ASYMPTOTIC LIMITS IN MULTI-PHASE SYSTEMS

Greg Knowles

In this note we consider the asymptotic behaviour of an inviscid fluid with heat conduction. This work has been done in conjunction with J. Ball [2]. The fluid is assumed homogeneous and to occupy a spatial region $\omega \subset R^n$, where ω is bounded and open. At time t and position $x \in \omega$ the fluid has density $\rho(x,t) \geq 0$, velocity $v(x,t) \in R^n$, and temperature $\theta(x,t) > 0$. For simplicity we assume there is no external body force or heat supply. The governing equations are then

$$\rho\mathring{v} = - \text{grad } p \tag{1}$$

$$\mathring{\rho} + \rho \, \text{div}(v) = 0 \tag{2}$$

$$\rho\mathring{U} + \rho \, \text{div}(v) + \text{div}(q) = 0 \tag{3}$$

where the dots denote material time derivatives, p is the pressure, U the internal energy density and q the (spatial) heat flux vector. The constitutive relations are given in terms of the Helmholtz free energy, $A(\rho,\theta)$ and specific entropy $\eta(\rho,\theta)$, by

$$p = \rho^2 \frac{\partial A}{\partial \rho} \quad , \quad \eta = -\frac{\partial A}{\partial \theta} \quad , \quad U = A + \eta\theta \tag{4}$$

$$q = q(\rho,\theta,\text{grad } \theta).$$

We impose the boundary conditions

$$v\cdot n\Big|_{\partial\omega} = 0 \tag{5}$$

$$\theta\Big|_{\partial\omega_2} = \theta_0 \quad , \quad q\cdot n\Big|_{\partial\omega/\partial\omega_2} = 0 \tag{6}$$

where $n = n(x)$ is the outward normal to $\partial\omega$ at x, and $\theta_0 > 0$ is constant.

We make the following hypotheses on A

(i) $A: (0,b)x(0,\infty) \to R$ is continuous, where $b > 0$ is a constant

(ii) for each fixed $\rho \in (0,b)$, $A(\rho,\cdot)$ is C^1

(iii) for each fixed $\theta \in (0,\infty)$, the function

$$f_\theta(\rho) \triangleq \rho A(\rho,\theta) \text{ satisfies } \lim_{\rho\to 0^+} f_\theta(\rho) = 0,$$

$$\lim_{\rho\to 0_+} \frac{f_\theta(\rho)}{\rho} = -\infty \quad \text{and} \lim_{\rho\to b_-} f_\theta(\rho) = +\infty$$

(iv) the function $L(\rho,\theta) = \rho[U(\rho,\theta) - \theta_0\eta(\rho,\theta)]$ attains a strict minimum in θ at $\theta = \theta_0$ for all $\rho \in [0,b]$, and $\lim_{\theta\to 0_+} L(\rho,\theta) = \lim_{\theta\to\infty} L(\rho,\theta) = \infty$ for all $\rho > 0$.

These hypotheses are satisfied by the classical van der Waals' fluid ([7]) for which

$$A(\rho,\theta) = -a\rho + k\theta\log[\frac{\rho}{b-\rho}] - c\theta\log\theta - d\theta + \text{const} \tag{7}$$

where a, k, c are positive constants.

The central mathematical tool in this study is the concept of a <u>Young measure</u> (originally called a generalized curve [9] or a parametrized measure). Namely if E is a compact subset of R^n and Y a locally compact Polish space, the Young measures $\mathcal{M}(E:Y)$ are just those Radon measures on ExY whose projection onto E is dx, Lebesgue measure. $\mathcal{M}(E:Y)$ is topologized with the vague topology. We can alternately view a Young measure $\mu = (\mu_x)$ as a mapping $x \to \mu_x$ from E into the probability measures on Y, $M_1^+(Y)$, measurable w.r.t. the vague topology on $M_1^+(Y)$ ([8]). Then $\mu^n = (\mu_x^n) \rightharpoonup \mu = (\mu_x)$ vaguely, if

$$\int_E \int_Y f(x,y) d\mu_x^n(y) dx \to \int_E \int_Y f(x,y) d\mu_x(y) dx$$

for every $f \in C_0^\infty(E\text{x}Y)$.

Given a measurable function $g{:}E \to Y$, we can associate it with the Young measure $\mu^g = (\delta_{g(x)})$, where δ is the Dirac measure. We say a sequence of such functions $g_n \rightharpoonup \mu$ vaguely, if $\mu^{g_n} \rightharpoonup \mu$. This is equivalent to $F(g_n) \to \int_Y F(\lambda) d\mu_x(\lambda)$ in $L^\infty(E)$ weak *, for every continuous $F{:}\ Y \to R$ with compact support ([8]).

