SOME PROBLEMS CONCERNING REFLEXIVE OPERATOR ALGEBRAS

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1. INTRODUCTION AND PRELIMINARIES

We discuss below some problems concerning a certain class of algebras of operators on complex Banach space. Each algebra of the class arises from a lattice of subspaces of the underlying space (in a way that will soon be made precise) and most of the problems are of the form: find conditions, additional to those specified à priori, on the lattice of subspaces, which are both necessary and sufficient for the corresponding algebra of operators to have a certain, specified, algebraic and/or topological property. It is more accurate to say, then, that these problems concern lattices of subspaces of certain types. Naturally, all the problems have partial solutions some of which will be described. While the interested reader can find complete proofs elsewhere, some brief proofs are included for the purpose of illustration. Most of these problems have arisen (in some cases, have re-surfaced) in joint work with S. Argyros and M.S. Lambrou.

Throughout, X denotes a complex non-zero Banach space and H denotes a complex non-zero Hilbert space. The topological dual of X is denoted by X^*. The terms 'operator' and 'subspace' will mean bounded linear mapping and closed linear manifold, respectively. For any family $\{M_\gamma\}$ of subspaces of X, $\vee M_\gamma$ denotes the closed linear span of $\{M_\gamma\}$. For any vectors $e \in X$ and $f^* \in X^*$, $f^* \otimes e$ denotes the operator on X given by $(f^* \otimes e)\, x = f^*(x)\, e$. The lattice of subspaces of X is denoted by $\mathcal{C}(X)$ and for any $L \in \mathcal{C}(X)$, $L^\perp$ denotes the annihilator of L, that is

$$L^\perp = \{f^* \in X^* : f^*(x) = 0\,, \quad \text{for every } x \in L\}\,.$$

For any subset $\mathcal{L} \subseteq \mathcal{C}(X)$ the set of operators on X that leave every member of $\mathcal{L}$ invariant is denoted by $\operatorname{Alg} \mathcal{L}$. Thus

$$\mathrm{Alg}\,\mathcal{L} = \{T \in \mathcal{B}(X) : TM \subseteq M\,, \quad \text{for every } M \in \mathcal{L}\}\,.$$

It is not difficult to verify that, for any $\mathcal{L}$, Alg $\mathcal{L}$ is a unital subalgebra of $\mathcal{B}(X)$ which is closed in the strong operator topology. So Alg $\mathcal{L}$ is a unital Banach algebra. Dually, for any subset $\mathcal{A} \subseteq \mathcal{B}(X)$, the set of invariant subspaces of $\mathcal{A}$ is denoted by Lat $\mathcal{A}$. Thus

$$\mathrm{Lat}\,\mathcal{A} = \{M \in \mathcal{C}(X) : TM \subseteq M\,, \quad \text{for every } T \in \mathcal{A}\}\,.$$

An algebra of operators $\mathcal{A} \subseteq \mathcal{B}(X)$ is called *reflexive* if $\mathcal{A} = \mathrm{Alg\,Lat}\,\mathcal{A}$. Thus an algebra is reflexive, if and only if it is completely determined by its invariant subspaces. The following easily obtainable characterization is more direct: the algebra $\mathcal{A} \subseteq \mathcal{B}(X)$ is reflexive, if and only if $\mathcal{A} = \mathrm{Alg}\,\mathcal{L}$ for some subset $\mathcal{L} \subseteq \mathcal{C}(X)$.

For example, every von Neumann algebra is reflexive (see [13]). In fact, an algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is reflexive and self–adjoint, if and only if it is a von Neumann algebra. Our problems concern non–self–adjoint reflexive algebras for the most part.

