The Relation between Subfactors arising from Conformal Nets and the Realization of Quantum Doubles

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ABSTRACT. We give a precise definition for when a subfactor arises from a conformal net which can be motivated by classification of defects. We show that a subfactor $N \subset M$ arises from a conformal net if there is a conformal net whose representation category is the quantum double of $N \subset M$.

CONTENTS

1. INTRODUCTION

Finite index subfactors $N \subset M$ generalize finite group fixed points and can be seen as describing quantum symmetries. An important invariant of a subfactor $N \subset M$ is its index $[M : N]$. It generalizes the index of a subgroup in the sense that for group-subgroup subfactors we have $[M^H : M^G] = [G : H]$. By Jones' index theorem [Jon83] the index takes values in:

$$[M : N] \in \left\{4\cos^2\left(\frac{\pi}{m}\right) : m = 3, 4, \ldots\right\} \cup [4 : \infty]$$

and all subfactors with index less than four have finite depth. For index greater than 4 there are some known exotic subfators with finite depth and the classification has been recently pushed to $5\frac{1}{4}$ [JMS14, AMP15]. Finite depth subfactors are rather algebraic objects, but since everything is defined on a Hilbert space this algebraic structure still has important positivity properties. It is an interesting question if and how they arise describing symmetries in models of quantum physics.

Using the Haag–Kastler axioms of algebraic quantum field theory [Haa96] one can describe quantum field theory directly using nets of local von Neumann algebras. Under natural assumptions the local algebras turn out to be factors and are in many cases isomorphic to the hyperfinite type III_1 factor [Con73, Haa87]. The theory of Doplicher–Haag–Roberts superselection sectors studies the representation theory of Haag–Kaster nets in terms of so-called localized endomorphisms. Each endomorphism gives a subfactor, but in higher dimensional QFT the index is rather boring and takes values in $\{n^2 : n \in \mathbb{N}\}$, indeed all subfactors come from a representation of a compact group [DR89]. On the other hand, in low-dimensional QFT the superselection theory gets interesting. The superselection sectors have braid group statistics [FRS89] and the index is in general not a square of an integer. For example all values $\left\{4\cos^2\left(\frac{\pi}{m}\right) : m = 3, 4, \ldots\right\}$ of

Supported in part by NSF Grant DMS-1362138.

the index can be realized by a loop group model $\mathcal{A}_{\mathsf{SU}(2)_{m-2}}$ of $\mathsf{SU}(2)$ at level $m-2$ [Was98]. But these subfactors always come with a braiding and there are known subfactors which do not admit such a braiding. Now there are two ways out. First: The quantum double of a subfactor gives a braided subfactor, namely it gives a unitary modular tensor category and one can try to construct quantum field theories realizing such quantum doubles as DHR category of superselection sectors. But these does in general not give the original subfactor back (at least not directly). Second: One can look into higher structures of the quantum field theory for example one can allow boundary conditions and defects. Here one needs to consider extensions and new subfactors arise which are not braided. The goal of this note is to show that these two directions are directly related.

The experience shows that it might be enough to consider completely rational Möbius covariant nets on the circle which describes chiral conformal field theory (CFT).[1] In this framework we want to make a precise statement what is meant by the following question.

Question 1.1 (cf. [Jon14])**.** Do all subfactors come from quantum field theory?

By a subfactor we mean from now on always a finite index, finite depth subfactor which is hyperfinite of type III_1. If we have a finite index and finite depth subfactor $N \subset M$ of type II_1 we can always pass to a hyperfinite III_1 subfactor. Namely, $\tilde{N} := N \subset A \subset \tilde{M} := M \otimes A$ with A the hyperfinite type III_1 factor has the same standard invariant.

Kawahigashi, Longo and Müger [KLM01] showed that under the rather natural assumption of complete rationality of a conformal net $\mathcal{A}$, its representation category $\mathrm{Rep}(\mathcal{A})$ is a **unitary modular tensor category (UMTC)**. Unitary modular tensor categories play a prominent role in topological quantum computing, they give 3 manifold invariants and topological quantum field theories via the Reshetikhin–Turaev construction [RT90, Tur94]. The natural question arises if one can find a general solution to the following problem (cf. [Kaw15]).

Problem 1.2. Given a unitary modular tensor category $\mathcal{C}$, find a conformal net with $\mathrm{Rep}(\mathcal{A})$ braided equivalent to $\mathcal{C}$.

The modular tensor category encodes topological information about the conformal net in terms of a three dimensional topological field theory. Note that nevertheless the conformal net has more information than just its representation category, because there infinitely many non-isomorphic conformal nets sharing the same UMTC as representation category. They arise by tensoring with a holomorphic net (see below).

Nevertheless, the problem does not seem completely hopeless. It is similar to inverse scattering problems in quantum field theory. One idea is to use all or some of the data of the UMTC to construct a statistical mechanical model which in a limit at critical temperature gives a conformal field theory. This way is full of technical difficulties and we will not further comment on it.

Since the quantum double $\mathcal{D}(N \subset M)$ of a finite index finite depth subfactor $N \subset M$ is a UMTC, it seems to be natural to consider the following subproblem of Problem 1.2 (cf. [EG11, Kaw15]).

