Eleventh International Conference on
Geometry, Integrability and Quantization
June 5–10, 2009, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2010, pp 119–126

LINEAR CONNECTION INTERPRETATION OF EXTENDED ELECTRODYNAMICS*

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Abstract. In this paper we give a presentation of the basic vacuum relations of Extended Electrodynamics in terms of linear connections.

1. Linear Connections

Linear connections are first-order differential operators in vector bundles. If such a connection ∇ is given and σ is a section of the bundle, then $\nabla\sigma$ is one-form on the base space valued in the space of sections of the vector bundle, so if X is a vector field on the base space then $i(X)\nabla\sigma = \nabla_X\sigma$ is a new section of the same bundle [2]. If f is a smooth function on the base space then $\nabla(f\sigma) = \mathrm{d}f \otimes \sigma + f\nabla\sigma$, which justifies the differential operator nature of ∇: the components of σ are differentiated and the basis vectors in the bundle space are linearly transformed.

Let e_a and ε^b, $a, b = 1, 2, \ldots, r$ be two dual local bases of the corresponding spaces of sections $< \varepsilon^b, e_a >= \delta^b_a$, then we can write

$$\sigma = \sigma^a e_a, \qquad \nabla = \mathbf{d} \otimes \mathrm{id} + \Gamma^b_{\mu a}\mathrm{d}x^\mu \otimes (\varepsilon^a \otimes e_b), \qquad \nabla(e_a) = \Gamma^b_{\mu a}\mathrm{d}x^\mu \otimes e_b$$

and therefore

$$\nabla(\sigma^m e_m) = \mathbf{d}\sigma^m \otimes e_m + \sigma^m \Gamma^b_{\mu a}\mathrm{d}x^\mu < \varepsilon^a, e_m > \otimes\, e_b = \left[\mathbf{d}\sigma^b + \sigma^a\Gamma^b_{\mu a}\mathrm{d}x^\mu\right] \otimes e_b$$

where $\Gamma^b_{\mu a}$ are the components of ∇ with respect to the coordinates $\{x^\mu\}$ on the base space and with respect to the bases $\{e_a\}$ and $\{\varepsilon^b\}$.

Since the elements $(\varepsilon^a \otimes e_b)$ define a basis of the space of (local) linear maps of the local sections, it becomes clear that in order to define locally a linear connection it

*Reprinted from JGSP **18** (2010) 13–22.

is sufficient to specify some one-form θ on the base space and a linear map $\phi = \phi^b_a \varepsilon^a \otimes e_b$ in the space of sections. Then

$$\nabla(\sigma) = \mathbf{d}\sigma^a \otimes e_a + \theta \otimes \phi(\sigma)$$

defines a linear connection with components $\Gamma^b_{\mu a} = \theta_\mu \phi^b_a$ in these bases. So, locally, any linear connection ∇ can be written as

$$\nabla = \mathbf{d} \otimes (\varepsilon^a \otimes e_a) + \Psi^b_{\mu a} \mathrm{d}x^\mu \otimes (\varepsilon^a \otimes e_b).$$

Let Ψ_1 and Ψ_2 be two one-forms valued in the space of linear maps in a vector bundle. A map $(\Psi_1, \Psi_2) \to (\wedge, \circledcirc)(\Psi_1, \Psi_2)$ is defined by (we shall write just $\circledcirc$ for $(\wedge, \circledcirc)$ and the usual $\circ$ will mean just composition)

$$\begin{aligned}
\circledcirc(\Psi_1, \Psi_2) &= (\Psi_1)^b_{\mu a} (\Psi_2)^n_{\nu m} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \left[\circ\, (\varepsilon^a \otimes e_b, \varepsilon^m \otimes e_n)\right] \\
&= (\Psi_1)^b_{\mu a} (\Psi_2)^n_{\nu m} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \left[< \varepsilon^a, e_n > (\varepsilon^m \otimes e_b)\right] \\
&= (\Psi_1)^b_{\mu a} (\Psi_2)^a_{\nu m} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes (\varepsilon^m \otimes e_b) \quad \mu < \nu.
\end{aligned}$$

In the case of trivial vector bundles, the curvature of ∇ is given by [2]

$$\left[\mathbf{d}(\Psi^b_{\mu a} \mathrm{d}x^\mu)\right] \otimes (\varepsilon^a \otimes e_a) + \circledcirc(\Psi, \Psi).$$

