω^i as functions of p_i , s_i and substituting these functions into the formula for the kinetic energy we obtain the kinetic energy expressed in the canonical terms.

Kinetic energy in the "polar" case

$$T = \frac{J_1(\xi s_1 + \alpha s_2)^2 + J_2(\alpha s_1 + \zeta s_2)^2}{2(\alpha^2 - \xi\zeta)^2 J_1 J_2} + \frac{(\xi^2 J_1 + \zeta^2 J_2) p_{\alpha}^2}{2(\xi + \zeta)^2 J_1 J_2} + \frac{(\alpha^2 J_1 + (\alpha^2 + (\xi + \zeta)^2) J_2) p_{\xi}^2 + ((\alpha^2 + (\xi + \zeta)^2) J_1 + \alpha^2 J_2) p_{\zeta}^2}{2(\xi + \zeta)^2 J_1 J_2} + \frac{(J_1 + J_2) (s_3 (s_3 - 2\alpha p_{\zeta}) + 2\alpha p_{\xi} (s_3 - \alpha p_{\zeta}))}{2(\xi + \zeta)^2 J_1 J_2} + \frac{2 (\xi J_1 - \zeta J_2) p_{\alpha} (\alpha p_{\zeta} - \alpha p_{\xi} + s_3)}{2(\xi + \zeta)^2 J_1 J_2}.$$

Kinetic energy in the "two-polar" case

$$T = \frac{1}{2J}p_{\lambda}^{2} + \frac{1}{2J}p_{\mu}^{2} + \frac{\lambda^{2} + \mu^{2}}{2J(\lambda^{2} - \mu^{2})^{2}}p_{\theta}^{2} + \frac{1}{2J\mu^{2}}s_{1}^{2} + \frac{1}{2J\lambda^{2}}s_{2}^{2} + \frac{\lambda^{2} + \mu^{2}}{2J(\lambda^{2} - \mu^{2})^{2}}s_{3}^{2} - \frac{2\lambda\mu}{J(\lambda^{2} - \mu^{2})^{2}}p_{\theta}s_{3}$$

The Hamiltonian (total energy) (6) has the following form $H = T + V(\alpha, \xi, \zeta)$ for the "polar" decomposition and $H = T + V(\lambda, \mu)$ for the "two-polar" decomposition, where V's are invariant under permutations of its arguments. Similarly as in the Newtonian description, the canonical equations of motion lead to the closed subsystem of equations imposed on parameters q^i , p_i , s_i

$$\frac{\mathrm{d}q^{i}}{\mathrm{d}t} = \left\{q^{i}, H\right\}, \qquad \frac{\mathrm{d}s_{i}}{\mathrm{d}t} = \left\{s_{i}, H\right\}, \qquad \frac{\mathrm{d}p_{i}}{\mathrm{d}t} = \left\{p_{i}, H\right\}$$

Poisson brackets on the right-hand sides can be easily calculated by the help of the standard rules

$$\begin{split} \{f,g\} &= -\left\{g,f\right\}, \qquad \{f,F(g)\} = \{f,g\} \; \frac{\mathrm{d}F}{\mathrm{d}g} \\ \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0. \end{split}$$

One should substitute here the following basic formula [2], [27], [1]

$$\{q^i, p_j\} = \delta^i{}_j, \quad \{s_i, s_j\} = -\varepsilon_{ijk}s_k, \quad \{p_i, s_j\} = 0, \quad \{q^i, s_j\} = 0$$

where ε_{ijk} is the completely antisymmetric tensor and $\varepsilon_{123} = 1$.

The equations of motions in canonical terms in the case of "two–polar" decomposition have the following form

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{p_{\lambda}}{J}, \quad \frac{d\mu}{dt} = \frac{p_{\mu}}{J}, \quad \frac{d\theta}{dt} = \frac{(\lambda^2 + \mu^2) p_{\theta}}{J(\lambda^2 - \mu^2)^2} - \frac{2\lambda\mu p_{\theta}}{J(\lambda^2 + \mu^2)^2} \\ \frac{ds_1}{dt} &= \frac{\lambda \left(2\lambda^3 p_{\theta} + \lambda \left(\mu^2 - 3\lambda^2\right) s_3\right) s_2}{J(\lambda^3 - \lambda\mu^2)^2} \\ \frac{ds_2}{dt} &= \frac{\lambda \left(2\mu^3 p_{\theta} + \lambda \left(\lambda^2 - 3\mu^2\right) s_3\right) s_1}{J(\mu^3 - \mu\lambda^2)^2} \\ \frac{ds_3}{dt} &= \frac{(\mu^2 - \lambda^2) s_1 s_2}{J\lambda^2 \mu^2}, \quad \frac{dp_{\theta}}{dt} = 0 \\ \frac{dp_{\lambda}}{dt} &= K + \frac{\lambda \left(\lambda^2 + 3\mu^2\right) p_{\theta}^2}{J(\lambda^2 - \mu^2)^3} + \frac{s_2^2}{J\lambda^3} - \frac{2 \left(3\lambda^2 \mu + \mu^3\right) p_{\theta} s_3}{J(\lambda^2 - \mu^2)^3} + \frac{\lambda \left(\lambda^2 + 3\mu^2\right) s_3^2}{J(\lambda^2 - \mu^2)^3} \\ \frac{dp_{\mu}}{dt} &= P - \frac{\left(3\lambda^2 \mu + \mu^2\right) p_{\theta}^2}{J(\lambda^2 - \mu^2)^3} + \frac{s_1^2}{J\mu^3} + \frac{2 \left(3\lambda\mu^2 + \lambda^3\right) p_{\theta} s_3}{J(\lambda^2 - \mu^2)^3} - \frac{\left(3\mu\lambda^2 + \mu^3\right) s_3^2}{J(\lambda^2 - \mu^2)^3} \end{aligned}$$