Problem 1.3. Given a finite index finite depth subfactor $N \subset M$,

(1) Find a conformal net $\mathcal{A}$, such that $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M)$.
(2) Find a conformal net $\mathcal{A}$, such that $\mathrm{Rep}(\mathcal{A}) \supset \mathcal{D}(N \subset M)$, i.e. $\mathcal{D}(N \subset M)$ is equivalent to a full subcategory of $\mathrm{Rep}(\mathcal{A})$. In this case $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M) \boxtimes \mathcal{C}$, where $\mathcal{C}$ is a modular tensor category.

[1] Probably one wants assume diffeomorphism covariance. But one the one hand for our structural results this is not necessary to assume. On the other hand, we are not aware that there is a known completely rational net which is not diffeomorphism covariant.

If we have a UMTC $\mathcal{C}$ we get a UMTC $\mathcal{C}^{\text{rev}}$ by replacing the braiding with the opposite braiding. In general $\mathcal{C}^{\text{rev}}$ is not braided equivalent to $\mathcal{C}$. Since the braiding is instrinisically defined and conformal nets have a posititvity of energy condition, there seems to be no easy way to get a net realizing the opposite braiding without destroying the positivity of energy condition. Therefore the following question naturally arises:

Question 1.4 ([Lon12])**.** Let $\mathcal{A}$ be a completely rational net. Is there a completely rational net $\tilde{\mathcal{A}}$ with $\operatorname{Rep}(\tilde{\mathcal{A}}) \cong \operatorname{Rep}(\mathcal{A})^{\text{rev}}$?

This question can be answered positively, if we solve the following problem (see Proposition 4.6).

Problem 1.5. Given a completely rational net $\mathcal{A}$, find a holomorphic net $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is normal and cofinite.

Motivated by the study of phase boundaries and topological defects, we say a subfactor $N \subset M$ arises from a conformal net $\mathcal{A}$, if there are two relatively local extensions $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$, and a sector $\beta\colon \mathcal{B}_a(I) \to \mathcal{B}_b(I)$ related to $\mathcal{A}$, such that $N \subset M \approx \beta(\mathcal{B}_a(I)) \subset \mathcal{B}_b(I)$, see Definition 4.1

Proposition 1.6. *Let $N \subset M$ be a finite index, finite depth subfactor.*

(1) If there is a conformal net with $\operatorname{Rep}(\mathcal{A})$ *braided equivalent to $\mathcal{D}(N \subset M)$, then $N \subset M$ arises from $\mathcal{A}$. Actually, it is enough that* $\operatorname{Rep}(\mathcal{A})$ *contains a full subcategory braided equivalent to $\mathcal{D}(N \subset M)$.*

(2) Conversely, if $N \subset M$ arises from $\mathcal{A}$, then there is a 2D conformal net $\mathcal{B}_2 \supset \mathcal{A} \otimes \mathcal{A}$ with $\operatorname{Rep}(\mathcal{B}_2) \cong \mathcal{D}(N \subset M)$*.*

(3) Further, if $N \subset M$ arises from $\mathcal{A}$, and there is a net $\tilde{\mathcal{A}}$, with $\operatorname{Rep}(\tilde{\mathcal{A}}) \cong \operatorname{Rep}(\mathcal{A})^{\text{rev}}$ *then there is a conformal net $\mathcal{B} \supset \mathcal{A} \otimes \tilde{\mathcal{A}}$ with* $\operatorname{Rep}(\mathcal{B})$ *braided equivalent to $\mathcal{D}(N \subset M)$.*

2. Subfactors and Unitary Fusion Categories

Let M be the hyperfinite type III_1 factor and $N \subset M$ a finite index and finite depth subfactor. We denote by $\iota\colon N \to M$ the canonical inclusion map, which is a morphism (normal $*$-homomorphism) $N \to M$. Then finite index of $N \subset M$ is equivalent to the existence of a morphism $\bar\iota\colon M \to N$, such that $\mathrm{id}_N \prec \bar\iota \circ \iota$ and $\mathrm{id}_M \prec \iota \circ \bar\iota$ cf. [Lon90]. Here we say a morphism $\rho\colon N \to M$ is contained in $\sigma\colon N \to M$, written $\rho \prec \sigma$ if and only if there is an isometry $e \in \operatorname{Hom}(\rho, \sigma) = \{t \in M : t\rho(N) = \sigma(N)t\}$.

All endomorphisms ρ of M, such that $\rho(M) \subset M$ have finite index, form a rigid C*-tensor category $\operatorname{End}_0(M)$. The arrows $t\colon \rho \to \sigma$ are given by $t \in \operatorname{Hom}(\rho, \sigma)$ as above and the tensor product is given by composition of endomorphisms. An endomorphism ρ is called irreducible if $\operatorname{Hom}(\rho, \rho) = \mathbb{C} \cdot 1$ and since $\operatorname{Hom}(\rho, \rho) = \rho(N)' \cap N$ this is equivalent with the subfactor $\rho(N) \subset N$ being irreducible. We denote by $[\rho]$ the sector of ρ which is the unitary equivalence class $\{u\rho(\,\cdot\,)u^* : u \in N \text{ unitary}\}$. There is a direct sum, well-defined on sectors, given by $\rho \oplus \sigma = v_1\rho(\,\cdot\,)v_1^* + v_2\sigma(\,\cdot\,)v_2^*$ with v_i isometries fulfilling the Cuntz algebra relations: $\sum_i v_i v_i^* = 1$ and $v_i^* v_j = \delta_{i,j}$.