2. Some Facts From the Clasical and the Extended Electrodynamics

We recall now some facts from the **Classical Electrodynamics** (CED) and from the **Extended Electrodynamics** (EED) [1]. The vector bundle under consideration is the (trivial) bundle $\Lambda^2(M)$ of two-forms on the Minkowski space-time M. Recall that if $(F, *F)$ is a CED **vacuum solution**, i.e., $\mathbf{d}F = 0, \ \ \mathbf{d} * F = 0$, then the combinations

$$\mathcal{F} = a\,F - b * F, \qquad \mathcal{F}^* = b\,F + a * F$$

where (a, b) are two arbitrary real numbers, also give a CED vacuum solution and, since on Minkowski space the corresponding **Hodge star** $*_2$ satisfies the relation $*^2 = -\,\mathrm{id}_{\Lambda^2(M)}$, we obtain $\mathcal{F}^* = *\mathcal{F}$. The two corresponding energy tensors are related by

$$T(\mathcal{F}, \mathcal{F}^*) = (a^2 + b^2)\, T(F, *F).$$

Recall the real representation of complex numbers $z = aI + bJ$ where I is the unit matrix in $\mathbb{R}^2$ and J is the standard complex structure matrix in $\mathbb{R}^2$ with columns $(0, -1)^T, (1, 0)^T$. So, we obtain an action of the linear group G of matrices $\alpha = aI + bJ$ on the CED vacuum solutions. This is a commutative group G and its Lie algebra $\mathcal{G}$ just adds the zero (2×2) matrix to G, and (I, J) define a natural basis of $\mathcal{G}$. So, having a CED vacuum solution, we have in fact a two-parameter family of vacuum solutions. Hence, we can define a $\mathcal{G}$-valued two-form Ω on M by $\Omega = F \otimes I + *F \otimes J$, and the equation $\mathbf{d}\Omega = 0$ is equivalent to $\mathbf{d}F = 0, \ \mathbf{d}{*F} = 0$.

Consider the new basis (I', J') in $\mathcal{G}$ given by

$$I' = aI + bJ, \qquad J' = -bI + aJ.$$

Accordingly, the "new" solution Ω', i.e., the old solution in the new basis of $\mathcal{G}$, will be

$$\begin{aligned}\Omega' &= F\otimes I' + *F\otimes J' = F\otimes(aI+bJ) + *F\otimes(-bI+aJ)\\ &= (a\,F - b\,*F)\otimes I + (b\,F + a\,*F)\otimes J\mathcal{F}\otimes I + \mathcal{F}^*\otimes J.\end{aligned}$$

In view of this we may consider this transformation as *nonessential*, i.e., we may consider $(F, *F)$ and $(\mathcal{F}, \mathcal{F}^*)$ as two different representations in corresponding bases of $\mathcal{G}$ of the same solution.

Such an interpretation is approporiate and useful if the field shows some *invariant properties* with respect to this class of transformations. For example, if the Lorentz invariants

$$I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\mathbf{B}^2 - \mathbf{E}^2), \qquad I_2 = \frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu} = 2\mathbf{E}.\mathbf{B}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the corresponding electric and magnetic components of F, are zero: $I_1 = I_2 = 0$, (the so called "null field case") then all the above transformations keep unchanged these zero-values of I_1 and I_2. In fact, under such a transformation $(F, *F) \rightarrow (\mathcal{F}, \mathcal{F}^*)$ the two Lorentz invariants transform to (I_1', I_2') in the following way

$$I_1' = (a^2 - b^2)\,I_1 + 2ab\,I_2, \qquad I_2' = -2ab\,I_1 + (a^2 - b^2)\,I_2$$

and the determinant of this transformation is $(a^2 + b^2)^2 \neq 0$. So, a null field, i.e., a field with zero invariants I_1 and I_2, stays a null field under these transformations. Moreover, NO non-null field can be transformed to a null field by means of these transformations, and, conversely, NO null field can be transformed to a non-null field in this way. Hence, the Lorentz invariance and the dual G-invariance of I_1 and I_2 hold simultanoiusly only in the null-field case. Further we are going to pay due respect to this invariance, keeping in mind the basic fact that only in this case the velocity of the energy propagation of the field is equal to "c" and follows straight lines, so this is intrinsic property of the field.