where K and P depend on the appropriate potentials. They are build of derivatives of the potential terms.

The resulting equations are strongly non–linear and terribly complicated. But there exist some special solutions - stationary ellipses as solutions on which the Green deformation tensor and angular velocities are constant. They are analogous to the stationary ellipsoids well–known in the astrophysics and geophysics as equilibrium figures, e.g., in the theory of the shape of the Earth.

We have shown these solutions related to components of canonical spins s_i in the cases of three kinds of potentials

•

$$\mathbf{V} = k \left(\lambda^2 + \mu^2\right) / 2 \tag{14}$$

This is the harmonic oscillator in deformation invariants. Non–limited extensions of the body are excluded. Although contraction is not forbidden, solutions of this kind are exceptional (measure–zero set), as usually solutions passing through the equilibrium are so in multidimensional oscillators. But some other requirements of macroscopic elasticity are not satisfied. Nevertheless, it may be applicable as some simplified model.

$$\mathbf{V} = c \left(\frac{1}{\lambda^2} + \lambda^2\right) + c \left(\frac{1}{\mu^2} + \mu^2\right)$$
(15)

With this potential both the unlimited extension and contraction are forbidden. This potential has the minimum at the natural non-deformed state $\lambda = \mu = 1$. Nevertheless, certain conditions from the macroscopic elasticity are not satisfied. Namely, there is no direct relationship between becoming longer in one direction and shorter in another.

$$\mathbf{V} = k \left(\frac{1}{\lambda \mu} + \frac{\lambda^2 + \mu^2}{2} \right) \tag{16}$$

This potential has some very interesting features, because in the purely twodimensional case with the isotropic inertial tensor, it admits separation of variables in the Hamilton–Jacobi equation. This admits solution in terms of quadratures, i.e., the resulting system is completely integrable. This is seen when we use the polar coordinates in (x, y)-plane

$$x = \rho \cos \varepsilon, \qquad y = \rho \sin \varepsilon$$

and λ , μ are replaced by their combinations

$$x = \frac{1}{\sqrt{2}} \left(\lambda + \mu\right), \qquad y = \frac{1}{\sqrt{2}} \left(\lambda - \mu\right).$$

3. Quantization Ideas

In previous section we discussed the mechanics of the affinely-rigid body with degenerate dimension from the classical point of view. The problem of dynamical systems on the manifolds of affine injections is also applicable in the theory of structured bodies, i.e., bodies consisting of small planes, like planar molecules and perhaps some submolecular elements. Applications in nanophysics are possible, e.g., flat or approximately flat molecules or the historical model of "Schwungrad" used in molecular dynamics [4]. Microstructure two-dimensional elements appear in condensed matter theory, e.g., as three-atomic molecules like CO_2 , S_3 , H_2O , or perhaps larger molecules, e.g., with some relatively stable two-dimensional skeleton. One-dimensional elements appear as constituents of liquid crystals.

Applications in nanophysics must be based on the quantum theory. Some very interesting phenomena may appear there, because one deals in such problems with some convolution, overlap of the classical and quantum levels.

Let us mention one-dimensional constituents of liquid crystals which are structure elements of degenerate dimension. They are well known in condensed matter theory. Two-dimensional objects appear as three-atomic molecules like H_2O , S_3 ,

•

 CO_2 and molecules consisting of a larger number of atoms, but having some almost "flat" core.