A finite index subfactor $N \subset M$ with $\iota\colon N \to M$ and conjugate $\bar\iota\colon M \to N$ gives two rigid C*-tensor categories,

- the dual even part ${}_N\mathcal{F}_N^{N\subset M} = \langle \rho \prec (\bar\iota \circ \iota)^n \rangle \subset \operatorname{End}_0(N)$ and
- the even part ${}_M\mathcal{F}_M^{N\subset M} = \langle \rho \prec (\iota \circ \bar\iota)^n \rangle \subset \operatorname{End}_0(M)$.

Here the $\langle S \rangle$ denotes the full and replete tensor subcategory generated by S and closed under taking direct sums.

The subfactor $N \subset M$ is called **finite depth** if and only if the set $\operatorname{Irr}({}_A\mathcal{F}_A^{N\subset M}) = \{[\rho] : \rho \in {}_A\mathcal{F}_A^{N\subset M} \text{ irreducible}\}$ with $A = N, M$ is finite. In this case, ${}_N\mathcal{F}_N^{N\subset M}$ and ${}_M\mathcal{F}_M^{N\subset M}$ are unitary fusion categories.

The subfactor actually generates a 2-category $\mathcal{F}_{N\subset M}$ with zero objects $\{N, M\}$, by taking all finite morphisms $\beta\colon \{N, M\} \to \{N, M\}$ contained in compositions of $\{\iota, \bar\iota\}$, such that ${}_N\mathcal{F}_N^{N\subset M}$ and ${}_M\mathcal{F}_M^{N\subset M}$ sit in the $N-N$ and $M-M$ corner, respectively, of $\mathcal{F}_{N\subset M}$.

We remark that a unitary fusion category given as a full and replete subcategory ${}_M\mathcal{F}_M \subset \mathrm{End}_0(M)$ is completely fixed by its finite set of sectors. Conversely, given a finite set of endomorphisms $\Delta = \{\rho_0 = \mathrm{id}, \rho_1, \ldots, \rho_n\}$ which is closed under

- conjugates, i.e. there is a permutation $i \mapsto \bar\imath$ on $\{1, \ldots, n\}$, such that $[\rho_{\bar\imath}] = [\bar\rho_i]$ and
- fusion, i.e. there are numbers N_{ij}^k, the so-called fusion rule coefficients, such that $[\rho_i \circ \rho_j] = \bigoplus_{\rho_k\in\Delta} N_{i,j}^k [\rho_k]$

there is a unique unitary fusion $\langle \rho \in \Delta\rangle \subset \mathrm{End}_0(M)$.

Every fusion category ${}_M\mathcal{F}_M$ can be seen as the even part of a subfactor $N \subset M$. For example we can simply take the subfactor $N = \rho_\oplus(M) \subset M$, where $[\rho_\oplus] = \bigoplus_{\rho\in\mathrm{Irr}({}_M\mathcal{F}_M)}[\rho]$. This particular subfactor actually has the special feature that the depth is two, which implies that there is a weak Kac algebra Q, such that $N = M^Q \subset M$ [Reh97, NSW98, NV00]. In this sense, one can see fusion categories as representation categories of weak Kac algebras, but the choice of Q with this property is not unique.

Further every abstract unitary fusion category $\mathcal{F}$ can be realized as a (as a concrete fusion category) in $\mathrm{End}_0(M)$, i.e. there is a full and replete ${}_M\mathcal{F}_M \subset \mathrm{End}_0(M)$, which is equivalent to $\mathcal{F}$. Namely, [HY00] gives a realization as bimodules of the hyperfine II_1 factor R and by tensoring with the hyperfinite type III_1 factor we get it realized as endomorphisms (cf. [Lon90]). Using Popa's theorem [Pop95] such a realization is unique, i.e. if we have another relation on $\tilde M$ then there is an isomorphism $\phi\colon M \to \tilde M$ which gives an equivalence of categories (cf. [KLM01, Proof of Corollary 35]).

Often one wants a spherical structure on a fusion category. In the unitary case, we don't need to worry, because there is always (a unique up unitary to equivalence) spherical structure [LR97].

The categorical dimension $d\rho$ of $\rho \in \mathrm{End}_0$ coincides with the square root of the minimal index $[M : \rho(M)]$. A unitary fusion category $\mathcal{F}$ is called **braided** if there is a natural family of unitaries $\varepsilon(\rho, \sigma) \in \mathrm{Hom}(\rho\sigma, \sigma\rho)$. It is called a **unitary modular tensor category (UMTC)** if $\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho) = 1 = 1_{\sigma\circ\rho}$ for all $\rho \in \mathcal{F}$ implies $[\sigma] = N[\mathrm{id}]$.