In the study of reflexive algebras Alg $\mathcal{L} \subseteq \mathcal{B}(X)$ we can assume that $\mathcal{L} \subseteq \mathcal{C}(X)$ satisfies

(1) $(0)\,,\ X \in \mathcal{L}$,

(2) $\mathcal{L}$ is a complete sublattice of $\mathcal{C}(X)$,

that is, for every family $\{M_\gamma\}$ of elements of $\mathcal{L}$ both $\cap M_\gamma$ and $\vee M_\gamma$ belong to $\mathcal{L}$. A subset $\mathcal{L} \subseteq \mathcal{C}(X)$ satisfying (1) and (2) is called a *subspace lattice on* X. A *nest* is a totally ordered subspace lattice.

2. SOME PROBLEMS

In 1965 J.R. Ringrose initiated the study of *nest algebras*, that is, those algebras of the form Alg $\mathcal{N}$ for some nest $\mathcal{N} \subseteq \mathcal{C}(X)$. We will come to one of his theorems presently. One of the early results in this study is the following.

THEOREM. (J.A. Erdos, 1968 [2]) *For every nest* $\mathcal{N}$ *on* H, *the subalgebra of* Alg $\mathcal{N}$ *generated by the rank one operators is dense in* Alg $\mathcal{N}$ *in the strong operator topology.*

DEFINITION. If $\mathcal{L}$ is a subspace lattice on X, say that $\mathcal{L}$ has the *strong rank one density property*, abbreviated SRO, if the subalgebra of Alg $\mathcal{L}$ generated by the rank one operators is dense in Alg $\mathcal{L}$ in the strong operator topology.

PROBLEM. Which subspace lattices on X have SRO?

If $\mathcal{L}$ is a subspace lattice on X, the subalgebra generated by the rank one operators of Alg $\mathcal{L}$ is just the set of finite sums ΣR_α where each R_α is an operator of Alg $\mathcal{L}$ of rank at most one. This algebra is an ideal of Alg $\mathcal{L}$, so $\mathcal{L}$ has SRO if and only if

> for every $\varepsilon > 0$ and for every finite set of vectors $x_1, x_2, \ldots, x_m$ of X, there exists $F = \Sigma R_\alpha$ (as described above) such that $\|x_j - Fx_j\| < \varepsilon$, for $j = 1,2,\ldots,m$.

If Alg $\mathcal{L}$ contains no rank one operators the subalgebra of it generated by the rank one operators is just $\{0\}$. This can happen, and there is a simple lattice–theoretic characterization of this phenomenon.

DEFINITION. Suppose $\mathcal{L}$ is a subspace lattice on X. For any $L \in \mathcal{L}$ define $L_- \in \mathcal{L}$ by $L_- = \vee\{M \in \mathcal{L} : L \not\subseteq M\}$.

In the above definition, and in what follows, we employ the conventions that $\vee\emptyset = (0)$ and $\cap\emptyset = X$; then $(0)_- = (0)$.

LEMMA. (W.E. Longstaff, 1975 [11]) *If* $\mathcal{L}$ *is a subspace lattice on* X *and* $e \in X$, $f^* \in X^*$, *then the operator* $f^* \otimes e$ *belongs to* Alg $\mathcal{L}$ *if and only if* $e \in L$ *and*

$f^* \in (L_-)^\perp$ *for some* $L \in \mathcal{L}$.

COROLLARY. Alg $\mathcal{L}$ *contains a rank one operator if and only if there is some* $L \in \mathcal{L}$ *with* $L \neq (0)$ *and* $L_- \neq X$.

For example, if K, L and M are non–trivial (that is, each is neither (0) nor X) subspaces of X satisfying $K \vee L = L \vee M = M \vee K = X$ and $K \cap L = L \cap M = M \cap K = (0)$, then there is no rank one operator leaving each of K, L and M invariant.

A subspace lattice $\mathcal{L}$ on X is *distributive* if $K \cap (L \vee M) = (K \cap L) \vee (K \cap M)$ holds identically in $\mathcal{L}$. Complete distributivity is a very much stronger condition than distributivity, though still purely lattice–theoretic. We need not be concerned with the actual definition here. For present purposes it is more than enough to note the following characterization of it.

