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AFFINELY-RIGID BODY AND OSCILLATORY TWO-DIMENSIONAL MODELS

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Abstract. Discussed are some classical and quantization problems of the affinely-rigid body in two dimensions. Strictly speaking, we consider the model of the harmonic oscillator potential and then discuss some natural anharmonic modifications. It is interesting that the considered doubly-isotropic models admit coordinate systems in which the classical and Schrödinger equations are separable and in principle solvable in terms of special functions on groups.

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1. Classical Description

Let us consider two Euclidean spaces (N, U, η) and (M, V, g), respectively the material and physical spaces. Here N and M are the basic point spaces, U and V are their linear translation spaces, and $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are their metric tensors.

The configuration space of the affinely-rigid body

$$Q := \mathrm{AfI}(N, M)$$

consists of affine isomorphisms of N onto M. The material labels $a \in N$ are parametrized by Cartesian coordinates a^K [1]. Cartesian coordinates in M will be denoted by y^i and the corresponding geometric points by y. The configuration $\Phi \in Q$ is to be understood in such a way that the material point $a \in N$ occupies the spatial position $y = \Phi(a)$. The Lagrange coordinates a^K in N will be always chosen in such a way that their origin

$$a^K = 0$$

coincides with the centre of mass $\mathcal{C}$

$$\int a^K \mathrm{d}\overline{\mu}(a) = 0$$

where $\overline{\mu}$ denote the comoving mass distribution in N. The configuration space may be identified then with $M \times \mathrm{LI}(U, V)$

$$Q = \mathrm{AfI}(N, M) \simeq M \times \mathrm{LI}(U, V) = M \times Q_{\mathrm{int}}$$

where $\mathrm{LI}(U, V)$ denotes the manifold of all linear isomorphisms of U onto V. The Cartesian product factors refer respectively to the translational motion (M) and the internal or relative motion $(\mathrm{LI}(U, V))$. The motion is described as a continuum of instantaneous configurations

$$\Phi(t, a)^i = \phi^i{}_K(t) a^K + x^i(t) \tag{1}$$

where $x(t)$ is the centre of mass position and $\phi(t)$ tells us how constituents of the body are placed with respect to the centre of mass. If we put $U = V = \mathbb{R}^n$, then Q_{int} reduces to $\mathrm{GL}(n, \mathbb{R})$ and Q becomes the semi-direct product $\mathbb{R}^n \times_s \mathrm{GL}(n, \mathbb{R})$; $\mathbb{R}^n$ is then interpreted as an Abelian group with addition of vectors as a group operation.

Inertia of affinely-constrained systems of material points is described by two constant quantities

$$m = \int \mathrm{d}\overline{\mu}(a), \qquad J^{KL} = \int a^K a^L \mathrm{d}\overline{\mu}(a)$$

i.e., the total mass m and the comoving second-order moment $J \in U \otimes U$. More precisely, it is so in the usual theory based on the d'Alembert principle, when the kinetic energy is obtained by summation (integration) of usual (based on the metric g) kinetic energies of constituents [12, 13]

$$T = \frac{1}{2} g_{ij} \int \frac{\partial \Phi^i}{\partial t} \frac{\partial \Phi^j}{\partial t} \mathrm{d}\overline{\mu}(a).$$

Substituting to this general formula the above affine constraints (1) we obtain

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{1}{2} g_{ij} \frac{\mathrm{d}\phi^i_A}{\mathrm{d}t} \frac{\mathrm{d}\phi^j_B}{\mathrm{d}t} J^{AB}.$$

Obviously, if we analytically identify U and V with $\mathbb{R}^n$ and $\mathrm{LI}(U, V)$ with $\mathrm{GL}(n, \mathbb{R})$, then

$$T_{\mathrm{int}} = \frac{1}{2} \mathrm{Tr} \left(\dot{\phi}^T \dot{\phi} J \right).$$

2. Some Two-Dimensional Models

The mechanics of the affinely-rigid body was discussed in various aspects in [3–7, 9, 12–15]. In this paper we concentrate on the "Flatland" physics, i.e., on the problems in two-dimensional world. The configuration space Q of two-dimensional affinely-rigid body may be analytically identified with the linear group $\mathrm{GL}(2,\mathbb{R})$. The description of degrees of freedom is based on the two-polar decomposition of matrices

$$\phi = ODR^T.$$

The matrices O, R are orthogonal and D is diagonal and positive. The natural parameterization of the problem is as follows

$$O = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \qquad D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}.$$

Here we consider symmetric model, where

$$J = \mu I$$

is isotropic, μ denoting a positive constant, and I is the 2×2 identity matrix. The isotropic kinetic energy is as follows

$$T = \frac{\mu}{2}\left[\left({D_1}^2 + {D_2}^2\right)\left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2\right) - 4D_1D_2\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\psi}{\mathrm{d}t} + \left(\frac{\mathrm{d}D_1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}D_2}{\mathrm{d}t}\right)^2\right].$$

In these coordinates the Hamilton-Jacobi equation is non-separable even in the interaction-free case, thus it is convenient to introduce new coordinates

$$\alpha = \frac{1}{\sqrt{2}}\left(D_1 + D_2\right), \quad \beta = \frac{1}{\sqrt{2}}\left(D_1 - D_2\right), \quad \eta = \varphi - \psi, \quad \gamma = \varphi + \psi.$$