One must be careful when applying concepts like "rigid body" or "affinely rigid body" or other constraints based on classical intuitions to objects in the molecular, atomic, or nuclear scale of physical phenomena. In any case one should express what is the meaning of the used terms on the level of Quantum Mechanics. In this paper we mean the following: The primary classical model is subject to ideal constraints and then the resulting mechanical system in a Riemannian configuration space of classical holonomic constraints is quantized according to the standard Schrödinger procedure. In this respect we follow some standard treatises like [12], [19], [20]. Nevertheless we are aware that such a procedure is not identical with that based on the reversed sequence: first to quantize and only then to impose constraints in a proper way. The first procedure, used below, is simpler and leads more directly to some results. The complete discussion of the problem goes fairly outside the scope of this paper and is a subject for the separate study of the general problem of quantum constraints. Roughly speaking, the idea is that the quantum Hamilton operator has the structure which approximately looks like

$$\widehat{H} = \widehat{H}_0 + W$$

where \hat{H}_0 is some background term and W is a confinement potential term. The peculiarity of W is that it vanishes on some submanifold Q of the original (unconstrained) configuration space M and grows very quickly (is very "tough") when departing from Q. So there are quantum vibrations in the "across-Q-variables" and "constrained" quantum dynamics in Q ("along-Q-variables"). There are some vibrational energy levels of the "across-Q-dynamics" and they split into some bands of energy levels of the "along-Q-dynamics". It is natural to expect that it is only ground state of quantized "across-Q-vibrations" that is relevant. Because of this there is some correspondence between two approaches

- 1. classically confine and then Schrödinger-quantize
- 2. Schrödinger–quantize and then investigate the bands of energy levels corresponding to the ground state of "across-Q-vibrations".

The quantum operator of the internal kinetic energy has the form proportional to the Laplace–Beltrami operator

$$\mathbf{T} = -\frac{\hbar^2}{2J}\,\Delta_{\Gamma}$$

where \hbar is the "crossed" Planck constant, J is the scalar moment of inertia of the isotropic disc, and Δ is the Laplace–Beltrami operator.

Laplace–Beltrami operator corresponding to the metric Γ is given by

$$\Delta \Psi = \sum_{\mu\nu} \frac{1}{\sqrt{|\Gamma|}} \frac{\partial}{\partial Q^{\mu}} \left(\sqrt{|\Gamma|} \, \Gamma^{\mu\nu} \frac{\partial \Psi}{\partial Q^{\nu}} \right)$$

where $|\Gamma| = |\det[\Gamma_{\mu\nu}]|$ and $\Gamma_{\mu\nu}(Q)$ is given by the underlying classical kinetic energy

$$T_{\rm int} = \frac{1}{2} \Gamma_{\mu\nu}(Q) \frac{\mathrm{d}Q^{\mu}}{\mathrm{d}t} \frac{\mathrm{d}Q^{\nu}}{\mathrm{d}t}$$

where Q^{ν} are generalized coordinates of internal motion. In a more geometric notation Laplace–Beltrami operator has a form

$$\Delta \Psi = \Gamma^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \Psi$$

where ∇ is the covariant differentiation in the Levi–Civita sense.

The Hilbert space of wave functions of internal degrees of freedom is $L^2(Q_{int}, \nu)$, where ν is the induced Γ -Riemannian measure

$$\mathrm{d}\nu(Q) = \sqrt{\Gamma(Q)} \,\mathrm{d}Q^1 \dots \mathrm{d}Q^6.$$

The scalar product is given by

$$\langle \Psi, \Phi \rangle = \int \overline{\Psi}(Q) \Phi(Q) \mathrm{d}\nu(Q).$$

In our problem the direct calculation would be very embarrassing and the result would not be readable. To calculate anything in detail is difficult, but there exists a simple expression based on the group-theoretic structure of the problem. Namely, one can replace the spin variables s_a by the quantum operators S_a generating right rotations of V

$$f(V(I+\varepsilon)) = f(V) + \varepsilon^{i} \mathcal{R}_{i} f(V) = f(V) + \frac{1}{\hbar} \varepsilon^{i} \mathbf{S}_{i} f(V) + \hbar \mathfrak{o}(\varepsilon)$$
(17)

where \mathcal{R}_i are generators

$$\varepsilon = \begin{bmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{bmatrix}$$

and $\frac{\mathfrak{o}(\varepsilon)}{\varepsilon} \to 0$ when $\varepsilon \to 0$. Quantum spin operators \mathbf{S}_i satisfy the relations in quantum Poisson brackets

$$\frac{1}{\mathrm{i}\hbar}\left[\mathbf{S}_{a},\mathbf{S}_{b}\right]=-\varepsilon_{abc}\mathbf{S}_{c}$$

where "i" is the imaginary unit.