One source of UMTCs are **quantum doubles of subfactors**. Given $N \subset M$, we take $S = M \otimes M^{\mathrm{op}}$ and ${}_S\mathcal{C}_S = \langle \rho\otimes\sigma^{\mathrm{op}} : \rho, \sigma \in {}_M\mathcal{F}_M^{N\subset M}\rangle$ which equivalent with the fusion category ${}_M\mathcal{F}_M^{N\subset M} \boxtimes ({}_M\mathcal{F}_M^{N\subset M})^{\mathrm{op}}$. Then there is a subfactor $\iota_{T\subset S}(T) \subset S$ and ${}_T\mathcal{C}_T := \langle \rho \prec \bar\iota_{T\subset S}\beta\iota_{T\subset S} : \beta \in {}_S\mathcal{C}_S\rangle$ can be identified with the category $Z({}_M\mathcal{F}_M^{N\subset M})$, which is a UMTC by [Müg03b]. Here $Z(\mathcal{F})$ denotes the **unitary (Drinfeld) center** of a unitary fusion category $\mathcal{F}$.

We could have started with N and ${}_N\mathcal{F}_N^{N\subset M}$ and obtain $Z({}_N\mathcal{F}_N^{N\subset M})$ which is braided equivalent to $Z({}_M\mathcal{F}_M)$. Indeed, ${}_N\mathcal{F}_N^{N\subset M}$ and ${}_M\mathcal{F}_M^{N\subset M}$ are (weakly) Morita equivalent and Morita equivalent fusion categories have braided equivalent Drinfeld centers by combining [Sch01] with [Müg03a]. Therefore we denote the obtained UMTC $Z({}_M\mathcal{F}_M^{N\subset M})$ by $\mathcal{D}(N \subset M)$ and call it **the quantum double** of $N \subset M$. There is a Galois correspondence between intermediate subfactors $S \subset A \subset T$ and subfusion categories $\mathcal{G} \subset {}_M\mathcal{C}_M$ [Izu00].

3. Conformal and Completely Rational Nets

A conformal net is a mathematical prescription of a chiral conformal quantum field theory on the circle using operator algebras. A well-behaving family of conformal nets are the so-called completely rational nets, which have a representation theory similar to finite groups and quantum groups at root of unity.

We denote by $\mathcal{I}$ the set of proper intervals $I \subset \mathbb{S}^1$ on the circle and by $I' = \mathbb{S}^1 \setminus \bar I$ the opposite interval. By a conformal net $\mathcal{A}$, we mean a local Möbius covariant net on the circle. It associates with every proper interval $I \in \mathcal{I}$ a von Neumann algebra $\mathcal{A}(I) \subset \mathrm{B}(\mathcal{H})$ on a fixed Hilbert space $\mathcal{H}$, such that the following properties hold:

A. Isotony. $I_1 \subset I_2$ implies $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

B. Locality. $I_1 \cap I_2 = \emptyset$ implies $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$.

C. Möbius covariance. There is a unitary representation U of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.

D. Positivity of energy. U is a positive energy representation, i.e. the generator L_0 (conformal Hamiltonian) of the rotation subgroup $U(z \mapsto \mathrm{e}^{\mathrm{i}\theta}z) = \mathrm{e}^{\mathrm{i}\theta L_0}$ has positive spectrum.

E. Vacuum. There is a (up to phase) unique rotation invariant unit vector $\Omega \in \mathcal{H}$ which is cyclic for the von Neumann algebra $\mathcal{A} := \bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

A conformal net $\mathcal{A}$ is called **completely rational** if it

F. fulfills the **split property**, i.e. for $I_0, I \in \mathcal{I}$ with $\overline{I_0} \subset I$ the inclusion $\mathcal{A}(I_0) \subset \mathcal{A}(I)$ is a split inclusion, namely there exists an intermediate type I factor M, such that $\mathcal{A}(I_0) \subset M \subset \mathcal{A}(I)$.

G. is **strongly additive**, i.e. for $I_1, I_2 \in \mathcal{I}$ two adjacent intervals obtained by removing a single point from an interval $I \in \mathcal{I}$ the equality $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ holds.

H. for $I_1, I_3 \in \mathcal{I}$ two intervals with disjoint closure and $I_2, I_4 \in \mathcal{I}$ the two components of $(I_1 \cup I_3)'$, the **μ-index** of $\mathcal{A}$

$$\mu(\mathcal{A}) := [(\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' : \mathcal{A}(I_1) \vee \mathcal{A}(I_3)]$$

is finite.

All known examples of completely rational nets also turn out to be covariant with respect to a projective representation of the diffeomorphism group of the circle and this leads to representation of the Virasoro algebra, but we just assume Möbius covariance, although the term conformal net often refers to diffeomorphism covariant nets.

