THE MODULI SPACE OF POINTS IN THE BOUNDARY OF QUATERNIONIC HYPERBOLIC SPACE

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Abstract

Let $\mathcal{F}_1(n, m)$ be the $\mathrm{PSp}(n, 1)$-configuration space of ordered m-tuple of pairwise distinct points in the boundary of quaternionic hyperbolic n-space $\partial\mathbf{H}_{\mathbb{H}}^n$, i.e., the m-tuple of pairwise distinct points in $\partial\mathbf{H}_{\mathbb{H}}^n$ up to the diagonal action of $\mathrm{PSp}(n, 1)$. In terms of Cartan's angular invariant and cross-ratio invariants, the moduli space of $\mathcal{F}_1(n, m)$ is described by using Moore's determinant. We show that the moduli space of $\mathcal{F}_1(n, m)$ is a real $2m^2 - 6m + 5 - \sum_{i=1}^{m-n-1} \binom{m-2}{n-1+i}$ dimensional subset of a algebraic variety with the same real dimension when $m > n + 1$.

1. Introduction

In the research of lattices in a Lie group, one of the most important topics is the study of the deformations of complex hyperbolic lattices in a Lie group L inside a larger Lie group $G \supset L$. Due to Margulis' superrigidity theorem and its generalization to $\mathrm{Sp}(n, 1)$ and F_4^{-20} by Corlette, it is interesting to expect possible deformations of a lattice in $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ inside the larger Lie group $\mathrm{Sp}(n, 1)$. It is well known that in complex hyperbolic geometry the group Γ generated by loxodromic elements is discrete if Γ contains only loxodromic elements, see [18]. Analogously, in quaternionic hyperbolic geometry, according to [11], the group Γ containing only loxodromic elements is discrete if there exists a Γ-invariant totally geodesic submanifold $M(\Gamma)$ such that the quaternionic dimension $\dim_{\mathbb{H}} M(\Gamma)$ is even. Hence the desired lattice can be found in Γ. Therefore, it is meaningful to investigate the deformation space of this kind of loxodromic groups. But in general, it is a difficult problem to consider the deformation space of a group generated by m loxodromic generators. As a starting point we could first understand the moduli space of m-tuple of points in projective spaces. For more information about moduli spaces we refer to [21] and [25].

In the complex projective space $\mathbb{CP}^n$, the moduli spaces of $\mathrm{PU}(n, 1)$-congruence classes of the ordered distinct points have been well studied. Brehm and Et-Taoui investigated in [5] the congruence classes and direct congruence classes of m-tuples in the complex hyperbolic space. For the moduli space of the ordered quadruples of distinct isotropic points in $\mathbb{CP}^2$, Parker and Platis [24] defined analogous Fenchel-Nielsen coordinates for complex hyperbolic quasi-conformal representations of surface groups and parameterized the deformation space of complex hyperbolic totally loxodromic quasi-Fuchsian groups by using traces and cross-ratio invariants. Falbel and Platis [16] showed the $\mathrm{PU}(2, 1)$-configuration space is a real four dimensional variety and proved the existence of natural complex and CR structures

on this space. Cunha and Gusevskii [12] investigated the cases of the ordered quadruples of distinct isotropic points in $\mathbb{CP}^n$ by using Gram matrices, Cartan's angular invariant and cross-ratio invariants; then they proceeded in [13] to the investigation of the moduli space of the ordered m-tuples ($m \geq 4$) of distinct isotropic points in $\mathbb{CP}^n$. Hakim and Sanler, [15], discussed the cases of isotropic points for $m = n + 1$. Moreover, Cunha et al., [14], explored the cases of positive points.

In the quaternionic projective space $\mathbb{HP}^n$, the moduli space of the space $\mathcal{F}_1(n, m)$ of the ordered distinct points also have been concerned by some experts. Brehm considered in [4] the triplets of distinct negative points in $\mathbf{H}_{\mathbb{H}}^n$. Then Brehm and Et-Taoui continue to investigate the congruence classes of m-tuples of points in the quaternionic elliptic space and the quaternionic hyperbolic space in [6]. The Riemannian metric in both elliptic space and hyperbolic space can be defined by Hermitian product, but this does not work for points in the boundary of quaternionic hyperbolic space. Cao [8] studied the moduli space of triplets and quadruples of isotropic points in $\mathbb{HP}^n$ without the use of quaternionic determinants.

