INDEFINITE INTEGRALS OF FUNCTIONS FROM $L^p(\mathbb{R})$

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ABSTRACT. In this paper we seek conditions under which the indefinite integrals of a function φ from $L^p(\mathbb{R})$ belong to $L^p(\mathbb{R})+\mathbb{C}$. We prove that if the spectrum $sp(\varphi)$ of φ is isolated from zero, then it is improperly integrable for $(1 \leq p < \infty)$ and its indefinite integrals belong to $L^p(\mathbb{R})+\mathbb{C}$. Also, we give applications to the differential equation $u'(x)+\lambda u(x)=\varphi(x)$.

§1. Introduction. Let $\varphi \in L^p(\mathbb{R})$ for some $1 \leq p \leq \infty$ and define

(1.1) $P\varphi(x)=\int_0^x \varphi(t)\,dt,\ x \in \mathbb{R}$.

We seek conditions under which $P\varphi \in L^p(\mathbb{R})+\mathbb{C}$. Similar problems have been studied extensively when φ belongs instead to certain classes of functions of almost periodic type. (See [1], [3], [4], [5], [7], [8], [10], [13], [16].) In particular, let $AP(\mathbb{R})$ denote the Banach space of complex-valued almost periodic functions defined on $\mathbb{R}$. Bohl-Bohr [5, p.58] proved that if $\varphi \in AP(\mathbb{R})$ and $P\varphi$ is bounded then $P\varphi \in AP(\mathbb{R})$. More generally, let X be a Banach space and $AP(\mathbb{R},X)$ the Banach space of X-valued almost periodic functions on $\mathbb{R}$. If $\varphi \in AP(\mathbb{R},X)$ and $P\varphi$ is bounded then $P\varphi$ does not necessarily belong to $AP(\mathbb{R},X)$. However, Kadets [10] showed that if X does not contain a subspace isomorphic to the Banach space c_0 then again $P\varphi \in AP(\mathbb{R},X)$.

Now let $C_{ub}(\mathbb{R},X)$ denote the space of uniformly continuous bounded functions from $\mathbb{R}$ to X, and recall that a function $\varphi \in C_{ub}(\mathbb{R},X)$ is called ergodic if there exists $a \in X$ such that $||lim_{T\to\infty} \sup_{x\in\mathbb{R}}||\frac{1}{2T}\int_{-T}^{T}[\varphi(t+x)-a]\,dt||=0$. Also let Λ be a closed translation invariant subspace of $C_{ub}(\mathbb{R},X)$. Basit [1] recently proved that if $\varphi \in \Lambda$ and if $P\varphi$ is

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A46 secondary 34K15.
Key words and phrases. Indefinite integral, spectrum and differential equations.

bounded and ergodic, then $P\varphi \in \Lambda$. This result is not true for $\Lambda = L^1(\mathbb{R}) + \mathbb{C}$. Indeed, consider the function defined by

(1.2) $\varphi(x) = x$ for $|x| \leq 1$ and $\varphi(x) = \frac{sign(x)}{x^2}$ for $|x| > 1$.

Then $\varphi \in L^1(\mathbb{R})$ and $P\varphi \in C_0(\mathbb{R}) + \mathbb{C}$. In particular $P\varphi$ is bounded and ergodic, yet $P\varphi \notin L^1(\mathbb{R}) + \mathbb{C}$.

In this paper we consider functions $\varphi \in L^p(\mathbb{R})$ and replace assumptions concerning $P\varphi$ by conditions on the spectrum of φ in order to conclude $P\varphi \in L^p(\mathbb{R}) + \mathbb{C}$. Spectra are defined in section 2 and the main result appears in section 3. In section 4 we discuss derivatives in place of indefinite integrals, and in section 5 we provide an application to differential equations.

§2. **Spectra.** Following Reiter[14,p.83] we call a function $w \in L^\infty_{loc}(\mathbb{R})$ a weight function on $\mathbb{R}$ if

(2.1) $w(x) \geq 1$ for all $x \in \mathbb{R}$, and

(2.2) $w(x + y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}$.

