SUBALGEBRAS OF AMENABLE ALGEBRAS

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It is well known that a closed ideal J of an amenable Banach algebra A is itself amenable if and only if it has a bounded approximate identity, and that if J and A/J are both amenable, then so is A. In this note we give some remarks concerning the question of whether or not anything can be said about closed subalgebras of amenable Banach algebras.

We begin by recalling some results from harmonic analysis which show, in various guises, that the "standard" amenable algebras, $L^1(\Gamma)$, for locally compact abelian groups Γ, always contain non-amenable closed subalgebras, provided that Γ is infinite. For convenience of notation, our discussion will be in terms of the dual group G, and the Fourier algebra $A(G) = L^1(\Gamma)^\wedge$.

Looking firstly at closed ideals, the remarks above show that the following gives a chacterization of amenable ideals in $A(G)$.

THEOREM 1 ([17], Theorem 13). *Let G be a locally compact abelian group, $I \subseteq A(G)$ a closed ideal. Then I has a bounded approximate identity if and only if $I = I(E) = \{ f \in A(G) : f(E) = (0) \}$ for some closed set $E \subseteq G$ which lies in the coset ring of G considered as a discrete group.*

In particular, if I has a bounded approximate identity then $E = hull(I)$ is a set of synthesis (though not conversely). Thus for G infinite and non-discrete, Malliavin's theorem shows the existence of E with $I(E)$ not amenable. Again, for G infinite and compact, spectral synthesis fails spectacularly in the sense that there exists $f \in A(G)$ such that the closed ideals generated by the positive powers of f are all distinct ([19]); of course these ideals cannot have bounded approximate identities. For G discrete, so every subset of G is of synthesis, take

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E to be any subset not in the coset ring. It is unknown to the present author whether it is possible for $I(E)^2 = I(E)$, or even $(I(E)^2)^- = I(E)$, when E is *not* of synthesis.

Coming now to subalgebras, there is a similar dichotomy between the discrete and non-discrete cases as in the above.

THEOREM 2 ([2]). *If G is a non-discrete locally compact abelian group, $A(G)$ contains a closed self-adjoint subalgebra B which is not finitely generated.*

The interest for us here is in the proof rather than the result – B is constructed to have infinite dimensional point derivation space at some maximal ideal.

THEOREM 3 ([3], Theorem 3.1]). *With G as above, there is $E \subset G$, a compact set of synthesis, such that $A(E) = A(G)/I(E)$ contains a regular closed self-adjoint algebra B isometrically isomorphic to an algebra of C^∞ functions on* $[-1, 1]$.

If B were amenable, then its isomorph $B' \subseteq C^\infty[-1, 1]$ would be also. Thus the ideal $I = \{f \in B' : f(0) = 0\}$ would have a bounded approximate identity, so would factor, whence any $f \in I$ would satisfy $f^{(n)}(0) = 0$ for $n \geq 0$, and so B' would contain no non-zero polynomials. But an examination of the proof from [3], in particular Lemma 2.9, shows explicitly that B' contains the Chebyshev polynomials (and hence all polynomials of course).

For infinite discrete abelian groups G there is another approach as follows. We will restrict ourselves to $G = \mathbb{Z}$, and its dual the circle group $\mathbb{T}$, though the ideas can be exploited in any discrete group.

For a closed subalgebra $A \subseteq A(\mathbb{Z})$, define an equivalence relation $\sim$ on $\mathbb{Z}$ by $n \sim m$ exactly when $f(n) = f(m)$ for all $f \in A$, and define the "zero" class $E_0 = \{n : f(n) = 0$ for all $f \in A\}$. Since $A(\mathbb{Z}) \subset c_0(\mathbb{Z})$, all equivalence classes except possibly E_0 are necessarily finite. Define A_0 to be the smallest closed subalgebra of A containing χ_F for each non-zero $\sim$-class F, A^0 to be the algebra of functions constant on each non-zero $\sim$-class, and zero on E_0. Then $A_0 \subseteq A \subseteq A^0$ and they each induce the equivalence relation $\sim$ on $\mathbb{Z}$. Further, A_0 is generated by the idempotents in A^0. An example is given in [18] where $A_0 \neq A^0$, and A^0 is (singly) generated by $\hat{f}$ for some $f \in \bigcap_{1<p<2} L^p(\mathbb{T})$.