It is a natural problem to study the other cases of the moduli space of the ordered distinct points in $\mathbb{HP}^n$. Let $\partial \mathbf{H}_{\mathbb{H}}^n$ be the boundary of quaternionic hyperbolic n-space. The PSp$(n, 1)$-configuration space $\mathcal{F}_1(n, m)$ is the quotient of the set of the ordered m-tuples of pairwise distinct points in $\partial \mathbf{H}_{\mathbb{H}}^n$ up to the isometric group PSp$(n, 1)$. The main purpose of this paper is to study $\mathcal{F}_1(n, m)$ and the moduli space of the discrete, faithful, totally loxodromic representations of groups into PSp$(2, 1)$.

In what follows we recall Gram matrices in an n-dimensional vector space V.

DEFINITION 1.1. Let v_i be vectors in V and let $\langle -, - \rangle$ be a Hermitian form on V. Then the *Gram matrix* of v_i is $G = (g_{ij}) = (\langle v_j, v_i \rangle)$, where $1 \leq i, j \leq n$.

The Gram matrices and invariants associated with a well defined determinant of these matrices are the main tools in the investigation of the moduli space of points in $\mathbb{CP}^n$. A further insight into the case of $\mathbb{HP}^n$, especially in the relationship between cross-ratio invariants and the moduli spaces, could be in terms of a suitable quaternionic determinant. There are three main definitions of quaternionic determinants given by Study, Moore and Dieudonné, see [1]. Among those we choose the Moore's determinant for Hermitian quaternionic matrices. Moore [22] defined the quaternionic determinant of Hermitian mastrices in a natural way, by preserving the character of quaternions.

Recently, Cao has also considered a similar problem in [10]. There, he directly parameterizes the moduli space without the use of quaternionic determinant.

We also note that Gongopadhyay and Kalane classify pairs of semisimple elements in the L-character variety in [17], where L is the Lie group SU$(n, 1)$ or Sp$(n, 1)$. They use the Sp$(n, 1)$-configuration space $\mathcal{F}(n, i, m - i)$ of the ordered m-tuple of points on $\overline{\mathbf{H}_{\mathbb{H}}^n}$, where i is the number of points in $\partial \mathbf{H}_{\mathbb{H}}^n$ and $m - i$ is the number of points in $\mathbf{H}_{\mathbb{H}}^n$, to classify the semisimple pairs. The main method they use in order to deal with the non-commutativity of quaternionic is the embedding of quaternions into a complex space; they also use certain spatial invariants. The method of embedding Sp$(n, 1)$ into GL$(2n + 2, \mathbb{C})$ was used by Study to define the Study determinant for quaternionic matrices.

We introduce some basic concepts before we state our results:

- *Special-Gram quaternionic matrices:* Let $\mathfrak{p} = (p_1, \ldots, p_m)$ be an ordered m-tuple of

pairwise distinct isotropic points in $\partial\mathbf{H}^n_{\mathbb{H}}$ and $\mathbf{p}_i \in \mathbb{H}^{n,1}$ be any lift of p_i, then we call $\tilde{\mathfrak{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)$ a lift of $\mathfrak{p}$. Then the special-Gram quaternionic matrix associated to $\mathfrak{p}$ is given by

$$G = G(\tilde{\mathfrak{p}}) = \left(\langle\mathbf{p}_j, \mathbf{p}_i\rangle_1\right),$$

where $\mathbf{p}_1$ is the standard lift of p_1. See Section 2.4 for details about lifts of points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$.

- *Cartan's angular invariant:* Let $\mathfrak{p} = (p_1, p_2, p_3)$ be an ordered triplet of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$. The quaternionic Cartan's angular invariant $\mathbb{A}_{\mathbb{H}}(\mathfrak{p})$ of $\mathfrak{p}$ is defined by

$$\mathbb{A}_{\mathbb{H}}(\mathfrak{p}) = \arccos\frac{\mathfrak{R}(-\langle\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle_1)}{|\langle\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle_1|},$$

where $\mathbf{p}_i \in \mathbb{H}^{n,1}$ are the lifts of p_i,

$$\langle\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle_1 = \langle\mathbf{p}_1, \mathbf{p}_2\rangle_1\langle\mathbf{p}_2, \mathbf{p}_3\rangle_1\langle\mathbf{p}_3, \mathbf{p}_1\rangle_1 \text{ and } \mathfrak{R}(-\langle\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\rangle_1) > 0.$$

One can prove that $0 \leq \mathbb{A}_{\mathbb{H}}(\mathfrak{p}) \leq \pi/2$ and $\mathbb{A}_{\mathbb{H}}(\mathfrak{p})$ is independent of the chosen lifts and the order of three points.