A weight function is symmetric if

(2.3) $w(x) = w(-x)$ for all $x \in \mathbb{R}$.

An important additional condition satisfied by many weights is the Beurling condition

(2.4) $\sum_{m=1}^{\infty} \frac{\log w(mx)}{m^2} < \infty$ for all $x \in \mathbb{R}$.

Given a weight w on $\mathbb{R}$ we define

(2.5) $L^1_w(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \|f\|_{1,w} = \int_{\mathbb{R}} |f(x)|w(x)\,dx < \infty\}$.

Then $L^1_w(\mathbb{R})$ is a subalgebra of $L^1(\mathbb{R})$ which is a Banach algebra under the norm $\|.\|_{1,w}$. The Banach space dual of $L^1_w(\mathbb{R})$ is

(2.6) $L^\infty_w(\mathbb{R}) = \{\varphi \in L^\infty_{loc}(\mathbb{R}) : \|\varphi\|_{\infty,w} = \operatorname{ess\,sup}_{x\in\mathbb{R}} \frac{|\varphi(x)|}{w(x)} < \infty\}$.

If w satisfies the Beurling condition, then the space of Fourier transforms $\hat{f}$ of functions

$f \in L^1_w(\mathbb{R})$ is a Wiener algebra. See Reiter [14, p.132 and p.19 remark]. In particular :

Lemma 2.1. *Let w be a weight function on $\mathbb{R}$ satisfying the Beurling condition. Given a neighbourhood V of a compact set W in $\mathbb{R}$, there exists $f \in L^1_w(\mathbb{R})$ such that $\hat{f} = 1$ on W and supp $\hat{f} \subset V$.*

If w is a symmetric weight function on $\mathbb{R}$, then for $f \in L^1_w(\mathbb{R})$ and $\varphi \in L^\infty_w(\mathbb{R})$, the convolution

(2.7) $f * \varphi(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t)\,dt$

is defined for almost every $x \in \mathbb{R}$ and $|f * \varphi(x)| \leq w(x)\|f\|_{1,w}\|\varphi\|_{\infty,w}$, a.e. For such w and φ we define a closed ideal of $L^1_w(\mathbb{R})$ by

(2.8) $I_w(\varphi) = \{f \in L^1_w(\mathbb{R}) : f * \varphi = 0, \text{ (a.e.) }\}$

and the w-spectrum of φ by

(2.9) $sp_w(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0 \text{ for all } f \in I_w(\varphi)\}$.

Since $\hat{f} \in C_0(\mathbb{R})$ for each $f \in L^1(\mathbb{R})$, the w-spectrum $sp_w(\varphi)$ is closed. For a list of further properties, see [2].

We also require a notion of spectrum for functions $\varphi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. For such φ, $f * \varphi \in L^p(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$ (see [9, corollary 20.14]) and hence we can define

(2.10) $I(\varphi) = \{f \in L^1(\mathbb{R}) : f * \varphi = 0\}$

and the spectrum of φ by

(2.11) $sp(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0 \text{ for all } f \in I(\varphi)\}$.

Once again $sp(\varphi)$ is a closed subset of $\mathbb{R}$.

Proposition 2.2. *Let $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$.*

(a) If $f \in L^1(\mathbb{R})$ then $f * \varphi \in L^p(\mathbb{R})$ and $sp(f * \varphi) \subset supp\hat{f} \cap sp(\varphi)$.

(b) $sp(\varphi) = \emptyset$ if and only if $\varphi = 0$.

Proof. (a) By [9, corollary 20.14] we have $f * \varphi \in L^p(\mathbb{R})$. Clearly $I(\varphi) \subset I(f * \varphi)$ and so

$sp(f * \varphi) \subset sp(\varphi)$. Finally, suppose $\lambda \in \mathbb{R} \setminus \operatorname{supp} \hat{f}$. By lemma 2.1, with $w = 1$, there exists $g \in L^1(\mathbb{R})$ such that $\hat{g} = 0$ on a neighbourhood of $\operatorname{supp} \hat{f}$ and $\hat{g}(\lambda) = 1$. As $\hat{g}\hat{f} = 0$, so $f * g = 0$ and therefore $g \in I(f * \varphi)$. But $\hat{g}(\lambda) \neq 0$ so $\lambda \notin sp(f * \varphi)$.