THEOREM 4 ([12, Lemma 3.3]). *If $A \cap L^p(\mathbb{T})^\wedge$ is dense in A for some* $1 < p < 2$, *then $A^m \subseteq A_0$ for some $m \geq 2$.*

From the example in [18] it follows immediately that there is A^0 which does not factor, and so is non-amenable.

We are interested in whether there are any vestiges of these phenomena in general amenable algebras, in particular for subalgebras which are large in some sense, and first give a further result involving notions of spectral synthesis.

THEOREM 5 ([3, Theorem 4.5]). *Suppose B is a commutative semi-simple regular self-adjoint Banach algebra. Suppose synthesis fails for principal ideals in B, that is, there is $f \in B$ such that $f \notin (Bf^2)^-$. Then B has a closed subalgebra admitting a non-zero continuous point derivation.*

In view of matters raised elsewhere in these proceedings, [9], it is of interest to remark, following [3], that if the real elements $\phi \in B$ satisfy $\|\exp(in\phi)\| = O(|n|^k)$, where $k = k(\phi)$, then the failure of synthesis in B implies the failure of synthesis for principal ideals. Indeed, as this is not detailed in [3], we give a proof here using a technique that has arisen in [1, 5, 9, 10]. Thus suppose synthesis fails in B, so there is a compact set $K \subset \Phi_B$ such that if $J(K) = \{f \in B : \hat{f}^{-1}(0)$ is a neighbourhood of $K\}$, then $J(K)^- \subsetneqq I(K)$; choose $f \in I(K) \setminus J(K)^-$ with $\hat{f}$ real without loss of generality. Then the coset $\mathbf{f} = f + J(K)^-$ lies in the radical R of the quotient algebra $B/J(K)^-$. Let $\exp(i\mathbf{f}) = \mathbf{1} + \mathbf{r}$ with $\mathbf{r} \in R$. The hypothesis on B shows that

$$\|(\mathbf{1} + \mathbf{r})^n\| = \| \exp(in\mathbf{f}) \| = O(|n|)^k = o(|n|)^{k+1}$$

for some $k \geq 1$. But then by [14, 4.10.1], $\mathbf{r}^{k+1} = \mathbf{0}$. Thus

$$\mathbf{0} = (\exp i\mathbf{f} - \mathbf{1})^{k+1} = \Big[\sum_{m=1}^{\infty} \frac{i^m}{m!} \mathbf{f}^m\Big]^{k+1} = \mathbf{f}^{k+1}\Big[\sum_{m=1}^{\infty} \frac{i^m}{m!} \mathbf{f}^{m-1}\Big]^{k+1},$$

whence $\mathbf{f}^{k+1} = \mathbf{0}$ since the second factor in the last expression is invertible, so that $f^{k+1} \in J(K)^-$. Choosing the least possible k with this property, and supposing $f \in (Bf^2)^-$, say $f = \lim_{n \to \infty} b_n f^2$, we have $\mathbf{f}^k = \lim_{n \to \infty} \mathbf{b}_n^k \mathbf{f}^{2k} = \mathbf{0}$, a contradiction. Thus synthesis fails for the principal ideal $(Bf)^-$. (In fact it follows that $I(K)/J(K)^-$ is nil, and hence nilpotent, so that $I(K)^m \subseteq J(K)^-$ for some $m \geq 2$.)