THEOREM. (see [11]) *Let* $\mathcal{L}$ *be a subspace lattice on* X. *The following are equivalent.*

(1) $\mathcal{L}$ *is completely distributive,*

(2) *for every* $L \in \mathcal{L}$, $L = \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$,

(3) *for every* $L \in \mathcal{L}$, $L = \cap\{M_- : M \in \mathcal{L} :$ *and* $M \not\subseteq L\}$.

The Alg of a completely distributive subspace lattice $\mathcal{L}$ is rich in rank one operators. So many are there that in fact $\mathcal{L} = \mathrm{Lat}\,\mathcal{R}$ where $\mathcal{R}$ is the set of rank one operators of Alg $\mathcal{L}$ [11] (see also [5]). We have the following necessary condition for SRO.

THEOREM. (W.E. Longstaff, 1976 [12], M.S. Lambrou, 1977 [5, 7]) *Let* $\mathcal{L}$ *be a subspace lattice on* X. *If* $\mathcal{L}$ *has* SRO, *then* $\mathcal{L}$ *is completely distributive.*

Proof. Suppose that $\mathcal{L}$ has SRO. By the preceding theorem, it is enough to show that $L \subseteq \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$, for every $L \in \mathcal{L}$ since, by the definition of M_-, the reverse inclusion is obvious.

Let $L \in \mathcal{L}$. We show that every rank one operator R of Alg $\mathcal{L}$ maps L into $\vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$.

By the earlier lemma, $R = f^* \otimes e$ where $e \in K$ and $f^* \in (K_-)^\perp$ for some $K \in \mathcal{L}$. For this K, if $RL \neq (0)$, then $L \not\subseteq K_-$ and $RL \subseteq K$. From this,

$$RL \subseteq \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}.$$

Now let $x \in L$ be arbitrary. Since $\mathcal{L}$ has SRO, for every $\varepsilon > 0$ there exists a finite sum $F = \Sigma R_\alpha$ of operators $R_\alpha \in$ Alg $\mathcal{L}$ each of rank at most one such that $\|x - Fx\| < \varepsilon$. But, by what we've just proved, $Fx \in \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$. Hence $x \in \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$ and $L \subseteq \vee\{M \in \mathcal{L} : L \not\subseteq M_-\}$ as required.

A subspace lattice $\mathcal{L}$ on H is called *commutative*, if the orthogonal projections onto its members pairwise commute. For example, every nest is commutative. For commutative subspace lattices on separable Hilbert space the converse of the preceding theorem is also true.

THEOREM. (C. Laurie and W.E. Longstaff, 1983 [10]) *Every completely distributive commutative subspace lattice on complex separable Hilbert space has* SRO.

Let $\mathcal{L}$ be a subspace lattice on X. An element $K \in \mathcal{L}$ is called an *atom* of $\mathcal{L}$ if $(0) \subseteq M \subset K$ and $M \in \mathcal{L}$ implies that $M = (0)$. We say that $\mathcal{L}$ is *atomic* if every non-zero element of $\mathcal{L}$ contains an atom and is the closed linear span of the atoms it contains. We say that $\mathcal{L}$ is *complemented* if for every $L \in \mathcal{L}$, there exists $L' \in \mathcal{L}$ such that $L \cap L' = (0)$ and $L \vee L' = X$. It is easily shown that, if $\mathcal{L}$ is distributive, then for every $L \in \mathcal{L}$ there is at most one element $L' \in \mathcal{L}$ satisfying $L \cap L' = (0)$ and

$L \vee L' = X$. By definition, an *atomic Boolean subspace lattice*, abbreviated ABSL, is an atomic, complemented and distributive subspace lattice. ABSL's are extreme opposites to nests, from a partial order point of view. Nevertheless every nest and every ABSL is completely distributive. The only ABSL on X with one atom is $\{(0), X\}$. Every ABSL on X with precisely two atoms is of the form $\{(0), K, L, X\}$ where K and L are non-trivial complementary subspaces (that is, $K \cap L = (0)$ and $K \vee L = X$). In any ABSL $\mathcal{L}$ we have $K_- = K'$ (the unique complement of K), for every atom K of $\mathcal{L}$ [11].