(b) If $\varphi = 0$ then $I(\varphi) = L^1(\mathbb{R})$ and $sp(\varphi) = \emptyset$. Conversely, if $sp(\varphi) = \emptyset$ then $f * \varphi = 0$ for all $f \in L^1(\mathbb{R})$. If $p = 1$, $f = \varphi$ gives $\hat{\varphi}^2 = 0$ and hence $\varphi = 0$. If $1 < p \leq \infty$ then $f * \varphi = 0$ for all $f \in C_c(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support. Since $C_c(\mathbb{R})$ is dense in $L^q(\mathbb{R})$, where $1/p + 1/q = 1$, and the mapping $f \to f * \varphi$ is continuous from $L^q(\mathbb{R})$ to $C_0(\mathbb{R})$, we conclude $f * \varphi = 0$ for all $f \in L^q(\mathbb{R})$. So $\int_{-\infty}^{\infty} f(x-t)\varphi(t)\,dt = 0$ for all $x \in \mathbb{R}$ and $f \in L^q(\mathbb{R})$. Taking $x = 0$ and applying the Hahn-Banach theorem we conclude that $\varphi = 0$.

Let $S(\mathbb{R})$ be the Schwartz space of rapidly decreasing infinitely differentiable complex-valued functions on $\mathbb{R}$. Let $S'(\mathbb{R})$ be the dual space of tempered distributions. If $\varphi \in L^p(\mathbb{R})$, then $T_\varphi(f) = \int_{\mathbb{R}} f(t)\varphi(t)\,dt$ for $f \in S(\mathbb{R})$, defines a distribution $T_\varphi \in S'(\mathbb{R})$. So $\widehat{T}_\varphi(g) = T_\varphi(\hat{g})$ for $g \in S(\mathbb{R})$, defines the Fourier transform $\widehat{T}_\varphi$ of T_φ. (See [17, p.146-152]).

Proposition 2.3. *Let $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$. Then $sp(\varphi) = \operatorname{supp} \widehat{T}_\varphi$.*

The proof is essentially the same as for [2, proposition 4.1]

§3. **Indefinite integrals.** Let $C_u(\mathbb{R})$ and $C_{ub}(\mathbb{R})$ denote respectively the spaces of uniformly continuous and uniformly continuous bounded functions on $\mathbb{R}$. To study indefinite integrals, we use the weight

(3.1) $w(x) = 1 + |x|$, $x \in \mathbb{R}$.

It is readily seen that w is a symmetric weight function satisfying condition (2.4).

Proposition 3.1. *If $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$ and w is given by (3.1), then $P\varphi \in C_u(\mathbb{R}) \cap L^\infty_w(\mathbb{R})$. Moreover,*

(3.2) $sp(\varphi) \subset sp_w(P\varphi) \subset sp(\varphi) \cup \{0\}$.

Proof. If $p = 1$, it is well-known that $P\varphi$ is absolutely continuous and hence uniformly continuous. For arbitrary p, and $x, h \in \mathbb{R}$, $|P\varphi(x+h) - P\varphi(x)| = |\int_0^h \varphi(x+t)\,dt| \le |h|^{1-1/p}\|\varphi\|_p$ showing that $P\varphi \in C_u(\mathbb{R})$. Moreover, $|P\varphi(x)| = |\int_0^x \varphi(t)\,dt| \le |x|^{1-1/p}\|\varphi\|_p$, showing that $P\varphi \in L_w^\infty(\mathbb{R})$. For $p = \infty$, (3.2) is given in [2, proposition 4.4]. The proof of (3.2) for other p is essentially the same.