The classical quantity p_{θ} will be replaced by the operator

$$\mathbf{p}_{\theta} = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \theta}.$$
 (18)

The operator $\mathbf{T}_{\text{int}} = -\frac{\hbar^2}{2J}\Delta$ may be expressed as

$$\Gamma_{\text{int}} = \frac{\mathbf{S}_{1}^{2}}{2J\mu^{2}} + \frac{\mathbf{S}_{2}^{2}}{2J\lambda^{2}} + \frac{(\lambda^{2} + \mu^{2})\mathbf{S}_{3}^{2}}{2J(\lambda^{2} - \mu^{2})^{2}} + \frac{(\lambda^{2} + \mu^{2})\mathbf{p}_{\theta}^{2}}{2J(\lambda^{2} - \mu^{2})^{2}} - \frac{2\lambda\mu\mathbf{p}_{\theta}\mathbf{S}_{3}}{J(\lambda^{2} - \mu^{2})^{2}} - \frac{\hbar^{2}}{2J}\frac{1}{\mathcal{P}}\left[\frac{\partial}{\partial\lambda}\mathcal{P}\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\mu}\mathcal{P}\frac{\partial}{\partial\mu}\right]$$
(19)

where the weight factor \mathcal{P} is given by

$$\mathcal{P} = \lambda \mu \left| \lambda^2 - \mu^2 \right|.$$

We can rewrite \mathcal{P} with the analogous expression in mechanics of the affine body with non–degenerate dimension [26–29]. When we write down explicitly the differential terms with respect to λ and μ , we obtain

$$\mathbf{T}_{\text{int}} = \frac{\mathbf{S}_{1}^{2}}{2J\mu^{2}} + \frac{\mathbf{S}_{2}^{2}}{2J\lambda^{2}} + \frac{\left(\lambda^{2} + \mu^{2}\right)\mathbf{S}_{3}^{2}}{2J\left(\lambda^{2} - \mu^{2}\right)^{2}} + \frac{\left(\lambda^{2} + \mu^{2}\right)\mathbf{p}_{\theta}^{2}}{2J\left(\lambda^{2} - \mu^{2}\right)^{2}} \\ - \frac{2\lambda\mu\mathbf{p}_{\theta}\mathbf{S}_{3}}{J\left(\lambda^{2} - \mu^{2}\right)^{2}} - \frac{\hbar^{2}}{2J}\frac{\partial^{2}}{\partial\lambda^{2}} - \frac{\hbar^{2}}{2J}\frac{\partial\ln\mathcal{P}}{\partial\lambda}\frac{\partial}{\partial\lambda} - \frac{\hbar^{2}}{2J}\frac{\partial^{2}}{\partial\mu^{2}} - \frac{\hbar^{2}}{2J}\frac{\partial\ln\mathcal{P}}{\partial\mu}\frac{\partial}{\partial\mu}.$$
(20)

Derivation of this formula is analogous to the one used in [10, 28, 29]. It is based on using some non-holonomic frame in the configuration space. The idea of this frame is to use the triple of basic left-invariant vector fields on $SO(3, \mathbb{R})$, i.e., on the manifold of V-type degrees of freedom. They are just given by the triple of first-order differential operators \mathcal{R}_i (17).

The remaining vector fields of the frame form, i.e., its "holonomic part", they given by $\partial/\partial\lambda$, $\partial/\partial\mu$, $\partial/\partial\theta$. An interesting feature of the formulas for T_{int} , T_{int} is the characteristic entanglement of deformation invariants λ , μ . Just like in the three– dimensional case, this is a consequence of the fact that SO(3, \mathbb{R}) is simple. In what follows we will assume that the potential energy is also spatially and materially isotropic, just like T_{int} , thus it has a form $\mathcal{V}(\lambda, \mu)$ and depends only on λ and μ . The total operator of the kinetic energy is given by

$$\mathbf{T} = \mathbf{T}_{\mathrm{tr}} + \mathbf{T}_{\mathrm{int}}$$

where the translational part is given by

$$\mathbf{T}_{\rm tr} = -\frac{\hbar^2}{2m} \,\Delta_{\rm tr} = -\frac{\hbar^2}{2m} \,g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = -\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial x^{i2}} \cdot$$

In the rectilinear coordinates Δ_{tr} is just the usual Laplace operator in \mathbb{R}^3 . The total volume element in the full configuration space $Q = Q_{tr} \times Q_{int}$ is given by

$$\mathrm{d}v_{\mathrm{vol}}(x, V, \lambda, \mu, \varphi) = \mathrm{d}_3 x \, \mathrm{d}\nu(V, \lambda, \mu, U(\theta))$$

where ν denotes the Γ -Riemannian volume measure on the internal configuration space Q_{int} .

When $J^{AB} = I\eta^{AB}$, i.e., the body is isotropic and the potential depends only on invariants λ , μ , then the solving procedure of Schrödinger equation may be partially algebraized. Namely, one can perform the Fourier analysis on $SO(3, \mathbb{R})$, $SO(2,\mathbb{R})$. What concerns the group $SO(2,\mathbb{R})$ it is the usual Fourier series expansion on the circle, i.e., the function series combining the functions $e^{ik\theta}$, $k \in \mathbb{Z}$. The dependence on V-arguments is expanded into series combining the matrix elements of irreducible representations of the group $SO(3,\mathbb{R})$, according to the Peter–Weyl Theorem [32]. These functions (found by Wigner) are traditionally denoted by $\mathfrak{D}^{j}_{mm'}$, where $j = 0, 1, 2, \dots$ (the set of non-negative integers) and $m, m' = -j, -j + 1, \dots, -1, 0, 1, \dots, j - 1, j$ (i.e., integers from -j to j). It is a good place here to mention that perhaps it might be reasonable to admit the situations with half-integer internal angular momentum, and so with j running over the range of non-negative multiples of number 1/2, i.e., $j = 0, 1/2, 1, 3/2, \dots$ The quantum numbers m, m' then run over the range from -j to j jumping by one. In both cases m, m' run over (2j + 1) values. Then the situations with half-integer j would occur only in coincidence with half-integer values of k in functions $e^{ik\theta}$. In any case, it is quite reasonable to proceed in this way when dealing with rigid and affine bodies in the dimension 3 [27–29]. However, here we do not go into details of this problem.