Examples of completely rational nets are:

- Diffeomorphism covariant nets with central charge $c < 1$ [KL04].
- The nets $\mathcal{A}_L$ where L is a positive even lattice [DX06] which contain as a special case [Bis12] loop group nets $\mathcal{A}_{G,1}$ at level 1 for G a compact connected, simply connected simply-laced Lie group.
- The loop group nets $\mathcal{A}_{\mathsf{SU}(n),\ell}$ for $\mathsf{SU}(n)$ at level ℓ. [Xu00].

Further examples of rational conformal nets come from standard constructions:

- Finite index extensions and subnets of completely rational nets. Namely, let $\mathcal{A} \subset \mathcal{B}$ be a finite subnet i.e. $[\mathcal{B}(I) : \mathcal{A}(I)] < \infty$ for some (then all) $I \in \mathcal{I}$, then $\mathcal{A}$ is completely rational iff $\mathcal{B}$ is completely rational [Lon03], in particular orbifolds $\mathcal{A}^G$ of completely rational nets $\mathcal{A}$ with G a finite group are completely rational.
- Let $\mathcal{A} \subset \mathcal{B}$ be a co-finite subnet, i.e. $[\mathcal{B}(I), \mathcal{A}(I) \vee \mathcal{A}^{\mathrm{c}}(I)] < \infty$ for some (then all) $I \in \mathcal{I}$, where the **coset net** $\mathcal{A}^{\mathrm{c}}$ is defined by $\mathcal{A}^{\mathrm{c}}(I) = \mathcal{A}' \cap \mathcal{B}(I)$ with $\mathcal{A}' = (\vee_{I\in\mathcal{I}}\mathcal{A}(I))'$. Then $\mathcal{B}$ is completely rational iff $\mathcal{A}$ and $\mathcal{A}^{\mathrm{c}}$ are completely rational [Lon03].

A **representation** π of $\mathcal{A}$ is a family of unital representations $\pi = \{\pi_I \colon \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}_{I\in\mathcal{I}}$ on a common Hilbert space $\mathcal{H}_\pi$ which are compatible, i.e. $\pi_J \restriction \mathcal{A}(I) = \pi_I$ for $I \subset J$. An example is the trivial representation $\pi_0 = \{\pi_{0,I} = \mathrm{id}_{\mathcal{A}(I)}\}$ on the defining Hilbert space $\mathcal{H}$. Let us fix through out an abitrary interval $I \in \mathcal{I}$. Every representation π with $\mathcal{H}_\pi$ separable turns out to be equivalent to a representation **localized** in I, i.e. ρ on $\mathcal{H}$, such that $\rho_J = \mathrm{id}_{\mathcal{A}(J)}$ for $J \cap I = \emptyset$. Then Haag duality implies that $\rho = \rho_I$ is an endomorphism of $\mathcal{A}(I)$. The **statistical dimension** of a localized endomorphism ρ is given by $d\rho = [N : \rho(N)]^{\frac{1}{2}}$ and we will restrict to endomorphisms with finite statistical dimension.

The category $\mathrm{Rep}^I(\mathcal{A})$ of representations of $\mathcal{A}$ with finite statistical dimension which are localized in I naturally is a full and replete subcategory of the rigid C^* tensor category of endomorphisms $\mathrm{End}_0(\mathcal{A}(I))$. In particular, this gives the representations of $\mathcal{A}$ the structure of a tensor category [DHR71]. It has a natural **braiding**, which is completely fixed by asking that if ρ is localized in I_1 and σ in I_2 where I_1 is left of I_2 inside I then $\varepsilon(\rho,\sigma) = 1$ [FRS89].

Proposition 3.1 ([KLM01])**.** *Let $\mathcal{A}$ be completely rational net, then* $\mathrm{Rep}^I(\mathcal{A})$ *is a UMTC and* $\mu_{\mathcal{A}} = \dim(\mathrm{Rep}^I(\mathcal{A}))$*, where* $\dim(\mathcal{C}) = \sum_{\rho\in\mathrm{Irr}(\mathcal{C})}(d\rho)^2$ *is the **global dimension**.*

We call a completely rational net $\mathcal{A}$ with $\mu(\mathcal{A}) = 1$ a **holomorphic net**. This means that every representation is equivalent to a direct set of the trivial representation π_0. Examples of holomorphic nets are conformal nets $\mathcal{A}_L$ associated with even self-dual lattices L constructed in [DX06], the the Moonshine net $\mathcal{A}_\natural$ [KL06] and certain framed nets [KS14].

Similar to the concept of subgroups, there is the notion of a subnet. We write $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \supset \mathcal{A}$ if there is a representation $\pi = \{\pi_I \colon \mathcal{A}(I) \to \mathcal{B}(I) \subset \mathrm{B}(\mathcal{H}_{\mathcal{B}})\}$ of $\mathcal{A}$ on $\mathcal{H}_{\mathcal{B}}$ and an isometry $V \colon \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{B}}$ with $V\Omega_{\mathcal{A}} = \Omega_{\mathcal{B}}$ and $VU_{\mathcal{A}}(g) = U_{\mathcal{B}}(g)V$. We ask that further that $Va = \pi_I(a)V$ for $I \in \mathcal{I}$, $a \in \mathcal{A}(I)$. Define p the projection on $\mathcal{H}_{\mathcal{A}_0} = \overline{\pi_I(\mathcal{A}(I))\Omega}$. Then pV is a unitary equivalence of the nets $\mathcal{A}$ on $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{A}_0$ defined by $\mathcal{A}_0(I) = \pi_I(\mathcal{A}(I))p$ on $\mathcal{H}_{\mathcal{A}_0}$.