In order to come to the equations of EED we can recall that every bilinear map $\varphi : \mathcal{G}\times\mathcal{G} \rightarrow W$, where W is some linear space with basis $\{e_i\}, i = 1, 2, \ldots,$ defines corresponding product in the $\mathcal{G}$-valued differential forms by means of the relation

$$\varphi(\Omega_1^i\otimes e_i, \Omega_2^j\otimes e_j) = \Omega_1^i\wedge\Omega_2^j\otimes\varphi(e_i, e_j).$$

Recall now the following identity in Minkowski space

$$I_1\delta_\mu^\nu \equiv \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\delta_\mu^\nu = F_{\sigma\mu}F^{\sigma\nu} - (*F)_{\sigma\mu}(*F)^{\sigma\nu}$$

and the standard energy-tensor Q_μ^ν of electromagnetic field

$$Q_\mu^\nu = -\frac{1}{2}\Big[F_{\sigma\mu}F^{\sigma\nu} + (*F)_{\sigma\mu}(*F)^{\sigma\nu}\Big].$$

We see that under $I_1 = 0$ the two fields F and $*F$ carry the same energy-momentum during propagation. Moreover, there is NO interaction stress-energy-momentum between F and $*F$ as it is seen from the expression for Q_μ^ν.

Corollary 1. *The two fields F and $*F$ may interact only in regime of dynamical equilibrium, i.e., any energy-momentum loss of $F/*F$ should be compensated by equal gain of $*F/F$: $F \rightleftarrows *F$.*

EED makes use of these facts assuming that the corresponding dynamical equations must have local energy-momentum exchange physical sense, so the symmetry $F \rightleftarrows *F$ must be respected.

Now, let $\varphi = \vee$, where "$\vee$" is the symmetrized tensor product in $\mathcal{G}$. We consider the expression $\vee(\Omega, *\,\mathbf{d}\Omega)$.

$$\begin{aligned}\vee(\Omega, *\mathbf{d}\Omega) &= (F \wedge *\mathbf{d}F) \otimes I \vee I(*F \wedge *\mathbf{d}*F) \otimes J \vee J \\ &\quad + (F \wedge *\mathbf{d}*F \otimes + *F \wedge *\mathbf{d}F) \otimes I \vee J.\end{aligned}$$

The vacuum EED equations are $\vee(\Omega, *\,\mathbf{d}\Omega) = 0$, or equivalently

$$F \wedge *\mathbf{d}F = 0, \qquad (*F) \wedge *\mathbf{d}*F = 0, \qquad F \wedge *\mathbf{d}*F + (*F) \wedge *\mathbf{d}F = 0.$$

In terms of the codifferential $\delta = *\mathbf{d}*$ these equations look like

$$\delta * F \wedge F = 0, \qquad \delta F \wedge *F = 0, \qquad \delta F \wedge F - \delta * F \wedge *F = 0.$$

Correspondingly in components we obtain

$$\begin{aligned}\frac{1}{2}F^{\alpha\beta}(\mathbf{d}F)_{\alpha\beta\mu} &\equiv (*F)_{\mu\nu}(\delta * F)^\nu = 0 \\ \frac{1}{2}(*F)^{\alpha\beta}(\mathbf{d}*F)_{\alpha\beta\mu} &\equiv F_{\mu\nu}(\delta F)^\nu = 0 \\ \frac{1}{2}(*F)^{\alpha\beta}(\mathbf{d}F)_{\alpha\beta\mu} + \frac{1}{2}F^{\alpha\beta}(\mathbf{d}*F)_{\alpha\beta\mu} &\equiv (\delta * F)^\nu F_{\nu\mu} + (\delta F)^\nu(*F)_{\nu\mu} = 0.\end{aligned}$$

3. Basic Property of the Nonlinear Solutions

All nonlinear solutions to the EED vacuum equations, namely, those satisfying $\mathbf{d}F \neq 0$, $\mathbf{d}*F \neq 0$, have zero invariants: $I_1 = I_2 = 0$, so, they minimize the quantity $I_1^2 + I_2^2 \geq 0$. Moreover, for any nonlinear solution defined by F there exists a canonical coordinate system on M, called further F-adapted, in which F and $*F$ look as follows

$$F = A \wedge \zeta, \qquad *F = A^* \wedge \zeta$$

$$A = u\,\mathrm{d}x + p\,\mathrm{d}y, \qquad A^* = p\,\mathrm{d}y - u\,\mathrm{d}z, \qquad \zeta = \varepsilon\mathrm{d}z + \mathrm{d}\xi, \qquad \varepsilon \pm 1$$

and (u, p) are two functions on M, satisfying the equation

$$u\,(u_\xi - \varepsilon u_z) + p\,(p_\xi - \varepsilon p_z) = 0.$$

As for the energy-momentum tensor $T_{\mu\nu}$ of the vacuum solutions, considered as a symmetric two-form on M, it is defined in terms of Ω as follows