Usually to parametrize the group $SO(3, \mathbb{R})$ one use the rotation vector \overline{k} , that is canonical coordinates of the first kind

$$V(\bar{k}) = \exp\left(k^a E_a\right), \qquad \left(E_a\right)^b{}_c = -\varepsilon_a{}^b{}_c$$

where E_a , a = 1, 2, 3, are basic antisymmetric matrices and ε is the completely antisymmetric Ricci symbol, and the shifting of indices is understood in a trivial sense of the Kronecker "delta" δ_{ab} . The length of vector \bar{k} , scalar $k = \sqrt{\bar{k} \cdot \bar{k}}$ is the angle of rotation, and the versor $\bar{n} = \frac{\bar{k}}{\bar{k}}$ is his axis oriented in accordance with the right–hand rule. The scalar k runs over range $[0, \pi]$ and $V(\pi \bar{n}) = V(-\pi \bar{n})$, $\bar{n} \cdot \bar{n} = 1$.

If we wish to admit "half integral spin", then instead of the group $SO(3, \mathbb{R})$ we would have to use its covering SU(2), i.e., unitary unimodular complex 2×2 matrices. Then

$$SU(2) \ni u(\bar{k}) = \exp\left(k^a e_a\right) = \cos\frac{k}{2} I_2 - i\frac{k^a}{k}\sin\frac{k}{2} \sigma_a$$

where σ_a are Pauli matrices, $e_a = \left(-\frac{i}{2}\right)\sigma_a$ and the "angle of rotation" runs over the range $[0, 2\pi]$ thus twice larger. For any versor \bar{n} ($\bar{n} \cdot \bar{n} = 1$), we have then $u(2\pi\bar{n}) = -I_2$, where I_2 is the identity matrix 2×2 . We find that the *j*-th irreducible representation of G = SU(2) (or $SO(3, \mathbb{R})$) is given by

$$\mathfrak{D}^{j}(\bar{k}) = \exp\left(\frac{\mathrm{i}}{\hbar}k^{a}S^{j}{}_{a}\right)$$

where $S^{j}{}_{a}$ are Wigner matrices of the angular momentum with the Casimir quantum number j and the square of magnitudes $\hbar^{2}j(j+1)$). In any case, \mathfrak{D}^{j} are unitary $(2j+1) \times (2j+1)$ matrices.

The operators \mathfrak{S}_a and \mathbf{S}_a are given by

$$\mathfrak{S}_a = \frac{\hbar}{\mathrm{i}} \mathcal{L}_a, \qquad \mathbf{S}_a = \frac{\hbar}{\mathrm{i}} \mathcal{R}_a$$
(21)

which are proportional to the generators of the left and right regular translations in $SO(3, \mathbb{R})$. They are self-adjoint in the sense of $L^2(SO(3, \mathbb{R}), \kappa)$ or $L^2(SO(2), \kappa)$, and for the functions $\Psi(\varphi) = \Psi(V, U, \lambda, \mu)$ in the sense of $L^2(Q, \nu)$. For "small" values of $\bar{\varepsilon}$ we have

$$f\left(V(\bar{k})V(\bar{\varepsilon})\right) = f\left(V(\bar{k})\right) + \varepsilon^{i}\mathcal{R}_{i}f\left(V(\bar{k})\right) + \mathfrak{o}(\varepsilon)$$
$$f\left(V(\bar{\varepsilon})V(\bar{k})\right) = f\left(V(\bar{k})\right) + \varepsilon^{i}\mathcal{L}_{i}f\left(V(\bar{k})\right) + \mathfrak{o}(\varepsilon)$$

where $\mathfrak{o}(\varepsilon)$ is small of higher order than ε are formally self-adjoint and may be expressed in terms of the radius vector \bar{k} by the formulas

$$\mathcal{L}_{a} = \frac{k}{2} \cot \frac{k}{2} \frac{\partial}{\partial k^{a}} + \left(1 - \frac{k}{2} \cot \frac{k}{2}\right) \frac{k_{a}}{k} \frac{k^{b}}{k} \frac{\partial}{\partial k^{b}} + \frac{1}{2} \varepsilon_{ab}{}^{c} k^{b} \frac{\partial}{\partial k^{c}}$$
(22)

$$\mathcal{R}_{a} = \frac{k}{2} \cot \frac{k}{2} \frac{\partial}{\partial k^{a}} + \left(1 - \frac{k}{2} \cot \frac{k}{2}\right) \frac{k_{a}}{k} \frac{k^{b}}{k} \frac{\partial}{\partial k^{b}} - \frac{1}{2} \varepsilon_{ab}{}^{c} k^{b} \frac{\partial}{\partial k^{c}}.$$
 (23)