Recall that a commutative semisimple Banach algebra is said to have the *Stone-Weierstrass property* if it has no proper, point separating, self-adjoint, closed subalgebras. That amenability is not sufficient to guarantee the Stone-Weierstrass property follows from an example of Katznelson and Rudin, [15], where it is shown that $A(\mathbb{T})$ fails to have the latter property – a closed subalgebra B is constructed, self-adjoint and point separating, but $B \neq A(\mathbb{T})$. B is certainly a "large" subalgebra of $A(\mathbb{T})$, and it would be of interest to know whether it is amenable. This is especially so since, as we now show, B is not generated by its doubly power bounded elements (cf. [10], §V).

Define functions $f_n : z \mapsto z^n$ on $\mathbb{T}$, and set $S = \{n \in \mathbb{Z} : f_n \in B\}$. Since B is a self-adjoint subalgebra, S is a group under addition, and as B is proper, S cannot contain 1 or -1. Choose the least $k \in S$ greater than 1. Then for $m \in S\backslash\{0\}$, there exist $p, q \in \mathbb{Z}$ such that $pm + qk = (m, k)$, so that $(m, k) \in S$.

But $|(m, k)| \le k$, whence $(m, k) = k$ and m is a multiple of k. It follows that either $S = \{0\}$ or $S = \{kn : n \in \mathbb{Z}\}$ for some $|k| > 1$. In either case the functions $\{f_n : n \in S\}$ do not separate the points of $\mathbb{T}$, and so they do not generate B. Finally, the Beurling-Helson theorem [20, Theorem 4.7.5] shows that the doubly power bounded elements of $A(\mathbb{T})$ are unimodular multiples of the functions $\{f_n : n \in \mathbb{Z}\}$, so that $\{\lambda f_n : n \in S, |\lambda| = 1\}$ is the set of doubly power bounded elements of B, which thus does not generate B.

We remark further that in this example, which is constructed from a totally disconnected subset $P \subset \mathbb{T}$ of positive measure, B will contain the set $I(P)$ if P is taken, as it may, to be a set of synthesis. $A(\mathbb{T})/I(P)$ is then an amenable algebra with P as its maximal ideal space, yet it is not generated by its idempotents, which all lie in $B/I(P)$. See [15] for details.

Pursuing these properties a little further, we note that if A is a commutative, semisimple Banach algebra generated by its idempotents, then A has the Stone-Weierstrass property, [15]. Further, A is weakly amenable, [6], but of course need not be amenable (for example, ℓ^2 with pointwise operations). If the idempotents are uniformly bounded, then certainly amenability is the case, even without the presupposition of semi-simplicity, since $A = C(\Phi_A)$ by an old result of Dunford (see [11], Proposition 5.43). Without the boundednesss of the idempotents, amenability may fail even if the primitive idempotents are bounded (the Feldman example).

With this evidence at hand we raise the following question.

QUESTION. Do there exist (infinite dimensional) amenable Banach algebras all of whose closed subalgebras are amenable ?

For suitably restricted algebras we can give a succinct answer.

THEOREM 6. *Let A be a unital uniform algebra with carrier space X. Then all closed subalgebras of A are amenable if and only if X is scattered.*

Proof. Suppose that A has the stated property. By Scheinberg's theorem, [16, Theorem 56], we have $A = C(X)$.

It suffices to consider the case when X infinite. Let Y be a nonempty perfect subset of X. By the Čech-Posposil theorem there is a continuous surjection of Y onto $[0, 1]$, and the Tietze extension theorem gives a continuous extension $f : X \to [0, 1]$. Then $g = e^{2\pi i f} \in A$, with $\sigma(g) = \{z : |z| = 1\}$, and so the closed unital subalgebra generated by g is isometrically isomorphic to the uniform closure of the polynomials on $\mathbb{T}$, which is not an amenable algebra. This contradicts the hypothesis, so that there can be no such Y, that is, X is scattered.