The kinetic energy becomes then

$$T = \frac{\mu}{2}\left[\alpha^2\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 + \beta^2\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)^2\right].$$

We consider doubly-isotropic models in which the potential energy given by

$$V(\varphi,\psi,\alpha,\beta) = \frac{V_\eta\left(\varphi-\psi\right)}{\alpha^2} + \frac{V_\gamma\left(\varphi+\psi\right)}{\beta^2} + V_\alpha(\alpha) + V_\beta(\beta)$$

does not depend on variables φ, ψ (equivalently η, γ). Performing the Legendre transformation we obtain the corresponding Hamiltonian as follows

$$H = \frac{1}{2\mu}\left(\frac{(p_\varphi - p_\psi)^2}{4\alpha^2} + {p_\alpha}^2\right) + \frac{1}{2\mu}\left(\frac{(p_\varphi + p_\psi)^2}{4\beta^2} + {p_\beta}^2\right) + V_\alpha(\alpha) + V_\beta(\beta)$$

where p_φ, p_ψ, p_α, p_β are the canonical momenta conjugate to φ, ψ, α, β.
It is convenient to use also other generalized coordinates in the affine kinematics. The most natural of them are just the variables α, β introduced above: they are obtained from D_1, D_2 by the rotation by $\pi/4$ in $\mathbb{R}^2$. The most natural of them are polar variables in the $\mathbb{R}^2$-plane of the pairs (α, β). In certain problems it is analytically convenient to use the modified "polar" variables r, ϑ given by

$$\alpha = \sqrt{r}\cos\frac{\vartheta}{2}, \qquad \beta = \sqrt{r}\sin\frac{\vartheta}{2}\cdot$$

Obviously, the "literal" polar variables ρ, ϵ are defined by

$$\alpha = \rho\cos\epsilon, \qquad \beta = \rho\sin\epsilon, \qquad \rho = \sqrt{r}, \qquad \epsilon = \frac{\vartheta}{2}\cdot$$

The natural metric on the manifold of 2×2 matrices,

$$\mathrm{d}s^2 = \mathrm{Tr}\left(\mathrm{d}\phi^T \mathrm{d}\phi\right)$$

becomes then

$$\begin{aligned}
\mathrm{d}s^2 &= r\cos^2\frac{\vartheta}{2}\mathrm{d}\eta^2 + r\sin^2\frac{\vartheta}{2}\mathrm{d}\gamma^2 + \frac{1}{4r}\mathrm{d}r^2 + \frac{r}{4}\mathrm{d}\vartheta^2 \\
&= \mathrm{d}\rho^2 + \rho^2\mathrm{d}\epsilon^2 + \rho^2\cos^2\epsilon\mathrm{d}\eta^2 + \rho^2\sin^2\epsilon\mathrm{d}\gamma^2 \\
&= \mathrm{d}\rho^2 + \frac{1}{4}\rho^2\mathrm{d}\vartheta^2 + \rho^2\cos^2\frac{\vartheta}{2}\,\mathrm{d}\eta^2 + \rho^2\sin^2\frac{\vartheta}{2}\mathrm{d}\gamma^2.
\end{aligned}$$

Obviously, kinetic energy is then expressed as follows

$$\begin{aligned}
T &= \frac{\mu}{2}\left(\frac{1}{4r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{r}{4}\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t}\right)^2 + r\cos^2\frac{\vartheta}{2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 + r\sin^2\frac{\vartheta}{2}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2\right) \\
&= \frac{\mu}{2}\left(\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 + \rho^2\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\right)^2 + \rho^2\cos^2\epsilon\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 + \rho^2\sin^2\epsilon\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2\right) \\
&= \frac{\mu}{2}\left(\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 + \frac{1}{4}\rho^2\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t}\right)^2 + \rho^2\cos^2\frac{\vartheta}{2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 + \rho^2\sin^2\frac{\vartheta}{2}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2\right).
\end{aligned}$$

For the completeness let us also mention about other orthogonal coordinates on the plane of deformation invariants

– elliptic variables (κ, λ)

$$\alpha = \sqrt{2}\cosh\kappa\cos\lambda, \qquad \beta = \sqrt{2}\sinh\kappa\sin\lambda$$

– parabolic variables (ξ, δ)

$$\alpha = \frac{1}{2}\left(\xi^2 - \delta^2\right), \qquad \beta = \xi\delta$$

– two-polar variables (e, f)

$$\alpha = \frac{c \sinh e}{\cosh e - \cos f}, \qquad \beta = \frac{c \sin f}{\cosh e - \cos f}$$

where c is a constant.