To study the asymptotic behaviour of solutions of (1) - (3) we recall a classical result of Duhem [4] (see also [3] for extensions to non-constant θ_0) that

$$E(\rho,v,\theta) \underset{=}{\Delta} \int_\omega \rho\Big[\frac{1}{2}|v|^2 + U(\rho,\theta) - \theta_0 \eta(\rho,\theta)\Big]dx \qquad (8)$$

is a Lyapunov function. Of course, we also have conservation of mass

$$\int_\omega \rho dx = M \tag{9}$$

So if $(\rho(t),v(t), \theta(t))$ is a solution, and $t_j \to \infty$, then $(\rho(t_j),v(t_j)), \theta(t_j))$ will be a minimizing sequence for E subject to (9), and hence to obtain information about the asymptotic limits of solutions of (1) - (3) we are led to firstly characterize the limits of all minimizing sequences of E, which is (non-convex) problem in the Calculus of Variations. It is an open (and much more difficult) question as to whether all of these limits are actually attained by solutions of (1) - (3) for varying initial conditions (partial results of this type are given in [1]).

We assume that $\partial\omega_2 \neq \phi$. Similar techniques can be used to analyse the Neumann problem, although it is somewhat more complicated. The details are given in [2]. The assumptions on L (assumption (iv)) imply that the integrand in (8) has a strict minimum, for fixed ρ, when $v = 0$ and $\theta = \theta_0$. Motivated by this we firstly consider the problem of minimizing

$$I(\rho) \underset{=}{\Delta} \int_\omega \rho[U(\rho,\theta_0) - \theta_0\eta(\rho,\theta_0)]dx \tag{10}$$

$$= \int_\omega f_{\theta_0}(\rho(x))dx \tag{11}$$

amongst measurable functions $\rho:\omega \to [0,b]$ satisfying (9), where $f_{\theta_0}(b)$ is defined to be $+\infty$ to match assumption (iii). Then we shall characterize the solution of the full problem in terms of the minimizers of (10), (11).

We denote by f^{**} the lower convex envelope of f_{θ_0}, i.e.

$$f^{**}_{\theta_0}(\rho) = \sup\{\alpha+\beta\rho : \alpha+\beta t \leq f_{\theta_0}(t), \text{ for all } t \in [0,b)\}, \tag{12}$$

and the subdifferential by,

$$\partial f^{**}_{\theta_0}(\rho) = \{\beta \in R: f^{**}_{\theta}(\rho) + \beta(t-\rho) \leq f^{**}_{\theta_0}(t), \ldots, \}, \tag{13}$$

and the Weierstrass set by

$$\mathcal{W} = \{\rho \in [0,b): f^{**}_{\theta_0}(\rho) = f_{\theta_0}(\rho)\},$$

($\mathcal{W}$ consists of the "points of convexity" of f_{θ_0}). Finally define

$$\bar{M} = M/\text{meas}(\omega) \tag{14}$$

(the mean mass) and

$$S(\bar{M}) = \{\rho \in (0,b): \partial f^{**}_{\theta_0}(\bar{M}) \subset \partial f_{\theta_0}(\rho)\} \tag{15}$$

It is easily seen that $S(\bar{M}) \subset \mathcal{W}$ and that $\bar{M}$ belongs to the convex hull of $S(\bar{M})$. For the van der Waals' fluid with $\frac{ab}{k\theta_0} > (\frac{3}{2})^3$ there exists one non-trivial common tangent to the graph f_{θ_0} with end points ρ_1, ρ_2 as shown in fig. 1. The Weierstrass set

$$\mathcal{W} = [0,\rho_1] \cup [\rho_1,b),$$

$$S(\bar{M}) = \begin{cases} \{\bar{M}\} & \text{for } \bar{M} \in (0,\rho_1) \cup (\rho_2,b) \\ \{\rho_1\} \cup \{\rho_2\} & \text{for } \rho_1 \leq \bar{M} \leq \rho_2. \end{cases}$$