4. Subfactors arising from Conformal Nets

If we have a completely rational net, then $A = \mathcal{A}(I)$ is the hyperfinite type III$_1$ factor. With ${}_A\mathcal{C}_A = \mathrm{Rep}^I(\mathcal{A}) \subset \mathrm{End}_0(A)$, every $\rho \in {}_A\mathcal{C}_A$ gives a subfactor $\rho(A) \subset A$.

But we are interested in taking all subfactors arising from ${}_A\mathcal{C}_A$ and Morita equivalent fusion categories.

The philosophy is similar to the following one: If we have one fusion category, e.g. the even part of a subfactor, we can look into all Morita equivalent fusion categories and then into all subfactors arising this way [GS15]. This is related to Ocneanu's maximal atlas [Ocn01] and the Brauer–Picard groupoid of a fusion category [ENO05].

An irreducible finite index overfactor $B \supset \iota(A)$ with $A = \mathcal{A}(I)$, where the dual canonical endomorphism $\bar\iota \circ \iota$ is in $\mathrm{Rep}^I(\mathcal{A})$ gives rise to a relatively local extension $\mathcal{B} \supset \mathcal{A}$ and all such extension arise in this way. The net $\mathcal{B}$ is itself is in general not local but just relatively local to $\mathcal{A}$. The net $\mathcal{B}$ is local and therefore itself a completely rational net [Lon03] if the extension comes from a commutative Q-system. Relatively local extensions arose by the study of boundary conformal field theory [LR04]; they give a boundary net by holographic projection. By removing the boundary [LR09] one obtains a full CFT on Minkowski space (see below).

This motivates the following definition.

Definition 4.1. Let $\mathcal{A}$ be a completely rational net. We say a **subfactor arises from** $\mathcal{A}$ if it is of the form $\beta(B_a) \subset B_b$, where $\beta \in {}_{B_b}\mathcal{C}_{B_a} = \langle \beta \prec \iota_b\rho\bar\iota_a : \rho \in \mathrm{Rep}^I(\mathcal{A})\rangle$, with $A = \mathcal{A}(I)$, $B_\bullet = \mathcal{B}_\bullet(I)$ and $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$ irreducible relatively local extensions.

Let $\mathcal{A}$ be a completely rational net on the circle. The net $\mathcal{A}$ describes the chiral symmetries of a full two-dimensional CFT. For example the Virasoro net with central charge $c < 1$ [KL04] is a completely rational net and the symmetries it describes are the the conformal transformations $\mathrm{Diff}_+(\mathbb{S}^1)$ on the circle. For $c \geq 1$ the Virasoro net is not completely rational but one can consider larger class of symmetries, for example the loop group net $\mathcal{A}_{G,k}$ which is known to be completely rational for $G = \mathsf{SU}(N)$ and level $k \in \mathbb{N}$ and which describes $\mathsf{SU}(N)$ gauge transformations.

A **full CFT based on** $\mathcal{A}$ on Minkowski space $\mathbb{R}^2$, is a maximal local extension $\mathcal{B}_2$ of the net $\mathcal{A}_2$ which is defined by

$$\mathcal{A}_2(I_+ \times I_-) = \mathcal{A}(I_+) \otimes \mathcal{A}(I_-), \quad I_+ \times I_- = \{(t,x) \in \mathbb{R}^2 : t \pm x \in I_\pm\}\,,$$

where we see $\mathcal{A}$ by restriction as a net on $\mathbb{R}$. Since $\mathcal{A}$ is completely rational, $\mathrm{Rep}^I(\mathcal{A})$ is a unitary modular tensor category $\mathcal{C}$. The category of representations of $\mathcal{A}_2$ is equivalent to the category $\mathcal{C} \boxtimes \bar{\mathcal{C}}$ and $\mathcal{B}_2$ is completely characterized by a commutative Q-system in $\mathcal{C} \boxtimes \bar{\mathcal{C}}$. With Kawahigashi and Longo, the author has obtained a classification of full CFTs in terms of $\mathcal{A}$:

Theorem 4.2 ([BKL15])**.** *Full CFTs based on $\mathcal{A}$, i.e. maximal local extensions $\mathcal{B}_2 \supset \mathcal{A}_2$ are in one-to-one correspondence with Morita equivalence classes of non-local extensions $\mathcal{B} \supset \mathcal{A}$.*

Given two full CFTs $\mathcal{B}_2^a, \mathcal{B}_2^b \supset \mathcal{A}_2$, there is a notion of a defect line or phase boundary [BKLR16, BKLR15] between the full conformal field theories $\mathcal{B}_2^a$ and $\mathcal{B}_2^b$ on 2D Minkowski space, which is invisible if restricted to $\mathcal{A}_2$, also called $\mathcal{A}$–topological. If a subfactor arises from $\mathcal{A}$ it comes from such an $\mathcal{A}$–topological $\mathcal{B}_2^a$–$\mathcal{B}_2^b$ defect.