$$\begin{aligned} T(X,Y) &= \frac{1}{2} * g\big[i(X)\Omega, * i(Y)\Omega\big] \\ &= -\frac{1}{2} X^\mu Y^\nu \Big[F_{\mu\sigma}F_\nu{}^\sigma + (*F)_{\mu\sigma}(*F)_\nu{}^\sigma\Big] = X^\mu Y^\nu T_{\mu\nu} \end{aligned}$$

where (X, Y) are two arbitrary vector fields on M, g is the metric in $\mathcal{G}$ defined by $g(\alpha, \beta) = \frac{1}{2}\,\mathrm{tr}(\alpha.\beta^*)$, and β^* is the transposed to β. Note that $g(I, J) = 0$, which elliminates the corresponding coefficient which reads

$$F_{\mu\sigma}(*F)^{\nu\sigma} + (*F)_{\mu\sigma}F^{\nu\sigma} = \frac{1}{2}F_{\alpha\beta}(*F)^{\alpha\beta}\delta^\nu_\mu$$

so, in a g-NONorthogonal basis of $\mathcal{G}$ this coefficient will appear.

Finally, recall the generalization of Lie derivative $\mathcal{L}_K$ with respect to the k-vector K, acting in the exterior algebra of differential forms according to the formula $\mathcal{L}_K = i(K)\mathbf{d} - (-1)^k \mathbf{d}i(K)$. Then, in view of the relations
$F_{\mu\nu}F^{\mu\nu} = (*F)_{\mu\nu}F^{\mu\nu} = 0$, the above equations acquire the form

$$\mathcal{L}_{\bar F}F = 0, \qquad \mathcal{L}_{*\bar F}(*F) = 0, \qquad \mathcal{L}_{\bar F}(*F) + \mathcal{L}_{(*\bar F)}F = 0$$

where $\bar F$ and $*\bar F$ are the η-corresponding two-vectors. In terms of the two-form Ω and
$\bar\Omega = \bar F \otimes e_1 + *\bar F \otimes e_2$ these three equations can be united in one as follows

$$\mathcal{L}^{\vee}_{\bar\Omega}\Omega = \mathcal{L}_{\bar F}F \otimes e_1 \vee e_1 + \mathcal{L}_{*\bar F} * F \otimes e_2 \vee e_2 + (\mathcal{L}_{\bar F} * F + \mathcal{L}_{*\bar F}F) \otimes e_1 \vee e_2 = 0.$$

4. Linear Connection Interpretation of the Nonlinear Part of the EED

If $\mathcal{I}$ is the identity in $\Lambda^2(M)$ and $\mathcal{J} = *$ is the complex structure in $\Lambda^2(M)$ then a representation ρ of G in $\Lambda^2(M)$ is given by

$$\rho(\alpha) = a\mathcal{I} + b\mathcal{J}.$$

Also, a representation ρ' of the corresponding Lie algebra $\mathcal{G}$ is defined by the same relation. So, if $\alpha : M \to \mathcal{G}$ is a map then $\rho'(\alpha)$ is a linear map in $\Lambda^2(M)$, and recalling our one-form $\zeta = \varepsilon\mathrm{d}z + \mathrm{d}\xi$ we define a linear connection ∇ in $\Lambda^2(M)$ by

$$\begin{aligned} \nabla &= \mathbf{d} \otimes \mathrm{id}_{\Lambda^2(M)} + \zeta \otimes \rho'(\alpha(u,p)) \\ &= \mathbf{d} \otimes \mathrm{id}_{\Lambda^2(M)} + \zeta \otimes (u\mathcal{I} + p\mathcal{J}), \qquad \alpha \in \mathcal{G}. \end{aligned}$$

Two other connections $\bar{\nabla}$ and ∇^* are defined by

$$\begin{aligned} \rho'(\bar{\alpha}(u,p)) &= \rho'(\alpha(u,-p)) = u\mathcal{I} - p\mathcal{J} \\ (\rho')^*(\alpha(u,p)) &= \rho'(\alpha.J) = \rho'(\alpha(-p,u)) = -p\mathcal{I} + u\mathcal{J} \end{aligned}$$

and we introduce for further use

$$\chi = u\mathcal{I} + p\mathcal{J}, \qquad \bar{\chi} = u\mathcal{I} - p\mathcal{J}, \qquad \chi^* = -p\mathcal{I} + u\mathcal{J}.$$