Proposition 3.2. *If $\varphi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ and $0 \notin sp(\varphi)$, then $P\varphi \in C_{ub}(\mathbb{R})$.*

Proof. Since $0 \notin sp(\varphi)$ there exists a neighbourhood $V = [-\delta, \delta]$ such that $sp_w(\varphi) \cap V = \emptyset$. Let $w(x) = 1 + |x|$. By lemma 2.1 there is a function $h \in L_w^1(\mathbb{R})$ such that $\hat{h} = 1$ for $|\lambda| \le \delta/4$ and $\hat{h} = 0$ for $|\lambda| \ge \delta/3$. By proposition 2.2, $h * \varphi = 0$. Similarly, by [2, proposition 3.12] and (3.2) $sp_w(h * P\varphi) \subset \text{supp } \hat{h} \cap sp_w(P\varphi) \subset \{0\}$. Since $\frac{d(h*P\varphi)}{dx} = h * \varphi = 0$ for all $x \in \mathbb{R}$, we conclude that $h * P\varphi$=c, a constant. If $\eta = P\varphi - c$ then $0 \notin sp_w(\eta)$. Indeed, $h * \eta = h * P\varphi - h * c = c - c = 0$. Thus $h \in I_w(\eta)$ and $\hat{h}(0) = 1$, showing $0 \notin sp_w(\eta)$. By proposition 3.1, $\eta \in C_u(\mathbb{R})$ and so by [2, theorem 9.5], η is bounded and so is $P\varphi$. This proves that $P\varphi \in C_{ub}(\mathbb{R})$.

Proposition 3.3. *If $\varphi \in L^p(\mathbb{R}) \cap C_u(\mathbb{R})$ where $1 \le p \le \infty$, then $\varphi \in C_0(\mathbb{R})$.*

Proof. Assume on the contrary that $\limsup_{t\to\infty} |\varphi(t)| \ge 3c > 0$. Choose a sequence $\{t_n\} \subset \mathbb{R}$ such that $t_{n+1} > 2 + t_n$ and $|\varphi(t_n)| \ge 2c$ for all $n \in \mathbb{N}$. Since φ is uniformly continuous, there exists $0 < \delta < 1$ such that $|\varphi(t)| \ge c$ whenever $|t_n - t| \le \delta$ for some $n \in N$. Hence $\int_{-\infty}^{\infty} |\varphi(t)|^p\,dt \ge \lim_{n\to\infty} 2nc^p\delta = \infty$, a contradiction.

Theorem 3.4. *If $\varphi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ and $0 \notin sp(\varphi)$, then $a + P\varphi \in L^p(\mathbb{R})$ for some $a \in \mathbb{C}$. If moreover $p < \infty$ then φ is improperly integrable and $a = -\lim_{|T|\to\infty} \int_0^T \varphi(t)\,dt$.*

Proof. For $p = \infty$ the result is contained in proposition 3.2. So assume $1 \le p < \infty$. Let

$\alpha_h(x) = P\varphi(x+h) - P\varphi(x) = \int_0^h \varphi(x+t)\,dt = \chi_{-h} * \varphi(x)$, where χ_{-h} is the characteristic function of the interval [-h,0] if $h \geq 0$, [0,-h] if $h < 0$. As each $\alpha_h \in L^p(\mathbb{R})$, we may define the function $\alpha(h) = \|\alpha_h\|_p$, $h \in \mathbb{R}$. It is easy to verify that α is a continuous function on $\mathbb{R}$ satisfying the property $\alpha(h_1 + h_2) \leq \alpha(h_1) + \alpha(h_2)$ for all $h_1, h_2 \in \mathbb{R}$. Therefore $\omega(h) = 1 + \alpha(h) + \alpha(-h)$, $h \in \mathbb{R}$ defines a symmetric weight function satisfying the Beurling condition (2.4). Choose $\delta > 0$ such that $[-\delta, \delta] \cap sp(\varphi) = \emptyset$. By lemma 2.1, there exists $f \in L^1_w(\mathbb{R})$ such that $\hat{f}(\lambda) = 1$ for $|\lambda| \leq \delta/4$ and $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \delta/3$. By proposition 3.2, $P\varphi \in C_{ub}(\mathbb{R})$ and so $f * P\varphi$ is defined and also belongs to $C_{ub}(\mathbb{R})$. Moreover, $\frac{d(f*P\varphi)}{dx} = f * \varphi = 0$, so $f * P\varphi = -a$ where $a \in \mathbb{C}$.