Using the operator which is the generator of rotations of the rotation vector, that is the internal automorphism in the group $SO(3, \mathbb{R})$

$$\mathcal{D}_a = \mathcal{L}_a - \mathcal{R}_a$$

we can rewrite (22) and (23) in the following form

$$\mathcal{L}_a = rac{k_a}{2}rac{\partial}{\partial k} - rac{1}{2}arepsilon_{ab}{}^c k^b \mathcal{D}_c + rac{1}{2}\mathcal{D}_a \ \mathcal{R}_a = rac{k_a}{2}rac{\partial}{\partial k} - rac{1}{2}arepsilon_{ab}{}^c k^b \mathcal{D}_c - rac{1}{2}\mathcal{D}_a.$$

The Casimir invariants have the form

$$\mathcal{L}^2 = \mathcal{R}^2 = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2 = \mathcal{R}_1^2 + \mathcal{R}_2^2 + \mathcal{R}_3^2$$
$$= \left(\frac{\partial^2}{\partial k^2} + \cot\frac{k}{2}\frac{\partial}{\partial k}\right) + \frac{1}{4\sin^2\frac{k}{2}}\mathcal{D}^2$$

where

$$\mathcal{D}^2=\mathcal{D}_1^2+\mathcal{D}_2^2+\mathcal{D}_3^2$$

is the Casimir invariant for \mathcal{D}_a . Let us remind that \mathfrak{S}_a (21) are operators of the internal angular momentum (spin) in the laboratory representation, and \mathbf{S}_a (21) are auxiliary operators, components of the spin in the system of axes connected with the moving top. The left–hand Poisson brackets have the following form

$$\frac{1}{i\hbar} \left[\mathfrak{S}_{a}, \mathfrak{S}_{b} \right] = \varepsilon_{ab}{}^{c} \mathfrak{S}_{c}, \qquad \frac{1}{i\hbar} \left[\mathbf{S}_{a}, \mathbf{S}_{b} \right] = -\varepsilon_{ab}{}^{c} \mathbf{S}_{c}.$$

Obviously

$$\mathfrak{S}^2 = \sum_a (\mathfrak{S}_a)^2 = \mathbf{S}^2 = \sum_a (\mathbf{S}_a)^2.$$

We have the following relations

$$\begin{split} \mathfrak{S}_{a}\mathfrak{D}^{j} &= S^{j}{}_{a}\mathfrak{D}^{j}, \qquad \mathbf{S}_{a}\mathfrak{D}^{j} = \mathfrak{D}^{j}S^{j}{}_{a}, \qquad \mathfrak{S}^{2}\mathfrak{D}^{j} = \mathbf{S}^{2}\mathfrak{D}^{j} = \hbar^{2}j(j+1)\mathfrak{D}^{j}\\ \mathfrak{S}_{3}\mathfrak{D}^{j}{}_{m\,m'} &= \hbar m\mathfrak{D}^{j}{}_{m\,m'}, \qquad \mathbf{S}_{3}\mathfrak{D}^{j}{}_{m\,m'} = \hbar m'\mathfrak{D}^{j}{}_{m\,m'}. \end{split}$$

Operators \mathfrak{S}_a , \mathbf{S}_a , \mathbf{p}_{θ} , $\mathbf{H} = \mathbf{T} + V(\lambda, \mu)$, where \mathbf{T} given by (19), (20) are formally self-adjoint.

Let us make the aforementioned Weyl-Peter expansion

$$\Psi(V;\lambda,\mu;\theta) = \sum_{j,m,m',k} f^{j,k}_{m',m}(\lambda,\mu) \,\mathfrak{D}^{j}_{m\ m'}(V) \,\mathrm{e}^{\mathrm{i}k\theta}.$$

Expansion expression has a compact form in matrix terms

$$\Psi(V;\lambda,\mu;\theta) = \sum_{j,k} \operatorname{Tr}\left(f^{j,k}(\lambda,\mu) \,\mathfrak{D}^{j}(V)\right) \,\mathrm{e}^{\mathrm{i}k\theta}.$$