- *Cross-ratio invariants:* This definition is used in [8]. Suppose that $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ is a quadruple of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$. Their cross-ratio is defined by

$$\mathbb{X}(p_1, p_2, p_3, p_4) = \langle\mathbf{p}_3, \mathbf{p}_1\rangle_1\langle\mathbf{p}_3, \mathbf{p}_2\rangle_1^{-1}\langle\mathbf{p}_4, \mathbf{p}_2\rangle_1\langle\mathbf{p}_4, \mathbf{p}_1\rangle_1^{-1},$$

where $\mathbf{p}_i$ are the lifts of p_i. This cross-ratio only depends on the lift of the first point. If we let first point be a standard lift, then it is independent of the chosen lifts of other points.

- *Fine moduli space:* the following definition comes from [25].

DEFINITION 1.2. A *fine moduli space* consists of a variety $\mathbb{M}$ and a family U parameterized by $\mathbb{M}$ such that, for every family X parameterized by a variety S, there is a unique morphism $\phi : S \to \mathbb{M}$ with $X \sim \phi^*U$. Such a family U is called a *universal family* for the given problem.

Let $\underline{G} = (g_{ij})$ be a Hermitian quaternionic $m \times m$-matrix, $m > 2$, with $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \ldots, m$ and $g_{ij} \neq 0$ for $i \neq j$. We call it *normalized special-Gram quaternionic matrix*. In Corollary 3.14, we have that the configuration space $\mathcal{F}_1(n, m)$ is in bijection with the space of normalized special-Gram quaternionic matrices. Therefore the study of $\mathcal{F}_1(n, m)$ is equivalent to the study the moduli space of normalized special-Gram quaternionic matrices.

By the properties of Moore's determinant we can simplify $\underline{G}$ and associate it with a Hermitian quaternionic $(m - 2) \times (m - 2)$-matrix $G^\star$. Then in Theorem 3.17, we show that $\underline{G}$ is a normalized special-Gram quaternionic matrix associated with some ordered m-tuple $\mathfrak{p} = (p_1, \ldots, p_m)$ of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$ if and only if $\mathrm{rank}(G^\star) \leq n - 1$ and all principal minors of $G^\star$ are non-negative. This result can help us to describe the moduli space of $\mathcal{F}_1(n, m)$ by the concrete algebraic expressions.

In Proposition 4.2, we give the relations between the normalized special-Gram quaternionic matrix $\underline{G}(\mathfrak{p}) = (g_{ij})$ and the parameters given by cross-ratios $\{\mathbb{X}_1, \ldots, \mathbb{X}_d\}$ with $d =$

$m(m-3)/2$, one Cartan's angular inviriant $\mathbb{A}_{\mathbb{H}}$, a unit pure quaternion $\mathfrak{u}$ and a positive number r. In Theorem 4.3, we see that $\mathcal{F}_1(n,m)$ is uniquely determined by the minimal parameter system $(\mathbb{X}_{2j}, \mathbb{X}_{3j}, \ldots, \mathbb{X}_{kj}, \mathfrak{u}, \mathbb{A}_{\mathbb{H}}, r)$.

Let $\det_{\mathbb{M}}(G^{\star}_{I^s_{m-2}})$ be principal minors of $G^{\star}$. Identify

$$w = (q_1, \ldots, q_d, \mathfrak{u}, t_1, t_2) \text{ with } (\mathbb{X}_{2j}, \mathbb{X}_{3j}, \ldots, \mathbb{X}_{kj}, \mathfrak{u}, \mathbb{A}_{\mathbb{H}}, r),$$

where $q_i \in \mathbb{H}$ is nonzero, $i = 1, \ldots, d$. Let $D^{\star}_{I^s_{m-2}}(w) = \det_{\mathbb{M}}(G^{\star}_{I^s_{m-2}})$.

Then the main results of this paper is concluded as following.

Theorem 1.3. *Let*

$$\mathbb{M}_1(n,m) = \{w \in \mathbb{H}^d \times \mathfrak{sp}(1) \times \mathbb{R}^2 \mid D^{\star}_{I^s_{m-2}}(w) \geq 0, s \leq n-1; D^{\star}_{I^s_{m-2}}(w) = 0, s > n-1\},$$

where $d = m(m-3)/2$. Then $\mathcal{F}_1(n,m)$ is homeomorphic to the set $\mathbb{M}_1(n,m)$.

Therefore, the moduli space for $\mathcal{F}_1(n,m)$ is described as $\mathbb{M}_1(n,m)$. For more details, see Section 4.2. Actually, based on the method used in [7], we can show that it is a fine moduli space.