PROBLEM. Does every ABSL on X have SRO?

This problem, when restricted to ABSL's on X with one-dimensional atoms, turns out to be (equivalent to) a well-known unsolved problem in the theory of bases (see [1]), namely (in the terminology of W.H. Ruckle [15]): Is every 1-series summable M-basis of X finitely series summable?

The general 1-atom case presents little difficulty; the finite-rank operators are dense in $\mathcal{B}(X) = \mathrm{Alg}\{(0), X\}$ in the strong operator topology. The 2-atom case has been solved only recently; every ABSL on X with precisely two atoms has SRO [1]. The n-atom case ($n \geq 3$) is still unsolved. Some partial solutions to the general problem are given in [1], including the following.

THEOREM. *Let* $\mathcal{L}$ *be an* ABSL *on* X . *For every finite set of vectors* $x_1, x_2, \ldots, x_m$ *of* X *each belonging to the linear span of the set of atoms of* $\mathcal{L}$ *(which is dense in* X *) there exists a finite sum* $F = \Sigma R_\alpha$ *of operators* $R_\alpha \in \mathrm{Alg}\,\mathcal{L}$, *each of rank at most one, such that* $x_j = Fx_j$, *for* $j = 1,2,\ldots,m$.

In actual fact, in [2] J.A. Erdos proved a stronger result than the one mentioned at the beginning of this section.

THEOREM. (J.A. Erdos, 1968 [2]) *For every nest* $\mathcal{N}$ *on* H *the unit ball of the subalgebra of* Alg $\mathcal{N}$ *generated by the rank one operators is dense in the unit ball of* Alg $\mathcal{N}$ *in the strong operator topology.*

DEFINITION. If $\mathcal{L}$ is a subspace lattice on X, say that $\mathcal{L}$ has the *metric strong rank one density property*, abbreviated metric SRO, if the unit ball of the subalgebra of Alg $\mathcal{L}$ generated by the rank one operators is dense in the unit ball of Alg $\mathcal{L}$ in the strong operator topology.

Clearly, the subspace lattice $\mathcal{L}$ on X has metric SRO if and only if

> for every $\varepsilon > 0$ and for every finite set of vectors $x_1, x_2, \ldots, x_m$ of X, there exists a finite sum $F = \Sigma R_\alpha$, where each R_α is an operator of rank at most one of Alg $\mathcal{L}$, such that $\|F\| \le 1$ and $\|x_j - Fx_j\| < \varepsilon$, for $j = 1,2,\ldots,m$.

If some subspace lattice on X has metric SRO, then X obviously has the metric approximation property (in the sense of A. Grothendieck).

PROBLEM. Which (necessarily completely distributive) subspace lattices, on Banach spaces with the metric approximation property, have metric SRO? Which ABSL's on such spaces have it?

On separable Hilbert space we have the following result. Note that, once again, the 1–atom case presents little difficulty. (If H is separable, the Alg of the ABSL $\{(0), H\}$ is $\mathcal{B}(H)$ and the unit ball of the algebra of all finite rank operators on H is dense in the unit ball of $\mathcal{B}(H)$ in the strong operator topology.)

THEOREM. (see [1]) *Every* ABSL *with precisely two atoms on complex separable Hilbert space has metric* SRO.

The proof of the above theorem given in [1] uses a result of K.J. Harrison and is surprisingly deep. An outline is as follows. If $\mathcal{L}$ is an ABSL on H it is not very difficult to show that the following are equivalent.