Conversely, suppose that X is scattered, and that A is a closed unital subalgebra of $C(X)$ with carrier space Φ_A. Then if $f \in A$ is invertible in $C(X)$, $f(X)$ is a countable compact set in $\mathbb{C}$ not containing 0, [19]. Thus $f(X)$ has no interior and does not separate $\mathbb{C}$, so by Lavrentiev's theorem there exist polynomials $\{p_n\}$ such that $p_n(z) \to z^{-1}$ uniformly on $f(X)$. But then the map $p_n(f) \mapsto f^{-1}$ in $C(X)$, so that $f^{-1} \in A$. It follows that Φ_A can be identified with a subset of X. A similar argument shows that $\hat{A}$ is closed under complex conjugation ([19]). But then the map $f \mapsto \hat{f}$ is an isometric, conjugate preserving isomorphism of A onto a point separating unital subalgebra of $C(\Phi_A)$. By the Stone-Weierstrass theorem, $\hat{A} = C(\Phi_A)$, which is amenable, so that A is amenable. Any non–unital closed subalgebra A is a maximal ideal in the amenable algebra $A \oplus \mathbb{C}1$, so is itself amenable.

The result for non–unital uniform algebras is an easy consequence, but what can be said more generally ? For any commutative amenable Banach algebra A, $(\hat{A})^-$ is an amenable uniform algebra, so Scheinberg's theorem shows that $\hat{A}$ is dense in $C_0(\Phi_A)$. Thus if all closed subalgebras of A are amenable, then in particular, for each $x \in A$, the closed subalgebra generated by x has Gelfand transform dense in $\{f \in C(\sigma(x)) : f(0) = 0\}$, so that $\sigma(x)$ has empty interior, and has connected complement. The following simple observation shows that these latter conditions also follow from a hypothesis of scatteredness.

THEOREM 7. *Suppose that B is commutative unital Banach algebra with scattered Silov boundary $\partial\Phi_B$. Let A be a closed unital subalgebra of B (possibly B itself). Then*

(i) A *is inverse closed,*
(ii) $\hat{A}$ *is regular on* $\Phi_A = \partial\Phi_A$ *which is totally disconnected,*
(iii) $\hat{A}$ *is dense in* $C(\Phi_A)$,
(iv) $A^{-1} = exp\ A$.

Proof. (i), (ii), and (iii) follow immediately from results of Glicksberg [13]. The argument there also shows that, for $x \in A$,

$$\partial\sigma_A(x) \subseteq \hat{x}(\partial\Phi_A) \subseteq \hat{x}(\partial\Phi_B) \subseteq \sigma_B(x) \subseteq \sigma_A(x) = \partial\sigma_A(x),$$

so that equality holds throughout, and $\sigma_A(x)$ is scattered in $\mathbb{C}$. Thus $\sigma_A(x)$ certainly has empty interior and does not separate the plane. If x is invertible we can thus define $log(x)$ via the functional calculus and (iv) follows.

Of course, these results are fragmentary, and certainly having a scattered carrier space is not sufficient for amenability of closed subalgebras even if the algebra itself is amenable; $A(\mathbb{Z})$ gives a counterexample. Scheinberg's result applied to finitely generated subalgebras shows that for any $\underline{x} = \{x_1, \ldots, x_k\} \subset A$, the closed subalgebra generated by $\{1, x_1, \ldots, x_k\}$ has Gelfand transform dense in $C(\sigma(\underline{x}))$, whence $\mathcal{P}(\sigma(\underline{x})) = C(\sigma(\underline{x}))$, where $\mathcal{P}(\sigma(\underline{x}))$ is the closure of polynomials on $\sigma(\underline{x})$. But in fact this gives no more information than the particular case $k = 1$ noted above. Indeed, taking the case $k = 2$, and supposing that elements x, y satisfy $\mathcal{P}(\sigma(x)) = C(\sigma(x))$ and $\mathcal{P}(\sigma(y)) = C(\sigma(y))$, the Stone-Weierstrass theorem shows that $\mathcal{P}(\sigma(x) \times \sigma(y)) = C(\sigma(x) \times \sigma(y))$, and since $\sigma(x, y)$ is a closed subset of $\sigma(x) \times \sigma(y)$, Tietze's theorem then gives $\mathcal{P}(\sigma(x, y)) = C(\sigma(x, y))$.