Theorem 4.3 ([BKLR16])**.** *$\mathcal{A}$–topological $\mathcal{B}_2^a$–$\mathcal{B}_2^b$ defects are in one-to-one correspondence with sectors $\beta \in {}_{B_a}\mathcal{C}_{B_b} = \langle \beta \prec \iota_b\, {}_A\mathcal{C}_A \bar{\iota}_a \rangle$, for ${}_A\mathcal{C}_A = \mathrm{Rep}^I(\mathcal{A})$ with $A = \mathcal{A}(I)$, $B_\bullet = \mathcal{B}_\bullet(I)$ and $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$ irreducible relatively local extension corresponding to the full CFT $\mathcal{B}_2^a, \mathcal{B}_2^b \supset \mathcal{A}_2$, respectively.*

Remark 4.4.
(1) The conditions that $B_\bullet$ comes from relatively local extension is equivalent to saying that ${}_A\mathcal{C}_A$ and ${}_{B_\bullet}\mathcal{C}_{B_\bullet}$ are weakly Morita equivalent in the sense of [Müg03a].
(2) ${}_{B_a}\mathcal{C}_{B_b}$ is a bimodule category over ${}_{B_a}\mathcal{C}_{B_a}$ and ${}_{B_b}\mathcal{C}_{B_b}$.
(3) Theorem 4.2 and 4.3 show that non-local extensions give full CFTs and endomorphisms between these extensions classify topological defects between this full CFTs. So if a subfactor arises from a conformal net $\mathcal{A}$ in the sense of Definition 4.1, then the subfactor describes a topological defect.

So far we have seen two sources of UMTCs:

- Quantum double of subfactors or equivalently Drinfeld centers of unitary fusion categories.
- Representation categories of completely rational nets.

They for sure don't give the same examples, e.g. $\mathcal{A}_{\mathsf{SU}(2)_1}$ has no local extensions and non-trivial representation category and we have the following characterization of nets having quantum doubles as representation category.

Proposition 4.5. *Let $\mathcal{A}$ be a completely rational conformal net. Then the following are equivalent:*

(1) $\mathrm{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$ *for some unitary fusion category $\mathcal{F}$.*
(2) There exists a holomorphic net $\mathcal{B} \supset \mathcal{A}$.

Every $\mathcal{F}$ with $\mathrm{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$ *gives a particular holomorphic net $\mathcal{B}_\mathcal{F} \supset \mathcal{A}$ and there is a Galois correspondence between full subcategories $\mathcal{G} \subset \mathcal{F}$ and intermediate nets $\mathcal{B}_F \supset \mathcal{B} \supset \mathcal{A}$.*

Proof. If $\mathrm{Rep}(\mathcal{A}) \cong Z(N \subset M)$ then there is an extension, such that $\mathcal{A}(I) \subset \mathcal{B}(I)$ is isomorphic to the Longo–Rehren subfactor of ${}_M\mathcal{F}_M$. Conversely, given $\mathcal{B} \supset \mathcal{A}$ we get that $\mathrm{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$ with $\mathcal{F} = \langle \beta \prec \alpha_\rho^+ : \rho \in \mathrm{Rep}(\mathcal{A}) \rangle$ the category coming from α^+-induction. See [Bis16] for details. □

To find a net $\mathcal{A}$ which realizes the quantum double $\mathcal{D}(N \subset M)$ is the mentioned Problem 1.3. We mention that if we find one net $\mathcal{A}$ with $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M)$ there are infinitely many examples since for every holomorphic net $\mathcal{B}$ we have $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Rep}(\mathcal{A} \otimes \mathcal{B})$ and there a infinitely man holomorphic nets.

Proposition 4.6. *Given a completely rational net $\mathcal{A}$, then the following are equivalent:*

(1) There is a completely rational net $\mathrm{Rep}(\tilde{\mathcal{A}}) \cong \mathrm{Rep}(\mathcal{A})^{\mathrm{rev}}$
(2) There is a holomorphic net $\mathcal{B}$, such that $\mathcal{A} \subset \mathcal{B}$ is normal and cofinite.

Proof. We sketch the proof, more details are in [Bis16]. If (1) is true we take $\mathcal{B}$ the Longo–Rehren extension of $\mathcal{A} \otimes \tilde{\mathcal{A}}$. Conversely, if (2) is true, we take $\tilde{\mathcal{A}}$ to be the coset of $\mathcal{A} \subset \mathcal{B}$. □

Proposition 4.7. *Let $N \subset M$ be a finite index, finite depth subfactor.*

(1) *If there is a conformal net with* $\mathrm{Rep}(\mathcal{A})$ *braided equivalent to* $\mathcal{D}(N \subset M)$*, then* $N \subset M$ *arises from* $\mathcal{A}$*. Actually it is enough that* $\mathrm{Rep}(\mathcal{A})$ *contains a full subcategory braided equivalent to* $\mathcal{D}(N \subset M)$*.*