(1) Alg $\mathcal{L}$ is self-adjoint,

(2) $\mathcal{L}$ is commutative,

(3) $K \cap (K')^{\perp} = K$, for every atom K of $\mathcal{L}$ (where K' is the complement of K in $\mathcal{L}$).

Thus the condition

(G) $K \cap (K')^{\perp} = (0)$, for every atom K of $\mathcal{L}$,

is extremely non-commutative, and non-self-adjoint. It can be shown that on a complex separable Hilbert space every ABSL has metric SRO if and only if every ABSL satisfying condition (G) does.

One way of obtaining an example of a 2-atom ABSL satisfying (G) on complex separable Hilbert space is as follows. Let H be separable and let $T \in \mathcal{B}(H)$ be a positive contraction satisfying $\ker T = \ker(I-T) = (0)$ (for example, $T = \frac{1}{2} I$). On $H \oplus H$ let $\mathcal{L}_T$ be the ABSL given by $\mathcal{L}_T = \{(0), \mathcal{G}(T), \mathcal{G}(-T), H \oplus H\}$, where for any operator $A \in \mathcal{B}(H)$, $\mathcal{G}(A) = \{(x, Ax) : x \in H\}$ denotes the graph of A. Verification of (G) is a routine exercise. For example, $(\mathcal{G}(T)')^{\perp} = \mathcal{G}(-T)^{\perp} = \{(Tx,x) : x \in H\}$. Thus if

$$(y,Ty) \in \mathcal{G}(T) \cap (\mathcal{G}(T)')^{\perp},$$

then $y = T^2 y$, and $(I-T)(I+T)y = 0$ gives $y = 0$.

In fact, P.R. Halmos [4] has shown that, up to unitary equivalence, the above way is the only way of obtaining such an example. By using the spectral theorem, applied to T, K.J. Harrison has proved that $\mathcal{L}_T$ has metric SRO. The n-atom case ($n \geq 3$) is still unsolved.

PROBLEM. For ABSL's on separable Hilbert space, with precisely n-atoms ($n \geq 3$) and satisfying condition (G), is there a representation theorem analogous to Halmos' for the case $n = 2$?

Next we discuss some automatic continuity results for reflexive algebras.

Before we come to these, we note the following result which characterizes ABSL's within the class of completely distributive subspace lattices.

THEOREM. (M.S. Lambrou, 1977 [5, 6, 8]) *Let* $\mathcal{L}$ *be a completely distributive subspace lattice on* X *and put* $\mathcal{J} = \{K \in \mathcal{L} : K \neq (0)$ *and* $K_{\_} \neq X\}$. *The following are equivalent.*

(1) Alg $\mathcal{L}$ *is semi–simple,*

(2) Alg $\mathcal{L}$ *is semi–prime,*

(3) $\mathcal{L}$ *is an* ABSL,

(4) *for every* $K \in \mathcal{J}$, $K \cap K_{\_} = (0)$.

Recall that a complex unital Banach algebra $\mathcal{A}$ is *semi–simple* if and only if it has no non–zero left ideals (or no non–zero right ideals) consisting entirely of quasinilpotent elements. Also, $\mathcal{A}$ is *semi–prime* if and only if it has no non–zero left ideal whose square is zero. Obviously, semi–simple Banach algebras are semi–prime.

Incidentally, the reader may be interested in the following result which is in the same vein as its predecessor.

THEOREM. (M.S. Lambrou and W.E. Longstaff, 1980 [9]) *Let* $\mathcal{L}$ *be a completely distributive subspace lattice on* X . *The following are equivalent.*

(1) Alg $\mathcal{L}$ *is abelian,*

(2) $\mathcal{L}$ *is an* ABSL *with* 1*–dimensional atoms.*

The proof given in [9] is for Hilbert spaces but, with minor modifications, it serves for Banach spaces as well (another proof is given in [7]).