Perhaps the obvious question is whether all closed subalgebras amenable necessitates scattered carrier space. Does the countability of the spectrum of every element imply that the carrier space is scattered ? (We note that the question of characterising algebras with this countable spectrum property was raised in §3.3 of [4]. Characterization of algebras satisfying the stricter condition, that the spectrum of each element has only limit point zero, had been given earlier in [7, Theorem 4.1], in particular the carrier space of such an algebra is discrete.) And, of course, what happens in the noncommutative case ? We finally remark that it is known that any non-type I amenable C*-algebra contains non-amenable C*-subalgebras, [8].

REFERENCES

1. G. R. Allan, Power-bounded elements in a Banach algebra and a theorem of Gelfand, *these proceedings*, 1 – 12.

2. A. Atzmon, Sur les sous-algèbres fermées de $A(G)$, *C. R. Acad. Sc. Paris, Série A*, 270 (1970), 946 – 948.

3. A. Atzmon, Spectral synthesis in regular Banach algebras, *Israel J. Math.*, 8 (1970), 197 – 212.

4. B. Aupetit, *Propriétés Spectrales des Algébres de Banach*, Lecture Notes in Math., 735, Springer-Verlag, Berlin and New York, 1979.

5. W. G. Bade, The Wedderburn decomposition for quotient algebras arising from sets of non-synthesis, *these proceedings*, 25 – 31.

6. W. G. Bade, P. C. Curtis, Jr. and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, *Proc. London Math. Soc.*, (3) 55 (1987), 359 – 377.

7. B. A. Barnes, On the existence of minimal ideals in a Banach algebra, *Trans. Amer. Math. Soc.*, 133 (1968), 511 – 517.

8. B. Blackadar, Nonnuclear subalgebras of C*-algebras, *J. Operator Theory*, 14 (1985), 347 – 350.

9. P. C. Curtis Jr., Complementation problems concerning the radical of a commutative amenable Banach algebra, *these proceedings*, 56 – 60.

10. P. C. Curtis Jr. and R. J. Loy, The structure of amenable Banach algebras, *J. London Math. Soc.*, to appear, 1989.

11. H. R. Dowson, *Spectral Theory of linear operators*, Academic Press, London, 1978.

12. S. Friedberg, Closed subalgebras of group algebras, *Trans. Amer. Math. Soc.*, 147 (1970), 117 – 125.

13. I. Glicksberg, Banach algebras with scattered structure spaces, *Trans. Amer, Math. Soc.*, 98 (1961), 518 – 526.

14. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ. **31**, Providence, R. I., 1957.

15. Y. Katznelson and W. Rudin, The Stone-Weierstrass property in Banach algebras, *Pac. J. Math.*, 11 (1961), 253 – 265.

16. A. Ya. Khelemskii, Flat Banach modules and amenable algebras, *Trans. Moscow Math. Soc.*, 47 (1984), (Amer. Math. Soc. Trans. (1985), 199 – 224).

17. T.-S. Liu, A. van Rooij and J.-K. Wang, Projections and approximate identities for ideals in group algebras, *Trans. Amer. Math. Soc.*, 175 (1973), 469 – 482.

18. D. Rider, Closed subalgebras of $L^1(\mathbb{T})$, *Duke Math. J.*, 36 (1969), 105 – 115.

19. W. Rudin, Continuous functions on compact spaces without perfect subsets, *Proc. Amer. Math. Soc.*, 8 (1957), 39 – 42.

20. W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

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