For our analysis of the deformative motion the parabolic and two-polar variables are non-useful, because the corresponding Hamilton-Jacobi equations are non-separable even in the non-physical geodetic models, i.e., ones with vanishing potentials. In the elliptic coordinates the metric underlying the kinetic energy takes on the form

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{Tr}\left(\mathrm{d}\phi^T \mathrm{d}\phi\right) &= \left(\cosh^2\kappa - \cos^2\lambda\right)\mathrm{d}\kappa^2 \\ &+ \left(\cosh^2\kappa - \cos^2\lambda\right)\mathrm{d}\lambda^2 + \cosh^2\kappa\cos^2\lambda \mathrm{d}\eta^2 + \sinh^2\kappa\sin^2\lambda\mathrm{d}\gamma^2. \end{aligned}$$

The general Stäckel-separable Hamiltonians

$$H = T + V$$

in variables $(\alpha, \beta, \eta, \gamma)$, $(r, \vartheta, \eta, \gamma)$ and $(\kappa, \lambda, \eta, \gamma)$ have the form, respectively

$$\begin{aligned} H &= \frac{1}{2\mu}\left(\left({p_\alpha}^2 + \frac{{p_\eta}^2}{\alpha^2}\right) + \left({p_\beta}^2 + \frac{{p_\gamma}^2}{\beta^2}\right)\right) \\ &\quad + V_\alpha(\alpha) + V_\beta(\beta) + \frac{V_\eta(\eta)}{\alpha^2} + \frac{V_\gamma(\gamma)}{\beta^2} \\ H &= \frac{1}{2\mu}\left(4r{p_r}^2 + \frac{1}{r}\left(\frac{{p_\varphi}^2 + {p_\psi}^2 + 2p_\varphi p_\psi \cos\vartheta}{\sin^2\vartheta} + 4{p_\vartheta}^2\right)\right) \\ &\quad + V_r(r) + \frac{V_\vartheta(\vartheta)}{r} + \frac{V_\eta(\eta)}{r\cos^2\frac{\vartheta}{2}} + \frac{V_\gamma(\gamma)}{r\sin^2\frac{\vartheta}{2}} \\ H &= \frac{1}{4\mu}\left(\frac{{p_\kappa}^2}{\left(\cosh^2\kappa - \cos^2\lambda\right)} + \frac{{p_\lambda}^2}{\left(\cosh^2\kappa - \cos^2\lambda\right)}\right. \\ &\quad \left. + \frac{{p_\eta}^2}{\cosh^2\kappa\cos^2\lambda} + \frac{{p_\gamma}^2}{\sinh^2\kappa\sin^2\lambda}\right) \\ &\quad + \frac{V_\kappa(\kappa)}{2\left(\cosh^2\kappa - \cos^2\lambda\right)} + \frac{V_\lambda(\lambda)}{2\left(\cosh^2\kappa - \cos^2\lambda\right)} \\ &\quad + \frac{V_\eta(\eta)}{2\cosh^2\kappa\cos^2\lambda} + \frac{V_\gamma(\gamma)}{2\sinh^2\kappa\sin^2\lambda}. \end{aligned} \tag{2}$$

We have quoted the general Stäckel-separable Hamiltonian in variables $(r, \vartheta, \varphi, \psi)$ (2). It is doubly-isotropic when the shape functions V_η, V_γ are put as constants. The corresponding terms $V_\eta/\cos^2(\vartheta/2)$, $V_\gamma/\sin^2(\vartheta/2)$ may be simply included into

$V_\vartheta(\vartheta)$. We have the following four constants of motion in involution, responsible for separability

– p_φ, p_ψ, i.e., equivalently p_η, p_γ

– $h_\vartheta = \frac{1}{2\mu}\frac{1}{\sin^2\vartheta}\left({p_\varphi}^2 + {p_\psi}^2 + 2p_\varphi p_\psi \cos\vartheta\right) + \frac{2}{\mu}{p_\vartheta}^2 + V_\vartheta(\vartheta)$

– $H = T + V = H_r + \frac{h_\vartheta}{r}$ where, however, the two indicated terms in H, namely

$$H_r = \frac{2}{\mu} r {p_r}^2 + V_r(r) \quad \text{and} \quad \frac{h_\vartheta}{r}$$

are not constants of motion when taken separately. The term V_r stabilizes the radial mode of motion which without this term would be unbounded, therefore physically non-applicable in elastic problems. The term V_ϑ is responsible for the shear dynamics. Particularly interesting is the following simple model

$$V = V_r(r) + \frac{V_\vartheta(\vartheta)}{r} = \frac{C}{2}r + \frac{2C}{r\cos\vartheta} = C\left(\frac{1}{D_1 D_2} + \frac{{D_1}^2 + {D_2}^2}{2}\right). \tag{3}$$

The model is perhaps phenomenological and academic, however, from the "elastic" point of view it has very physical properties: it prevents the collapse to the point or straight-line, because the term $1/D_1D_2$ is singularly repulsive there, and at the same time it prevents the unlimited expansion, because the harmonic oscillatory term $C({D_1}^2 + {D_2}^2)/2 = C(\alpha^2 + \beta^2)/2$ grows infinitely then. There is a stable continuum of relative equilibria at the non-deformed configurations when $D_1 = D_2 = 1$. Expansion along some axis results in contraction along the perpendicular axis, because

$$\frac{\partial^2 V}{\partial D_1 \partial D_2} > 0$$

at $D_1 = D_2 = 1$. This qualitatively physical potential of nonlinear hyperelastic vibrations is separable, therefore, at the same time it is also in principle analytically treatable.

In the chapter below we begin with some problems concerning the harmonic oscillator potential,

$$V(\alpha,\beta) \;=\; \frac{C}{2}\left(\alpha^2 + \beta^2\right) = \frac{C}{2}\left({D_1}^2 + {D_2}^2\right) = \frac{C}{2}\mathrm{Tr}\left(\phi^T\phi\right), \quad C > 0 \tag{4}$$

and then discuss some natural anharmonic modifications.