THEOREM. (J.R. Ringrose, 1966 [14]) *For* $j = 1,2$ *let* $\mathcal{N}_j$ *be a nest on the complex Hilbert space* H_j *and let* $\varphi : \mathrm{Alg}\,\mathcal{N}_1 \longrightarrow \mathrm{Alg}\,\mathcal{N}_2$ *be an algebraic isomorphism. Then* φ *is spatial in the sense that there exists a bicontinuous linear bijection* $T : H_1 \longrightarrow H_2$ *such that* $\varphi(A) = TAT^{-1}$ *, for every* $A \in \mathrm{Alg}\,\mathcal{N}_1$ *. In particular* φ *is continuous.*

Among nests only the trivial nest $\{(0), X\}$ has semi-simple Alg. So the (automatic) continuity of the map φ in the statement of the above theorem seldom follows from B.E. Johnson's well-known theorem about epimorphisms onto semi-simple Banach algebras. Now nests are completely distributive and so are ABSL's; moreover, the Alg of an ABSL is always semi-simple. Thus any algebraic isomorphism $\varphi : \mathrm{Alg}\,\mathcal{L}_1 \longrightarrow \mathrm{Alg}\,\mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are ABSL's, will be automatically continuous by Johnson's theorem. Even more is true in these circumstances.

THEOREM. (M.S. Lambrou, 1977 [5, 8]) *For* $j = 1,2$ *let* $\mathcal{L}_j$ *be an* ABSL *on the complex Banach space* X_j *and let* $\mathcal{A}_j$ *be a closed subalgebra of* $\mathrm{Alg}\,\mathcal{L}_j$ *containing every rank one operator of* $\mathrm{Alg}\,\mathcal{L}_j$ *. If* $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ *is an algebraic monomorphism whose range contains every rank one operator of* $\mathrm{Alg}\,\mathcal{L}_2$ *, then* φ *is quasi-spatial. In particular* φ *is continuous.*

Here, by 'φ is quasi-spatial' we mean the following. There exists a closed linear densely defined injective mapping $T : \mathrm{Dom}\,T(\subseteq X_1) \longrightarrow \mathrm{Ran}\,T(\subseteq X_2)$ with dense range and with $\mathrm{Dom}\,T$ an invariant linear manifold of $\mathcal{A}_1$ such that $\varphi(A)y = TAT^{-1}y$, for every $A \in \mathcal{A}_1$ and every $y \in \mathrm{Ran}\,T$. Under these circumstances we say that T *implements* φ . In the preceding theorem 'quasi-spatial' cannot be replaced with 'spatial' (in the sense described in Ringrose's theorem). In fact, there are ABSL's $\mathcal{L}_1$ and $\mathcal{L}_2$ on complex Banach spaces X_1 and X_2 respectively, each with 1-dimensional atoms (so with abelian Alg's) and an algebraic isomorphism $\varphi : \mathrm{Alg}\,\mathcal{L}_1 \longrightarrow \mathrm{Alg}\,\mathcal{L}_2$ which is not implemented by any continuous T.

Now nests, on Hilbert spaces, are also commutative subspace lattices. For commutative subspace lattices we have the following result.

THEOREM. (F. Gilfeather and R.L. Moore, 1986 [3]) *For* $j = 1,2$ *let* $\mathcal{L}_j$ *be a commutative subspace lattice on a complex separable Hilbert space* H *and let* $\varphi : \mathrm{Alg}\,\mathcal{L}_1 \longrightarrow \mathrm{Alg}\,\mathcal{L}_2$ *be an algebraic isomorphism. Then* φ *is continuous.*

Such a map φ, as in the statement of the preceding theorem, can fail to be quasi–spatial, even if both $\mathcal{L}_1$ and $\mathcal{L}_2$ are also completely distributive.

Finally, we mention two problems related to von Neumann's celebrated double commutant theorem.

PROBLEM. Which ABSL's $\mathcal{L}$ on X have the double commutant property: $(\mathrm{Alg}\,\mathcal{L})'' = \mathrm{Alg}\,\mathcal{L}$?