Denoting

$$\Psi = \zeta \otimes \chi, \qquad \bar{\Psi} = \zeta \otimes \bar{\chi}, \qquad \Psi^* = \zeta \otimes \chi^*$$

we have (because $\zeta \wedge \zeta = 0$)

$$\circledcirc(\Psi, \Psi) = \circledcirc(\Psi, \bar{\Psi}) = \circledcirc(\Psi, \Psi^*) = 0.$$

Now, since

$$\begin{aligned} \Psi &= u\,\zeta \otimes \mathcal{I} + p\,\zeta \otimes \mathcal{J} \\ \bar{\Psi} &= u\,\zeta \otimes \mathcal{I} - p\,\zeta \otimes \mathcal{J} \\ \Psi^* &= -\,p\,\zeta \otimes \mathcal{I} + u\,\zeta \otimes \mathcal{J} \end{aligned}$$

we obtain for the corresponding curvatures

$$\begin{aligned} \mathcal{R} &= \mathbf{d}(u\,\zeta) \otimes \mathcal{I} + \mathbf{d}(p\,\zeta) \otimes \mathcal{J} \\ \bar{\mathcal{R}} &= \mathbf{d}(u\,\zeta) \otimes \mathcal{I} - \mathbf{d}(p\,\zeta) \otimes \mathcal{J} \\ \mathcal{R}^* &= \mathbf{d}(-p\,\zeta) \otimes \mathcal{I} + \mathbf{d}(u\,\zeta) \otimes \mathcal{J}. \end{aligned}$$

By direct calculation we obtain also

$$\begin{aligned} *\frac{1}{6}\,\mathrm{Tr}\left[\circledcirc(\bar{\Psi}, *\mathbf{d}\Psi)\right] &= -\varepsilon\big[u\,(u_\xi - \varepsilon u_z) + p\,(p_\xi - \varepsilon p_z)\big]\mathrm{d}z \\ &\quad -\big[u\,(u_\xi - \varepsilon u_z) + p\,(p_\xi - \varepsilon p_z)\big]\mathrm{d}\xi \\ \frac{1}{6}\,\mathrm{Tr}\left[\circledcirc(\Psi^*, *\mathbf{d}\Psi)\right] &= \varepsilon\Big[p\,(u_\xi - \varepsilon u_z) - u\,(p_\xi - \varepsilon p_z)\Big]\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &\quad +\Big[p\,(u_\xi - \varepsilon u_z) - u\,(p_\xi - \varepsilon p_z)\Big]\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\xi \\ &= \delta F \wedge F = \delta * F \wedge *F = \varepsilon\frac{1}{6}\,\mathrm{Tr}\left[\circledcirc(\Psi, *\mathbf{d}\Psi^*)\right] \end{aligned}$$

where δ is the coderivatie. Denoting by $|\mathcal{R}|^2$ the quantity $\frac{1}{6}|*\,\mathrm{Tr}\left[\circledcirc(\mathcal{R} \wedge *\bar{\mathcal{R}})\right]|$ we obtain (in the F-adapted coordinate system)

$$|\mathcal{R}|^2 = \frac{1}{6}|*\,\mathrm{Tr}\left[\circledcirc(\mathbf{d}\Psi, *\mathbf{d}\bar{\Psi})\right]| = (u_\xi - \varepsilon u_z)^2 + (p_\xi - \varepsilon p_z)^2 = |\delta F|^2 = |\delta * F|^2.$$

Finally we note the relations

$$\frac{1}{6}\,\mathrm{tr}(\chi) = \frac{1}{6}\,\mathrm{tr}(u\mathcal{I} + p\mathcal{J}) = u$$