The action of the operators \mathfrak{S}_a and \mathbf{S}_a on Ψ is algebraically represented in such a way that the reduced amplitudes $f_{m',m}^{j,k}$ interpreted with the fixed values of j, k as $(2j+1) \times (2j+1)$ matrices with indices m', m are transformed as follows

$$f^{j,k} \mapsto S^{j}{}_{a}f^{j,k}, \qquad f^{j,k} \mapsto f^{j,k}S_{a}{}^{j}.$$

and \mathbf{p}_{θ} (18) acts on Ψ by a rule

$$f^{j,k} \mapsto \hbar k f^{j,k}.$$

If we use the isotropic internal Hamiltonian

 $\mathbf{H} = \mathbf{T} + \mathcal{V}(\lambda, \mu)$

then the stationary Schrödinger equation

$$\mathbf{H}\Psi = E\Psi$$

becomes reduced to the system of independent equations for the matrix amplitudes $f^{j,k}(\lambda,\mu)$

$$H^{j,k}f^{j,k} = E^{j,k}f^{j,k}$$

where

$$\begin{split} H^{j,k}f^{j,k} &= \frac{S_1^{j^2}}{2J\mu^2}f^{j,k} + \frac{S_2^{j^2}}{2J\lambda^2}f^{j,k} + \frac{\lambda^2 + \mu^2}{2J\left(\lambda^2 - \mu^2\right)^2}S_3^{j^2}f^{j,k} \\ &+ \frac{\lambda^2 + \mu^2}{2J\left(\lambda^2 - \mu^2\right)^2}\hbar^2k^2f^{j,k} - \frac{2\lambda\mu}{J\left(\lambda^2 - \mu^2\right)^2}S_3^{j^2}\hbar kf^{j,k} \\ &- \frac{\hbar^2}{2J}\frac{1}{\mathcal{P}}\frac{\partial}{\partial\lambda}\left(\mathcal{P}\frac{\partial}{\partial\lambda}f^{j,k}\right) - \frac{\hbar^2}{2J}\frac{1}{\mathcal{P}}\frac{\partial}{\partial\mu}\left(\mathcal{P}\frac{\partial}{\partial\mu}f^{j,k}\right) + Vf^{j,k}. \end{split}$$

We obtain, the family of reduced Schrödinger equations for the system of matrix– valued amplitudes $f^{j,k}(\lambda,\mu)$. These amplitudes are dependent on deformation invariants.

In this way the number of degrees of freedom of internal motion of our model is effectively reduced from six to two. The price we pay is that we obtain the system of Schrödinger equations for multicomponent complex amplitudes, however, depending only on two variables. Surprisingly enough, at least on this general level, the quantum theory is simpler than the classical one. On the classical level there is no direct way to reduce effectively the number of degrees of freedom and perform the partial reduction to relatively autonomous dynamics of deformation invariants. This reduction is possible only for models with high symmetries, when both the inertial tensor and the potential energy are isotropic. For example for the potentials which we use in the classical case (14), (15) and (16).

Comments: And what about the possibility of half-integer spin? In non-degenerate three-dimensional models it appeared in a natural way by taking instead of $GL(3,\mathbb{R})$ its covering group $\overline{GL}(3,\mathbb{R})$. In the two-polar decomposition the orthogonal group $SO(3,\mathbb{R})$ had to be replaced by the universal covering group SU(2). The same may be done here. Namely, the $SO(3,\mathbb{R})$ -factor of the decomposition must be replaced by SU(2). And the resulting wave functions must satisfy a condition that they combine expressions $\mathfrak{D}^{j}_{mm'}(u)e^{ik\theta}$ in such a way that either both j and k in the admissible superposition are integers, or both of them are half-integers.

Acknowledgements

This paper partially contains results obtained within the framework of the research project N N501 049 540 financed from the Scientific Research Support Fund in the years 2011-2014. We thank to the Polish Ministry of Science and Higher Education for this financial support.

We are thankful also to Professor Ivaïlo M. Mladenov and to all Organizers of the Sixteenth International Conference on Geometry, Integrability and Quantization Dedicated to 70-th Anniversary of Professor Jan J. Sławianowski, June 6-11, 2014, Sts. Constantine and Elena, Varna, Bulgaria, for their cordial hospitality.