The stationary Hamilton-Jacobi equation has the following form [9]

$$\left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2}\right)\left(\left(\frac{\partial S}{\partial\varphi}\right)^2 + \left(\frac{\partial S}{\partial\psi}\right)^2\right) + \left(\frac{1}{2\beta^2} - \frac{1}{2\alpha^2}\right)\frac{\partial^2 S}{\partial\varphi\partial\psi} \tag{5}$$

$$+\left(\frac{\partial S}{\partial \alpha}\right)^2+\left(\frac{\partial S}{\partial \beta}\right)^2=2\mu\left(E-\left(V_\alpha(\alpha)+V_\beta(\beta)\right)\right)$$

where E is a fixed value of the energy. Due to the fact that the variables φ, ψ have the cyclic character, we may write

$$S=S_\varphi(\varphi)+S_\psi(\psi)+S_\alpha(\alpha)+S_\beta(\beta)=a\varphi+b\psi+S_\alpha(\alpha)+S_\beta(\beta).$$

Then, the action variables are as follows

$$J_\varphi=\oint p_\varphi \mathrm{d}\varphi=2\pi a, \quad J_\alpha=\pm\oint\sqrt{2\mu\left(E_\alpha-V_\alpha(\alpha)\right)-\frac{\left(J_\varphi-J_\psi\right)^2}{16\pi^2\alpha^2}}\mathrm{d}\alpha$$

$$J_\psi=\oint p_\psi \mathrm{d}\psi=2\pi b, \quad J_\beta=\pm\oint\sqrt{2\mu\left(E_\beta-V_\beta(\beta)\right)-\frac{\left(J_\varphi+J_\psi\right)^2}{16\pi^2\beta^2}}\,\mathrm{d}\beta$$

where E_α, E_β, a, b are separation constants.

3. Harmonic Oscillator and Certain Anharmonic Modifications

Let us consider the model of the harmonic oscillator potential (4) [9]. This potential describes only the attractive forces which prevent the unlimited expansion of the body. It does not prevent the collapse, i.e., the contraction to the null-dimensional singularity. It attracts to the configuration $D_1=D_2=0$ instead than to the non-deformed state $D_1=D_2=1$.

Here we obtain the dependence of the energy on the action variables as follows

$$E=\frac{\omega}{4\pi}\left[4J+|J_\varphi-J_\psi|+|J_\varphi+J_\psi|\right], \qquad J=J_\alpha+J_\beta \tag{6}$$

where $\omega=\sqrt{C/\mu}$ and

$$\begin{aligned} E_\alpha &= \frac{\omega}{4\pi}\left(4J_\alpha+|J_\varphi-J_\psi|\right)\\ E_\beta &= \frac{\omega}{4\pi}\left(4J_\beta+|J_\varphi+J_\psi|\right). \end{aligned}$$

On the purely classical level of the action variables we have the following formulas

i) in the phase space region where $|J_\varphi|>|J_\psi|$

$$E=\frac{\omega}{2\pi}\left(2J\pm J_\varphi\right)=\frac{\omega}{2\pi}\left(2J_\alpha+2J_\beta\pm J_\varphi\right) \tag{7}$$

ii) in the region where $|J_\varphi|<|J_\psi|$

$$E=\frac{\omega}{2\pi}\left(2J\pm J_\psi\right)=\frac{\omega}{2\pi}\left(2J_\alpha+2J_\beta\pm J_\psi\right) \tag{8}$$

iii) on the submanifold where $J_\varphi=J_\psi$

$$E=\frac{\omega}{2\pi}\left(2J\pm J_\varphi\right)=\frac{\omega}{2\pi}\left(2J\pm J_\psi\right). \tag{9}$$

The total degeneracy of the doubly invariant model with the potential (4) is a priori obvious. If we use coordinates $(D_1, D_2, \varphi, \psi)$, or equivalently $(\alpha, \beta, \varphi, \psi)$, then the total degeneracy is visualized by the fact that the action variables J_α, J_β, J_φ, J_ψ enter (7) with integer coefficients, J_ψ with the vanishing one. Similarly in (8) they are also combined with integer coefficients, but now the coefficient at J_φ does vanish. The third case (9) is, so-to-speak, the seven-dimensional "separatrice" submanifold.

Then performing the Bohr-Sommerfeld quantization procedure, i.e., supposing that $J = nh,\ J_\varphi = mh,\ J_\psi = lh$, where h is the Planck constant and $n = 0, 1, \ldots$; $m, l = 0, \pm 1, \ldots$, we obtain the energy spectrum in the following form

$$E = \frac{1}{2}\hbar\omega\left[4n + |m - l| + |m + l|\right]. \tag{10}$$

We may rewrite this formula as follows

i) if $|m| > |l|$, then $m^2 > l^2$ and

$$E = \hbar\omega\,(2n \pm m) \tag{11}$$

ii) if $|m| < |l|$, then $m^2 < l^2$ and

$$E = \hbar\omega\,(2n \pm l) \tag{12}$$

iii) if $|m| = |l|$, then $m^2 = l^2$ and

$$E = \hbar\omega\,(2n \pm m) = \hbar\omega\,(2n \pm l)\,. \tag{13}$$

The quasiclassical degeneracy of the Bohr-Sommerfeld energy levels is due to the fact that the quantum numbers may be combined in a single one, although in slightly different ways in three possible ranges. Let us observe that in (11) the quantum number l still does exist although does not explicitly occur in the formula for E. It runs the range $|l| < |m|$ and labels quasiclassical states within the same energy levels. And analogously in the remaining cases (12), (13).