and
$$\frac{1}{6}\operatorname{tr}\left[(\chi\circ\bar{\chi})\right]=\frac{1}{6}\operatorname{tr}\left[(u\mathcal{I}+p\,\mathcal{J})\circ(u\mathcal{I}-p\,\mathcal{J})\right]=u^2+p^2.$$
These relations allow to introduce two characteristic functions for any nonlinear solution: *the phase-function* ψ and *the scale factor* $\mathcal{L}$ as
$$\psi=\arccos\frac{\frac{1}{6}\operatorname{tr}\chi}{\sqrt{\frac{1}{6}\operatorname{tr}(\chi\circ\bar{\chi})}},\qquad \mathcal{L}=\frac{\sqrt{\frac{1}{6}\operatorname{tr}(\chi\circ\bar{\chi})}}{\sqrt{\frac{1}{6}}|\mathcal{R}|}=\frac{\sqrt{\operatorname{tr}(\chi\circ\bar{\chi})}}{|\mathcal{R}|}.$$
Since the nonlinear solutions of the two equations
$$\delta F\wedge *F=0,\qquad \delta * F\wedge F=0$$
i.e., those satisfyiing $\delta F\neq 0$ and $\delta * F\neq 0$, are parametrized by two functions (u,p) and satisfy the relations
$$u\,(u_\xi-\varepsilon u_z)+p\,(p_\xi-\varepsilon p_z)=0,\qquad u_\xi-\varepsilon u_z\neq 0,\qquad p_\xi-\varepsilon p_z\neq 0$$
in the corresponding F-adapted coordinate system, we obtain that on those solutions the following relation holds
$$\operatorname{Tr}\left[\circledcirc(\bar{\Psi},*\mathbf{d}\Psi)\right]=0$$
and that the equation
$$\delta F\wedge F=\delta * F\wedge *F$$
is equivalent to
$$\operatorname{Tr}\left[\circledcirc(\Psi,*\mathbf{d}\Psi^*)\right]=\varepsilon\ \operatorname{Tr}\left[\circledcirc(\Psi^*,*\mathbf{d}\Psi)\right].$$
It can easily be shown (we leave this to the reader) that the non-zero value of the squared curvature invariant $|\mathcal{R}|^2$ guarantees availability of rotational component of propagation.

5. Conclusion

The linear connection ∇ is defined through $\zeta\otimes\rho'(\alpha(u,p))$. We note that the two-form $F_o=dx\otimes\zeta$ gives the possibility to consider a nonlinear solution $F(u,p)$ as an appropriately defined linear map
$$\rho'(\alpha(u,p))=u\mathcal{I}+p\mathcal{J}$$
in $\Lambda^2(M)$, since the action of $\rho'(\alpha(u,p))$ on F_o gives F.
This special importance of ζ is based on the fact that it intrinsically defines the translational part of the dynamical befavior of the solution, and its uniqueness is determined by the fact that all nonlinear solutions of EED equations have zero invariants: $F_{\mu\nu}F^{\mu\nu}=F_{\mu\nu}(*F)^{\mu\nu}=0$. As for the rotational part of the dynamical behavior of the solution it is available only if the curvature $\mathcal{R}$ iz nonzero and is locally represented by any of the two three-forms $F\wedge\delta F=*F\wedge\delta * F\neq 0$. For

all nonlinear solutions we have $\delta F \neq 0$ and $\delta * F \neq 0$, and all finite nonlinear solutions have finite energy density

$$0 < \phi^2 = \frac{1}{6}\operatorname{tr}(F \circ \bar{F}) = \frac{1}{6}\operatorname{tr}(\chi \circ \bar{\chi}) = (u^2 + p^2) < \infty.$$

The nonzero finite scale factor $0 < \mathcal{L} < \infty$ separates those finite nonlinear solutions which carry spin momentum, and this happens only when $|\delta F| = |\mathcal{R}| \neq 0$. The spin momentum is carried by any of the two three-forms $\delta F \wedge F = \delta * F \wedge *F$, determining the energy-momentum exchange between F and $*F$. Clearly, on the linear Maxwell solutions these three-forms are zero.

Hence, in terms of curvature we can say that *the nonzero curvature invariant* $|\mathcal{R}|$, *is responsible for availability of rotational component of propagation*, in other words, *the spin properties of a nonlinear solution require non-zero curvature.*

From physical viewpoint the corresponding dynamical process that generates these spin properties is the mutual energy-momentum exchange between the two field components F and $*F$ during propagation. It has the following three characteristic properies:

- it is *permanent*, i.e., it occurs constantly during propagation
- it is *simultanious* in the both directions: $F \rightleftarrows *F$
- it is in *equal quantities.*

It follows that F and $*F$ live in a *permanent dynamical equilibrium.* They carry always the same quantities of energy-momentum

$$\left[F_{\mu\nu}F^{\mu\nu} = 0\right] \Rightarrow F_{\mu\sigma}F^{\nu\sigma} = (*F)_{\mu\sigma}(*F)^{\nu\sigma}.$$

This dynamical equilibrium is quantitatively described by the equation

$$\delta F \wedge F = \delta * F \wedge *F$$

or by

$$\operatorname{Tr}\left[\circledcirc(\Psi, *\mathbf{d}\Psi^*)\right] = \varepsilon \operatorname{Tr}\left[\circledcirc(\Psi^*, *\mathbf{d}\Psi)\right].$$

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