(2) *Conversely, if* $N \subset M$ *arises from* $\mathcal{A}$*, then there is a 2D conformal net* $\mathcal{B}_2 \supset \mathcal{A}_2$ *with* $\mathrm{Rep}(\mathcal{B}_2) \cong \mathcal{D}(N \subset M)$*.*

(3) *Further, if* $N \subset M$ *arises from* $\mathcal{A}$*, and there is a net* $\tilde{\mathcal{A}}$*, with* $\mathrm{Rep}(\tilde{\mathcal{A}}) \cong \mathrm{Rep}(\mathcal{A})^{\mathrm{rev}}$ *then there is a conformal net* $\mathcal{B} \supset \mathcal{A} \otimes \tilde{\mathcal{A}}$ *with* $\mathrm{Rep}(\mathcal{B}) \cong \mathcal{D}(N \subset M)$*.*

Proof. The dual of the Longo–Rehren subfactor applied to ${}_N\mathcal{F}_N$ gives an extension $\mathcal{B}$ of $\mathcal{A}$ and it follows that ${}_B\mathcal{C}_B \supset {}_N\mathcal{F}_N$. We remark, that in the case $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M)$ the net $\mathcal{B}$ is a holomorphic net. We can take an overfactor $\tilde{B} \supset B$ equivalent to $M \supset N$. This does in general not give a relatively local extension of $\mathcal{B}$ but it gives a relatively local extension of $\tilde{\mathcal{B}} \supset \mathcal{A}$ and the inclusion $\iota(B) \subset \tilde{B}$ does the job.

That $N \subset M$ arises from $\mathcal{A}$ means that there are two extensions $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}$ and for $B_\bullet = \mathcal{B}(I)_\bullet$ there is a morphism $\beta\colon B_a \to B_b$, such that $\beta(B_a) \subset B_b$ is isomorphic to $N \subset M$. But this means that the dual category ${}_{B_a}\mathcal{C}_{B_a}$ contains ${}_N\mathcal{F}_N^{N\subset M}$ as a full subcategory. Since $\mathrm{Rep}(\mathcal{A}_2) \cong Z(\mathrm{Rep}(\mathcal{A})) \cong Z({}_{B_a}\mathcal{C}_{B_a})$ it follows from Galois correspondence that there is a local extension $\mathcal{B}_2 \supset \mathcal{A}_2$ with $\mathrm{Rep}(\mathcal{B}_2) \cong Z({}_N\mathcal{F}_N^{N\subset M}) \equiv \mathcal{D}(N \subset M)$.

The net $\mathcal{A} \otimes \tilde{\mathcal{A}}$ is a conformal net with $\mathrm{Rep}(\mathcal{A} \otimes \tilde{\mathcal{A}}) \cong Z(\mathrm{Rep}(\mathcal{A})$, then by exactly the same argument as before, there is a local extension $\mathcal{B} \supset \mathcal{A} \otimes \tilde{\mathcal{A}}$ with $\mathrm{Rep}(\mathcal{B}) \cong \mathcal{D}(N \subset M)$. □

In [Bis16] we used (3) and well-known constructions a to identify net $\mathcal{A}_{N\subset M}$ with $\mathrm{Rep}(\mathcal{A}_{N\subset M}) \cong \mathcal{D}(N \subset M)$ for all subfactors with index less than 4. It seems to be interesting to generalize this to other families of subfactors and fusion categories. Particular interesting are near group categories [EG14], since all subfactors in the small index classifications besides extended Haagerup [Haa94] seem to be related to near group fusion categories. The double of the 2221 subfactor or equivalently the $\mathbb{Z}_3 + 3$ near group category is realized by the loop group net $\mathcal{A}_{G_{2,3}} \otimes \mathcal{A}_{\mathsf{SU}(3)_1}$ and the 2^41 subfactor or the $\mathbb{Z}_4 + 4$ near category is related to a unitary fusion category coming from the conformal inclusion $\mathcal{A}_{\mathsf{SU}(3)_5} \subset \mathcal{A}_{\mathsf{SU}(6)_1}$ cf. [Liu15]. This gives hope that near group categories all come from rational nets.

We hope that we convinced the reader that the following are interesting problems.

- Finding interesting finite index subnets $\mathcal{A} \subset \mathcal{B}$ for $\mathcal{B}$ a holomorphic net which give new interesting subfactors/unitary fusion categories
- For interesting subfactors, find a completely rational net $\mathcal{A}$ with $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M)$. Evans and Gannon argue that for the Haagerup subfactor such a subnet of the conformal net associated with the E_8 lattice [EG11] should exist, but so far it has not been constructed.
- Find a general construction for every finite index, finite depth subfactor $N \subset M$ which gives a conformal net $\mathcal{A}$ with $\mathrm{Rep}(\mathcal{A}) \cong \mathcal{D}(N \subset M)$. This would show that all finite index, finite depth subfactors come from conformal nets.

Acknowledgements. I would like thank Yasuyuki Kawahigashi, Roberto Longo and Zhengwei Liu for discussions.

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