Let us now consider the anharmonic modifications of the harmonic model of affine vibrations (4). They are based on the use of variables $(\alpha, \beta, \varphi, \psi)$ or $(\rho, \vartheta, \varphi, \psi)$. The corresponding potentials are given by

$$V(\alpha, \beta) = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right) + \frac{C}{2}\beta^2 = \frac{C}{2}\left(\alpha^2 + \beta^2\right) + \frac{2C}{\alpha^2} \tag{14}$$

$$V(\rho, \vartheta) = \frac{C}{2}\left(\rho^2 + \frac{4}{\rho^2}\right) + \frac{2C}{\rho^2}\tan^2\frac{\vartheta}{2} = \frac{C}{2}\rho^2 + \frac{2C}{\rho^2}\frac{1}{\cos^2\frac{\vartheta}{2}} \tag{15}$$

where C is some positive constant.

Using the former symbols we have

$$V_\alpha = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right), \quad V_\beta = \frac{C}{2}\beta^2, \quad V_r = \frac{C}{2}r, \quad V_\vartheta = \frac{2C}{\cos^2\frac{\vartheta}{2}}.$$

An important peculiarity of these models is that they have the stable equilibria in the natural configuration $D_1 = D_2 = 1$, so they are viable from the elastic point of view. And both of them are separable ((14) in the obvious additive sense), therefore, the corresponding Hamiltonian systems are integrable.

For the model (14) we obtains

$$E = \frac{\omega}{4\pi}\left(4(J_\alpha + J_\beta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right).$$

It is seen that the collapse-preventing term C/α^2 in V_α partially removes the degeneracy. Evidently, there is no longer resonance between φ and ψ. The resonance between α and β obviously survives; their conjugate actions J_α, J_β enter the energy formula through the rational combination $J = J_\alpha + J_\beta$ and the corresponding frequencies are equal

$$\nu_\alpha = \nu_\beta = \frac{\omega}{\pi}\cdot$$

We use the standard formulas

$$\nu_\alpha = \frac{\partial E}{\partial J_\alpha}, \qquad \nu_\beta = \frac{\partial E}{\partial J_\beta}, \qquad \nu_\varphi = \frac{\partial E}{\partial J_\varphi}, \qquad \nu_\psi = \frac{\partial E}{\partial J_\psi}\cdot$$

There are two phase-space regions given respectively by $J_\varphi + J_\psi > 0$ and $J_\varphi + J_\psi < 0$. In any of these regions there is a resonance between $\gamma = \varphi + \psi$ and α, β. This is seen from the formulas

$$J_\varphi = J_\eta + J_\gamma, \qquad J_\psi = -J_\eta + J_\gamma.$$

In the mentioned regions we have respectively

$$E = \frac{\omega}{4\pi}\left(4J_\alpha + 4J_\beta \pm 2J_\gamma + \sqrt{16\mu\pi^2 C + {J_\eta}^2}\right).$$

We can write the following independent resonances

$$\nu_\alpha - \nu_\beta = 0, \qquad \nu_\alpha \mp 2\nu_\gamma = 0$$

or, equivalently,

$$\nu_\alpha - \nu_\beta = 0, \qquad \nu_\beta \mp 2\nu_\gamma = 0.$$

Thus, in any of the mentioned regions, where $J_\gamma > 0$ or $J_\gamma < 0$, the system is twice degenerate and the closures of its trajectories are two-dimensional isotropic tori in the eight-dimensional phase space.

Using the primary variables φ, ψ, we have the following expressions for ν_φ, ν_ψ

$$\begin{aligned}
\nu_\varphi &= \frac{\omega}{4\pi}\left(\pm 1 + \frac{2(J_\varphi - J_\psi)}{\sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}}\right)\\
\nu_\psi &= \frac{\omega}{4\pi}\left(\pm 1 + \frac{2(J_\psi - J_\varphi)}{\sqrt{64\mu\pi^2 C + (J_\psi - J_\varphi)^2}}\right)
\end{aligned}$$

the $\pm$ signs respectively in the regions where $J_\varphi + J_\psi > 0$ or $J_\varphi + J_\psi < 0$. Then, taking into account that

$$\omega = \pi\nu_\alpha = \pi\nu_\beta = \pi\nu = \frac{\partial E}{\partial J}$$

we have the following degeneracy conditions

$$\nu_\alpha - \nu_\beta = 0, \qquad \nu_\alpha \mp 2\nu_\varphi \mp 2\nu_\psi = 0$$

respectively in the regions where $J_\alpha + J_\beta > 0$ or $J_\alpha + J_\beta < 0$. In the second equation, ν_α may be equivalently replaced by ν_β.