Next consider $a + P\varphi(x) = \int_{-\infty}^{\infty}[P\varphi(x) - P\varphi(x-t)]f(t)\,dt = -\int_{-\infty}^{\infty} \alpha_{-t}(x)f(t)\,dt$. We have $\|\alpha_{-t}\|_p = \alpha(-t) \leq \omega(t)$ and since $f \in L^1_w(\mathbb{R})$, $w|f| \in L^1(\mathbb{R})$. The function $t \to \psi(t) = \alpha_{-t}f(t) : \mathbb{R} \to L^p(\mathbb{R})$ is weakly measurable and its range is separable, as $1 \leq p < \infty$. Hence ψ is strongly measurable ([17, p.131]). As the function $t \to \|\psi(t)\|_p$ is integrable, Bochner's theorem [17, p.133] yields that ψ is Lebesgue-Bochner integrable and its integral is an element of $L^p(\mathbb{R})$. So $a + P\varphi \in L^p(\mathbb{R})$. Finally, by proposition 3.3, $a + P\varphi \in L^p(\mathbb{R}) \cap C_{ub}(\mathbb{R}) \subset C_0(\mathbb{R})$. Hence $\lim_{|T|\to\infty} \int_0^T \varphi(t)\,dt = -a$.

Corollary 3.5. *Let $\varphi \in L^p(\mathbb{R})$. Then $f * P\varphi \in L^p(\mathbb{R}) + \mathbb{C}$ for each $f \in L^1(\mathbb{R})$ with $0 \notin supp\ \hat{f} = sp(f)$.*

Proof. By theorem 3.4 the function Kf defined by $Kf(x) = \int_{-\infty}^{x} f(t)\,dt$, belongs to $L^1(\mathbb{R})$. By [9, corollary 20.14], we conclude $\varphi * Kf \in L^p(\mathbb{R})$. Since $(f * P\varphi)' = (Kf * \varphi)'$, there exists $a \in \mathbb{C}$ such that $f * P\varphi = a + Kf * \varphi \in L^p(\mathbb{R}) + \mathbb{C}$.

Remark 3.6. *Let X be a Banach space and $L^1(\mathbb{R}, X)$ the space of Lebesgue-Bochner integrable X-valued functions on $\mathbb{R}$. Define $L^p(\mathbb{R}, X)$ similarly for $1 < p \leq \infty$. Then $sp(\varphi)$ for $\varphi \in L^p(X, \mathbb{R})$ is again defined by (2.10) and (2.11). Theorem 3.4 remains true in this* *more general setting.*

§4. **Derivatives.** In this section we briefly consider derivatives in place of indefinite integrals.

Theorem 4.1. *Let $\varphi \in L^1(\mathbb{R})$. If $\varphi' \in L^\infty(\mathbb{R})$ then φ' is improperly integrable, and if $\varphi' \in C_{ub}(\mathbb{R})$ then $\varphi' \in C_0(\mathbb{R})$.*

Proof. Of course $\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)\,dt$. If $\varphi' \in L^\infty(\mathbb{R})$ then $\varphi \in C_{ub}(\mathbb{R})$ and by proposition 3.3, $\varphi \in C_0(\mathbb{R})$. Hence $\lim_{|x|\to\infty} \int_0^x \varphi'(t)\,dt = -\varphi(0)$. If $\varphi' \in C_{ub}(\mathbb{R})$, then $n[\varphi(x+1/n) - \varphi(x)] = n\int_0^{1/n} \varphi'(t+x)\,dt = \varphi'(x+\theta/n)$ for some $\theta = \theta(x,n)$, $0 < \theta < 1$. Hence $\lim_{|x|\to\infty} |\varphi'(x)| = 0$.