By von Neumann's theorem every commutative ABSL on H has the double commutant property (its Alg is a von Neumann algebra). Let $\mathcal{L}$ be an ABSL on X. By a result of M.S. Lambrou [7], $(\mathrm{Alg}\,\mathcal{L})'' = \mathrm{Alg}\,\mathcal{M}$ for a unique ABSL $\mathcal{M}$ on X. Moreover, each atom of $\mathcal{M}$ is a closed linear span of atoms of $\mathcal{L}$. Thus the atoms of $\mathcal{L}$ 'coalesce' in a certain way to form the atoms of $\mathcal{M}$ on passing to the double commutant.

PROBLEM. Given two atoms K and L of $\mathcal{L}$ what determines whether or not they will coalesce (that is, be or not be contained in the same atom of $\mathcal{M}$)?

This question has an algebraic answer, which we will describe briefly, but what is sought is a geometric one. The condition $T \in (\mathrm{Alg}\,\mathcal{L})'$ is very strong. In particular, it implies that T acts like a scalar on each atom of $\mathcal{L}$, that is, it implies that for every atom $K \in \mathcal{L}$ there exists a scalar λ such that $T|_K = \lambda I$. In [7] it is shown that the atoms K and L of $\mathcal{L}$ coalesce if and only if, for every operator $T \in (\mathrm{Alg}\,\mathcal{L})'$, the equations $T|_K = \lambda I$ and $T|_L = \lambda I$ hold simultaneously.

For example, if $K+L$ is not closed then K and L coalesce. Indeed, in this case, let $T \in (\mathrm{Alg}\,\mathcal{L})'$ and let $T|_K = \lambda I$ and $T|_L = \mu I$ for λ,μ scalars. Choose a vector $z \in (K \vee L)\backslash(K+L)$. Then there exist sequences (x_n) and (y_n), of vectors of K and L respectively, such that $x_n + y_n \longrightarrow z$. Applying T gives $\lambda x_n + \mu y_n \longrightarrow Tz$. But $\lambda x_n + \lambda y_n \longrightarrow \lambda z$ and $\mu x_n + \mu y_n \longrightarrow \mu z$. Subtraction gives $(\lambda-\mu)x_n \longrightarrow (T-\mu I)z$ and $(\mu-\lambda)y_n \longrightarrow (T-\lambda I)z$. From this, $(T-\mu I)z \in K$ and $(T-\lambda I)z \in L$ so $(\lambda-\mu)z = (T-\mu I)z - (T-\lambda I)z \in K+L$. Since $z \notin K+L$, $\lambda = \mu$. Thus K and L coalesce.

On the other hand, if K is an atom of $\mathcal{L}$ and $K+K'$ is closed, then K does not coalesce with any other atom. For, let Q be the projection onto K along K' in this case. Then $Q \in (\mathrm{Alg}\,\mathcal{L})'$. If L is an atom of $\mathcal{L}$ different from K, then $L \subseteq K'$, so $Q|_K = I$ and $Q|_L = 0$. Thus K and L don't coalesce.

It can happen that two atoms of $\mathcal{L}$ have a closed vector sum, yet nevertheless, they coalesce (an example is given in [1]).

THEOREM. (see [1]) *Let $\mathcal{L}$ be an* ABSL *on* X. *If* $K+K'$ *is a closed vector sum, for every atom* K *of* $\mathcal{L}$, *then* $\mathcal{L}$ *has the double commutant property.*

Thus for example every ABSL with finite–dimensional atoms has the double commutant property. The converse of the above theorem holds for finite ABSL's but not in general. In fact, there exists an ABSL $\mathcal{L}$ on separable Hilbert space with $L+L'$ not closed, for every non–trivial $L \in \mathcal{L}$, yet nevertheless having the double commutant property (see [1]).

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