The corresponding Bohr-Sommerfeld spectrum is as follows

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right). \tag{16}$$

Another interesting model is (15), separable in the variables (ρ, ϑ), i.e., equivalently (r, ϑ). Then we obtain

$$\begin{aligned} E &= \frac{\omega}{4\pi}\left(4(J_r + J_\vartheta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right) \\ &= \frac{\omega}{4\pi}\left(4(2J_\rho + J_\vartheta) + |J_\varphi + J_\psi| + \sqrt{64\mu\pi^2 C + (J_\varphi - J_\psi)^2}\right). \end{aligned}$$

Again there is only a two-fold degeneracy and the system is not periodic. Trajectories are dense in two-dimensional isotropic tori. Degeneracy is described by the following pair of independent equations

$$\nu_\rho - 2\nu_\vartheta = 0, \qquad \nu_\vartheta \mp 2\nu_\varphi \mp 2\nu_\psi = 0$$

respectively in the phase-space regions where $J_\varphi + J_\psi > 0$ or $J_\varphi + J_\psi < 0$. Obviously, the second equation may be alternatively replaced by

$$\nu_\rho \mp 4\nu_\varphi \mp 4\nu_\psi = 0.$$

The corresponding quasiclassical spectrum is given by

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right)$$

where the quantum numbers n, m, l, refer respectively to the action variables J, J_φ, J_ψ, and the system is twice degenerate. Quasiclassical energy levels are labelled by two effective quantum numbers, namely, $(4n + m + l)$ and $(m - l)$, and there is also an obvious degeneracy with respect to the simultaneous change of signs of m and l.

4. Schrödinger Quantization

Now we formulate the quantized version of our model. Wave mechanics is formulated in $\mathrm{L}^2(Q,\widetilde{\mu})$ [2], the space of square-integrable functions on Q with the scalar product meant as follows

$$\langle\Psi|\Phi\rangle := \int \overline{\Psi}(q)\Phi(q)\mathrm{d}\widetilde{\mu}(q).$$

The invariant measure $\widetilde{\mu}$ is given by

$$\mathrm{d}\widetilde{\mu}(q) = \sqrt{|G(q)|}\mathrm{d}q^1\dots\mathrm{d}q^n$$

where the components of G with respect to some local coordinates $q^1,\dots,q^n$ will be denoted by G_{ij}. The determinant of the matrix $[G_{ij}]$ will be briefly denoted by the symbol $|G|$, obviously, this determinant is an analytic representation of some scalar density of weight two; the square root $\sqrt{|G|}$ is a scalar density of weight one. Operators of the covariant differentiation induced in the Levi-Civita sense by G will be denoted by ∇_i. The corresponding Laplace-Beltrami operator Δ is analytically given by

$$\Delta = G^{ij}\nabla_i\nabla_j$$

or explicitly, when acting on scalar fields

$$\Delta\Psi = \frac{1}{\sqrt{|G|}}\sum_{i,j}\frac{\partial}{\partial q^i}\left(\sqrt{|G|}G^{ij}\frac{\partial\Psi}{\partial q^j}\right)$$

where Ψ denoting a twice differentiable complex function on Q.

The operator Δ is symmetric with respect to this product, and ∇_i are skew-symmetric [16–18]. The metric G underlies the classical kinetic energy/Hamiltonian function

$$H = \frac{\mu}{2}G_{ij}(q)\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\mathrm{d}q^j}{\mathrm{d}t} + V(q) = \frac{1}{2\mu}G^{ij}(q)p_ip_j + V(q).$$

The Hamiltonian operator is as follows

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\Delta + V.$$

Denoting and ordering our coordinates q^i as $(\varphi,\psi,\alpha,\beta)$ in the Cartesian case we have

$$[G_{ij}] = \begin{bmatrix} \alpha^2+\beta^2 & \beta^2-\alpha^2 & 0 & 0 \\ \beta^2-\alpha^2 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

and

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\Delta + V(\alpha, \beta). \tag{18}$$

After some calculations we obtain

$$\begin{aligned}\Delta\Psi &= \frac{\partial^2\Psi}{\partial\alpha^2} + \frac{\partial^2\Psi}{\partial\beta^2} + \frac{1}{\alpha}\frac{\partial\Psi}{\partial\alpha} + \frac{1}{\beta}\frac{\partial\Psi}{\partial\beta} + \left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2}\right)\left(\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial\psi^2}\right) \\ &\quad + \left(\frac{1}{2\beta^2} - \frac{1}{2\alpha^2}\right)\frac{\partial^2\Psi}{\partial\varphi\partial\psi}\cdot \end{aligned} \tag{19}$$

Separable solutions of the stationary Schrödinger equation

$$\widehat{H}\Psi = E\Psi$$

have the form

$$\Psi(\varphi, \psi, \alpha, \beta) = f_\varphi(\varphi) f_\psi(\psi) f_\alpha(\alpha) f_\beta(\beta) \tag{20}$$

where $f_\varphi(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}$, $f_\psi(\psi) = \mathrm{e}^{\mathrm{i}l\psi}$; m, l are integers.