References

- [1] Abraham R. and Marsden J., *Fundations of Mechanics*, 2nd Edn, The Benjamin/Cumming, London 1978.
- [2] Arnold V., Mathematical Methods of Classical Mechanics, 2nd Edn, Springer, New York 1989.
- [3] Bogoyavlensky O., Methods of Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamic, Springer, Berlin 1985.
- [4] Born M., Vorlesungen über Atommechanik, von Max Born. Hrsq. unter Mitw. von Friedrich Hund, Springer, Berlin 1925.
- [5] Burov A. and Chevallier D., On the Variational Principle of Poincare, the Poincare-Chetayev Equations and the Dynamics of Affinely Deformable Bodies., Cahier de C.E.R.M.I.C.S. 4 (May 1996).
- [6] Burov A. and Stepanov S., On Geometry of Masses in Dynamics of Deformable Bodies (in Russian), In: Problems of investigation on Stability and Stabilization of motion, CCRA, Moscow 1995.
- [7] Calogero F. and Marchioro C., Exact Solution of a One–Dimensional Three–Body Scattering Problem with Two–Body and/or Three–Body Inverse–Square Potentials, J. Math. Phys. 15 (1974) 1425–1430.
- [8] Chandrasekhar S., *Ellipsoidal Figures of Equilibrium*, Yale University, New Haven 1969.
- [9] Chevallier D., On the Foundations of Ordinary and Generalized Rigid Body Dynamics and the Principle of Objectivity, Arch. Mech. 56 (2004) 313–353.
- [10] Godlewski P., Quantization of Anisotropic Rigid Body, Int. J. Theor. Phys. 42 (2003) 2863–2875.
- [11] Gołubowska B., Kovalchuk V. and Sławianowski J., Constraints and Symmetry in Mechanics of Affine Motion, J. Geom. Phys 78 (2014) 59–79.
- [12] Landau L. and Lifshitz E., *Quantum Mechanics. Non–Relativistic Theory*, Pergamon Press, New York 1977.
- [13] Moser J., *Dynamical Systems Theory and Applications*, Lect. Notes Phys. **38**, Springer, Berlin 1975.
- [14] Moser J., Three Integrable Hamiltonian Systems Connected with Isospectral Deformations, Adv. Math. 16 (1975) 197–220.
- [15] Roberts M., Wulff C. and Lamb J., *Hamiltonian Systems Near Relative Equilibria*, J. Differ. Equations 179 (2002) 562–604.
- [16] Rożko E. and Gobcewicz E., Quantization of Systems with Internal Degrees of Freedom in Two–Dimensional Manifolds, Rep. Math. Phys. 73 (2014) 325–343.

- [17] Rożko E., Dynamics of Affinely-Rigid Bodies with Degenerate Dimension, Rep. Math. Phys. 56 (2005) 311–332.
- [18] Rożko E., The Dynamical Systems on the Homogeneous Spaces and Theirs Applications to Continuum Mechanics (in Polish), PhD Thesis, Warsaw 2006.
- [19] Slater J., Quantum Theory of Matter, McGraw-Hill, New York 1951.
- [20] Slater J., Quantum Theory of Molecules and Solids vol I, McGraw–Hill, New York 1963.
- [21] Sławianowski J., The Mechanics of the Homogeneously–Deformable Body. Dynamical Models with High Symmetries, Z. Angew. Math. Mech. 62 (1982) 229–240.
- [22] Sławianowski J., Analytical Mechanics of Deformable Bodies (in Polish), PWN, Warszawa-Poznań 1982.
- [23] Sławianowski J., *Quantum and Classical Models Based on* $GL(n, \mathbb{R})$ *Symmetry*, In: Proceedings of the Second International Symposium on Quantum Theory and Symmetries, Kraków, Poland, July 18–21, 2001, E. Kapuścik and A. Horzela (Eds), World Scientific 2002, Singapore pp 582–588.
- [24] Sławianowski J., Classical and Quantum Collective Dynamics of Deformable Objects. Symmetry and Integrability Problems, Geometry, Integrability & Quantization 5 (2004) 81–108.
- [25] Sławianowski J. and Gołubowska B., Motion of Test Bodies with Internal Degrees of Freedom in Non–Euclidean Spaces, Rep. Math. Phys. 65 (2010) 379–422.
- [26] Sławianowski J., Kovalchuk V., Gołubowska B., Martens A. and Rożko E., Dynamical Systems with Internal Degrees of Freedom in Non-Euclidean Space, IFTR Reports 8, Warsaw 2006.
- [27] Sławianowski J., Kovalchuk V., Sławianowska A., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., *Invariant Geodetic Systems on Lie Groups and Affine Models of Internal and Collective Degrees of Freedom*, IFTR Reports 7, Warsaw 2004.
- [28] Sławianowski J., Kovalchuk V., Sławianowska A., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., Affine Symmetry in Mechanics of Collective and Internal Modes. Part I. Classical Models, Rep. Math. Phys. 54 (2004) 373–427.
- [29] Sławianowski J., Kovalchuk V., Sławianowska A., Gołubowska B., Martens A., Rożko E. and Zawistowski Z., Affine Symmetry in Mechanics of Collective and Internal Modes. Part II. Quantum Models, Rep. Math. Phys. 55 (2005) 1–45.
- [30] Sławianowska A. and Sławianowski J., Quantization of Affinely Rigid Body in N Dimensions, Rep. Math. Phys. 29 (1991) 297–320.
- [31] Tolmachev V., Quasiclassical Approximation in Quantum Mechanics (in Russian), Moscow University Press, Moscow 1980.
- [32] Weyl H., The Theory of Groups and Quantum Mechanics, Dover, New York 1931.
- [33] Wulff C. and Roberts M., Hamiltonian Systems Near Relative Periodic Orbits, SIAM J. Dyn. Syst. 1 (2002) 1–43.