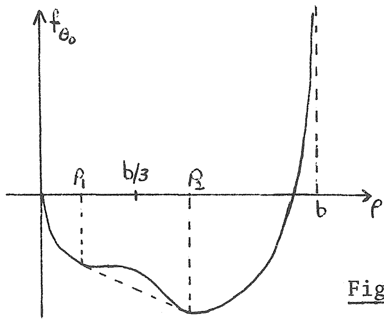

Figure 1

Since $f_{\theta_0}(\cdot)$ is, in general, not convex, we introduce a relaxed problem which is convex, and which has the same minimum as I. We use the theory of Young measures introduced by L.C. Young [9], which are now playing an increasing role in the study of non-linear partial differential equations, (Tartar [8]). This approach is motivated by the following consideration, if $\delta_{\rho(x)}$ denotes the Dirac measure supported at $\rho(x)$, $0 \leq x < b$, then

$$I(\rho) = \int_\omega f_{\theta_0}(\rho(x))dx = \int_\omega \int_0^b f_{\theta_0}(\rho)d\delta_{\rho(x)}(\rho)dx. \tag{16}$$

consequently, if $\nu = (\nu_x) \in \mathcal{M}(0,b)$ is a Young measure, and we define

$$\hat{I}(\nu) = \int_\omega \int_0^b f_{\theta_0}(\rho)d\nu_x(\rho)dx \tag{17}$$

then $\hat{I}(\delta_{\rho(x)}) = I(\rho)$, and the functional I is now linear in ν. Similarly, the constraint (9) can be generalized as

$$\int_\omega \int_0^b \rho d\nu_x(\rho)dx = M \tag{18}$$

The characterization of the minimizers and minimizing sequences for (10), (11), (17), (18) can now be stated (for proof see [2], also [5] for related results).

Theorem

(a) The minimum of $\hat{I}(\nu)$ subject to (18) is attained. The minimizing Young measures $\bar{\nu}$ are exactly those satisfying (18) for which $\text{supp}\bar{\nu}_x \subset S(\bar{M})$ a.e. $x \in \omega$.

(b) The minimum value of I subject to (9) is the same as that of $I(\nu)$ subject to (18), and is attained exactly by the functions ρ satisfying (9) and such that $\rho(x) \in S(\bar{M})$ a.e. $x \in \omega$.

(c) Let $\{\rho_i\}$ be any minimizing sequence for I subject to (9), then there exists a subsequence $\{\rho_\mu\}$ and a minimizing Young measure $\bar{\nu}$ for $\hat{I}$ subject to (18) such that $\rho_\mu \rightharpoonup \bar{\nu}$ in the sense of Young measures.
Conversely, given any minimizing Young measure $\bar{\nu}$ for $\hat{I}$ subject to (18) there exists a minimizing sequence $\{\rho_\mu\}$ of I subject to (9) converging to $\bar{\nu}$ in the sense of Young measures.

Note that part (b) of the Theorem states that only values $\rho \in \mathcal{W}$ can be observed in an absolute minimizer, this is the classical Weierstrass condition of the calculus of variations. Sometimes it is asserted that because of this 'stability' condition f must be convex; the correct interpretation has been pointed out, by Ericksen [6].

We are now in a position to state our characterization of the minimizers and minimizing sequences of E.

Theorem 2 The absolute minimizers of E in the space of bounded measurable functions subject to (9) are of the form $(\rho^*, 0, \theta_0)$ where $\rho^*(x) \in S(\bar{M})$ a.e. $x \in \omega$. For any minimizing sequence $(\rho_j(x), v_j(x), \theta_j(x)$ of E subject to (9), there holds, $v_j(x) \to 0$, $\theta_j(x) \to \theta_0$ a.e. $x \in \omega$, and there exists a subsequence $\rho_\mu \rightharpoonup \bar{\nu}$ vaguely, where $\text{supp}\ \bar{\nu} \subset S(\bar{M})$, a.e. $x \in \omega$.

There is an interesting possible "physical" explanation of the convergence of the densities $\rho_\mu \rightharpoonup \bar{\nu}$, and the limit "density" $\bar{\nu}$, in the case it is measure. It could represent the creation of "mist" where the phases are mixed more and more finely as $t_\mu \to \infty$, as energy is transferred to higher and higher modes by the non-linear dynamics. Hence in the limit we can really only talk about the probability of the different material phases in a given region of ω. Young measures would seem a natural way to analyse this energy transfer in non-linear systems. Of course, as pointed out earlier, showing that such limits are <u>actually realized</u> from certain initial conditions by <u>solutions</u> of <u>the pde</u> is much more difficult, and seemingly, as yet, unresolved problem. Numerical studies could illuminate this point

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Department of Electrical Engineering
Imperial College
London