The stationary Schrödinger equation with an arbitrary potential $V(\alpha, \beta)$ leads after the standard separation procedure to the following system of one-dimensional eigenequations [9]

$$\frac{\mathrm{d}^2 f_\alpha(\alpha)}{\mathrm{d}\alpha^2} + \frac{1}{\alpha}\frac{\mathrm{d}f_\alpha(\alpha)}{\mathrm{d}\alpha} - \frac{(m-l)^2}{4\alpha^2} f_\alpha(\alpha) + \frac{2\mu}{\hbar^2}\left(E_\alpha - V_\alpha(\alpha)\right) f_\alpha(\alpha) = 0$$

$$\frac{\mathrm{d}^2 f_\beta(\beta)}{\mathrm{d}\beta^2} + \frac{1}{\beta}\frac{\mathrm{d}f_\beta(\beta)}{\mathrm{d}\beta} - \frac{(m+l)^2}{4\beta^2} f_\beta(\beta) + \frac{2\mu}{\hbar^2}\left(E_\beta - V_\beta(\beta)\right) f_\beta(\beta) = 0.$$

Let us now quote some formulas for quantized problems separable in polar coordinates. We assume the doubly-isotropic separable potential energy (3), i.e.,

$$V = V_r(r) + \frac{V_\vartheta(\vartheta)}{r} = V_\rho(\rho) + \frac{V_\vartheta(\vartheta)}{\rho^2}\cdot$$

The corresponding Schrödinger equation separates and, taking into account the cyclic character of angular variables φ, ψ, we put

$$\Psi(\varphi, \psi, r, \vartheta) = \mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}l\psi} f_r(r) f_\vartheta(\vartheta) = \mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}l\psi} f_\rho(\rho) f_\vartheta(\vartheta) \tag{21}$$

where m, l are integers.

Quantum integration constants responsible for this separability are given by operators

- $\widehat{p}_\varphi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi} = \widehat{S}$ – spin

- $\widehat{p}_\psi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\psi} = \widehat{V}$ – vorticity

- $\widehat{h}_\vartheta = \frac{1}{2\mu\sin^2\vartheta}\left(\widehat{p}_\varphi^2 + 2\cos\vartheta\,\widehat{p}_\varphi\widehat{p}_\psi + \widehat{p}_\psi^2\right) - \frac{4\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\frac{\partial}{\partial\vartheta}\right) + V_\vartheta$

- $\widehat{H} = \widehat{H}_r + \widehat{H}_\vartheta = \widehat{H}_r + \frac{1}{r}\widehat{h}_\vartheta = \widehat{H}_\rho + \frac{1}{\rho^2}\widehat{h}_\vartheta$ – energy

where the "radial energy" is given by

$$\widehat{H}_r = \widehat{H}_\rho = -\frac{\hbar^2}{2\mu}\left(4r\frac{\partial^2}{\partial r^2} + 8\frac{\partial}{\partial r}\right) + V_r(r) = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho}\frac{\partial}{\partial \rho}\right) + V_\rho(\rho).$$

The four mentioned constants of motion $\widehat{p}_\varphi$, $\widehat{p}_\psi$, $\widehat{h}_\vartheta$, $\widehat{H}$ are pairwise commuting and therefore they represent co-measurable physical quantities.

The stationary Schrödinger equation for the factorized wave function (21) reduces to the following pair of one-dimensional eigenequations

$$\frac{\mathrm{d}^2 f_\vartheta}{\mathrm{d}\vartheta^2} + \cot\vartheta\frac{\mathrm{d}f_\vartheta}{\mathrm{d}\vartheta} - \left(\frac{m^2 + 2ml\cos\vartheta + l^2}{4\sin^2\vartheta} + \frac{\mu}{2\hbar^2}(V_\vartheta - e_\vartheta)\right) f_\vartheta = 0 \qquad (22)$$

$$4r\frac{\mathrm{d}^2 f_r}{\mathrm{d}r^2} + 8\frac{\mathrm{d}f_r}{\mathrm{d}r} + \frac{2\mu}{\hbar^2}\left(E - \left(V_r + \frac{e_\vartheta}{r}\right)\right) f_r = 0 \qquad (23)$$

where m, l are integers. Let us now divide by 4 the nominator and denominator in the bracket expression (22) and formally admit half-integer coefficients. We can rewrite our equations as follows

$$\frac{\mathrm{d}^2 f_\vartheta}{\mathrm{d}\vartheta^2} + \cot\vartheta\frac{\mathrm{d}f_\vartheta}{\mathrm{d}\vartheta} - \left(\frac{m^2 + 2ml\cos\vartheta + l^2}{\sin^2\vartheta} + \frac{\mu}{2\hbar^2}(V_\vartheta - e_\vartheta)\right) f_\vartheta = 0 \qquad (24)$$

$$\frac{\mathrm{d}^2 f_\rho}{\mathrm{d}\rho^2} + \frac{3}{\rho}\frac{\mathrm{d}f_\rho}{\mathrm{d}\rho} + \frac{2\mu}{\hbar^2}\left(E - \left(V_\rho + \frac{e_\vartheta}{\rho^2}\right)\right) f_\rho = 0 \qquad (25)$$

where now the numbers m, l are assumed to run over the set of non-negative integers and half-integers, i.e., $m, l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

5. Quantized Harmonic Oscillator and Certain Anharmonic Modifications

Let us consider the model of the harmonic oscillator potential (4) [9]. Applying the Sommerfeld polynomial method [4, 6–11] we obtain the energy levels as follows