Remark 4.2. *It can happen that $\varphi \in L^1(\mathbb{R})$ and $\varphi' \in C_{ub}(\mathbb{R})$ yet $\varphi' \notin L^1(\mathbb{R})$. For example, let $\varphi(x) = \sum_{n=4}^{\infty} n[(x-n)^2 - 1/n]^2 g_n(x)$, where g_n is the characteristic function of the interval $I_n = [n - 1/n^{1/2}, n + 1/n^{1/2}]$. Then $\varphi, \varphi' \in C_0(\mathbb{R})$ with $\varphi \in L^1(\mathbb{R})$ and $\varphi' \notin L^1(\mathbb{R})$*

§5. **Application to a differential equation.** Consider the following differential equation.

(5.1) $u'(x) + \lambda u(x) = \varphi(x), x \in \mathbb{R}$.

Given $\lambda \in \mathbb{C}$ and $\varphi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$, we seek solutions $u \in L^p(\mathbb{R})$. The general solution of (5.1) is

(5.2) $u(x) = e^{-\lambda x}[c + \int_0^x e^{\lambda t}\varphi(t)\,dt]$,

where c is a constant. When $\mathcal{R}e(\lambda) \neq 0$ it is easy to see that (5.1) has a unique solution $u \in L^p(\mathbb{R})$ given by

(5.3) $u(x) = \int_{-\infty}^x e^{-\lambda(x-t)}\varphi(t)\,dt = g_\lambda * \varphi(x)$ if $\mathcal{R}e(\lambda) > 0$,

(5.4) $u(x) = -\int_x^\infty e^{-\lambda(x-t)}\varphi(t)\,dt = h_\lambda * \varphi(x)$ if $\mathcal{R}e(\lambda) < 0$.

Here, $g_\lambda(x) = e^{-\lambda x}\chi_+(x)$ and $h_\lambda(x) = -e^{-\lambda x}(1-\chi_+(x))$ where χ_+ is the characteristic function of the interval $[0,\infty[$. The case $\mathcal{R}e(\lambda) = 0$ is more delicate.

Theorem 5.1. *Suppose $\mathcal{R}e(\lambda) = 0$, $\varphi \in L^p(\mathbb{R})$ and $i\lambda \notin sp(\varphi)$. If $1 \le p < \infty$ then (5.1) has a unique solution $u \in L^p(\mathbb{R})$ given by*

(5.5) $u(x) = \lim_{T\to\infty} \int_{-T}^{x} e^{-\lambda(x-t)}\varphi(t)\,dt.$

If $p = \infty$ then (5.1) has infinitely many solutions $u \in L^\infty(\mathbb{R})$ given by (5.2).

Proof. Let $\psi(x) = e^{\lambda x}\varphi(x)$ and $v(x) = e^{\lambda x}u(x)$. Then u is a solution in $L^p(\mathbb{R})$ of (5.1) if and only if v is a solution in $L^p(\mathbb{R})$ of

(5.6) $v'(x) = \psi(x)$, $x \in \mathbb{R}$.

Further, $sp(\psi) = -i\lambda + sp(\varphi)$, so $0 \notin sp(\psi)$. If $1 \le p < \infty$, then by theorem 3.4, the equation (5.6) has a unique solution $v \in L^p(\mathbb{R})$ given by $v(x) = \lim_{T\to\infty} \int_{-T}^{x} \psi(t)\,dt$. If $p = \infty$, the same theorem shows $v(x) = c + \int_0^x \psi(t)\,dt$ defines a bounded solution of (5.6) for each constant c.

Remark 5.2. *If $\varphi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ then $sp(\varphi) \subset \mathbb{R}$. Hence if $\mathcal{R}e(\lambda) \ne 0$, then $i\lambda \notin sp(\varphi)$. On the other hand if $\mathcal{R}e(\lambda) = 0$ and $i\lambda \in sp(\varphi)$ then (5.1) may have no solution $u \in L^p(\mathbb{R})$. For example, if $\lambda = 0$ and φ is a non-zero constant, then (5.1) has no solution $u \in L^\infty(\mathbb{R})$. Again, if $\lambda = 0$ and φ is defined by (1.2), then (5.1) has no solution $u \in L^1(\mathbb{R})$.*

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