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m - l| + |m + l|\right) \qquad (26)$$

where

$$E_\alpha = \frac{\hbar\omega}{2}\left(4n_\alpha + 2 + |m - l|\right), \qquad E_\beta = \frac{\hbar\omega}{2}\left(4n_\beta + 2 + |m + l|\right) \qquad (27)$$

and $\omega = \sqrt{C/\mu}$, $n = n_\alpha + n_\beta$, $n = 0, 1, \ldots$, $m, l = 0, \pm 1, \ldots$. We may write

i) if $|m| > |l|$, then $m^2 > l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm m\right)$$

ii) if $|m| < |l|$, then $m^2 < l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm l\right)$$

iii) if $|m| = |l|$, then $m^2 = l^2$ and

$$E = \hbar\omega\left(2n + 2 \pm m\right) = \hbar\omega\left(2n + 2 \pm l\right).$$

The deformative wave functions $f_\alpha(\alpha)$ and $f_\beta(\beta)$ are as follows

$$f_\alpha(\alpha) = \alpha^\sigma \kappa^{\frac{1}{4}+\frac{\sigma}{2}} \mathrm{e}^{-\frac{\kappa}{2}\alpha^2} F_2\left(-n_\alpha; 1+\sigma; \kappa\alpha^2\right)$$

$$f_\beta(\beta) = \beta^\gamma \kappa^{\frac{1}{4}+\frac{\gamma}{2}} \mathrm{e}^{-\frac{\kappa}{2}\beta^2} F_2\left(-n_\beta; 1+\gamma; \kappa\beta^2\right)$$

where $\sigma = \frac{1}{2}|m-l|$ $\kappa = \sqrt{C\mu/\hbar^2}$, $\gamma = \frac{1}{2}|m+l|$.

The constant term 4 occurring in the rigorous quantum formula (26) and absent in the quasiclassical one (10) resembles the difference between Schrödinger and Bohr-Sommerfeld-quantized harmonic oscillators. This is an essentially quantum effect.

On the classical and quasiclassical level we discussed the potential (14), i.e.,

$$V(\alpha,\beta) = \frac{C}{2}\left(\alpha^2 + \frac{4}{\alpha^2}\right) + \frac{C}{2}\beta^2.$$

The model may be rigorously solved on the quantum level and we obtain the following formula for the energy levels

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right). \tag{28}$$

The energy in (28) depends on an integer combination of the quantum numbers, i.e., $n = n_\alpha + n_\beta$. The wave functions are as follows

$$f_\alpha(\alpha) = \alpha^\chi \kappa^{\frac{1}{4}+\frac{\chi}{2}} \mathrm{e}^{-\frac{\kappa}{2}\alpha^2} F_2\left(-n_\alpha; 1+\chi; \kappa\alpha^2\right)$$

$$f_\beta(\beta) = \beta^\gamma \kappa^{\frac{1}{4}+\frac{\gamma}{2}} \mathrm{e}^{-\frac{\kappa}{2}\beta^2} F_2\left(-n_\beta; 1+\gamma; \kappa\beta^2\right)$$

where

$$\chi = \frac{1}{2}\sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\cdot$$

It is seen that the formula for the energy levels is structurally "almost" identical with the quasiclassical one (16), i.e.,

$$E = \frac{1}{2}\hbar\omega\left(4n + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right)\cdot$$

This is rather typical for systems invariant under "large" symmetry groups and based on interesting geometric structures. There is a characteristic shift of energy levels, corresponding to the "null vibrations" of the harmonic part of the system.

Just like on the classical and quasiclassical levels, the system is two-fold degenerate and its energy levels are essentially controlled by two effective quantum numbers: $n_\alpha + n_\beta + |m+l|$ and $|m-l|$.

Using the formulas (22), (23), i.e., (24), (25), we can also quantize the model (15), i.e.,

$$V(r,\vartheta) = \frac{C}{2}\left(r + \frac{4}{r}\right) + \frac{2C}{r}\tan^2\frac{\vartheta}{2}\cdot$$

Then, we obtain the expression for the energy levels as follows

$$E = \frac{1}{2}\hbar\omega\left(4n + 4 + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right)$$

where $n = n_r + n_\vartheta$. The functions $f_r(r)$, $f_\vartheta(\vartheta)$ have the form

$$f_r(r) = r^{-\frac{1}{2}+\varepsilon}\kappa^{\frac{1}{2}+\varepsilon}\mathrm{e}^{-\frac{\kappa}{2}r}F_2\left(-n_r; 1+2\varepsilon; \kappa r\right)$$

$$f_\vartheta(\vartheta) = \left(\cos\frac{\vartheta}{2}\right)^\chi\left(\sin\frac{\vartheta}{2}\right)^\gamma F_1\left(-n_\vartheta, 1 + n_\vartheta + \gamma + \chi; 1 + \chi; \cos^2\frac{\vartheta}{2}\right)$$

where

$$\varepsilon = \frac{1}{2}\sqrt{1 + \frac{2\mu}{\hbar^2}e_\vartheta + \frac{2C\mu}{\hbar^2}}$$

$$e_\vartheta = \frac{\hbar^2}{8\mu}\left(\left(4n_\vartheta + 2 + |m+l| + \sqrt{(m-l)^2 + \frac{16C\mu}{\hbar^2}}\right)^2 - 4 - \frac{16C\mu}{\hbar^2}\right).$$

Rigorous solutions for two-dimensional problems may be useful in microscopic physical problems (vibrations of planar molecules like S_8, C_6H_6) and in macroscopic elasticity (cylinders with homogeneously-deformable cross-sections).

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