Definition 1.4. A *family of* $PSp(n,1)$-*configuration space* $\mathcal{F}_1(n,m)$ of the ordered m-tuple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$ over a topological space S is a subspace $X \subset S \times \mathcal{F}_1(n,m)$ such that projection onto the first coordinate is a (continuous, proper) fiber bundle projection and such that the fiber over each $s \in S$ is (after projection onto the second coordinate) a $PSp(n,1)$-orbit of the ordered m-tuple $\mathfrak{p}$, denoted by $[\mathfrak{p}]_s$. A *morphism* between family $X \to S$ and $X' \to S'$ is a continuous map $f : S \to S'$ such that for each $s \in S$ there exists an isometry $g \in PSp(n,1)$ such that the diagonal action $g([\mathfrak{p}]_s) = [\mathfrak{p}]_{f(s)}$.

There exists a universal family $U \to \mathbb{M}_1(n,m)$ over $\mathbb{M}_1(n,m)$. The total space is

$$U = \{(w, s(w)) \in \mathbb{M}_1(n,m) \times \mathcal{F}_1(n,m) : w \in \mathbb{M}_1(n,m)\},$$

where the map $s : \mathbb{M}_1(n,m) \to \mathcal{F}_1(n,m)$ is given by

$$w \mapsto s(w) = [\mathfrak{p}]_s.$$

By Theorem 4.3, s is a bijection map. Therefore any family $X \to S$ of $PSp(n,1)$-orbit of m-tuple in $\partial \mathbf{H}^n_{\mathbb{H}}$ over S determines a cnonical map

$$g : S \to \mathbb{M}_1(n,m)$$

which send $s \in S$ to the parameters $w \in \mathbb{M}_1(n,m)$ of the fiber $[\mathfrak{p}]_s \in \mathcal{F}_1(n,m)$. Hence there is a unique isomorphism of X with the pullback bundle g^*U, i.e.,

$$X \cong g^*U.$$

By definition 1.2, we conclude the following theorem.

Theorem 1.5. $\mathbb{M}_1(n,m)$ *is a fine moduli space.*

Then we apply Theorem 1.3 to describe the moduli space of the representation family of groups into $PSp(2,1)$ and we have the following theorem which gives an embedding of the $PSp(2,1)$-character variety.

Theorem 1.6. *Let* Γ *be a finitely generated group with a set of* k *generators,* $\mathrm{Lox}(\Gamma, \mathrm{PSp}(2,1))$ *be the discrete, faithful, totally loxodromic representation family of* Γ *into* $\mathrm{PSp}(2,1)$. *Then* $\mathrm{Lox}(\Gamma, \mathrm{PSp}(2,1))$ *can be identified with an open subset of*

$$\mathbb{M}_1(2,2k) \times \Lambda^k \times R^{2k},$$

where

$$\Lambda = \{\lambda \in \mathbb{R} \mid \lambda > 0, \ \lambda \neq 1\}, \quad R = [0,\pi].$$

This paper is organised as follows. Section 2 contains some basic facts about quaternion, Moore's determinant, the rank of quaternionic matrix and quaternionic hyperbolic geometry. Section 3 provides a characterization of special-Gram quaternionic matrices of the ordered m-tuple of distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$. Finally, in Section 4, we construct and describe the moduli space and prove the main results. Then we apply our results to study of the moduli space of discrete, faithful, totally loxodromic representation family of groups into $\mathrm{PSp}(2,1)$.

2. Preliminaries

In this section, we shortly review some facts about quaternions and quaternionic hyperbolic space. We also introduce Moore's determinant on quaternonic Hermitian matrices.

2.1. Quaternions. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ be the field of real, complex and quaternion numbers, respectively. We denote

$$\mathbb{H} = \{q = t + ix + jy + kz \mid t,x,y,z \in \mathbb{R}\},$$

where

$$i^2 = j^2 = k^2 = -1, \ ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik.$$

Any $q \in \mathbb{H}$ can also be written in the form $q = z_1 + jz_2$, where $z_1, z_2 \in \mathbb{C}$. Note that $zj = j\bar{z}$ for $z \in \mathbb{C}$, since the centre of $\mathbb{H}$ is $\mathbb{R}$. Conjugation and modulus is given respectively by

$$\bar{q} = \overline{t + ix + jy + kz} = t - ix - jy - kz, \quad |q| = \sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2}.$$

Then

$$\overline{q_1 q_2} = \overline{q_2}\, \overline{q_1} \quad \text{for} \quad q_1, q_2 \in \mathbb{H},$$

$$\mathfrak{R}(q) = \frac{1}{2}(q + \bar{q}) = t,$$

$$\mathfrak{R}(q_1 q_2) = \mathfrak{R}(q_2 q_1) = \mathfrak{R}(\bar{q}_1 \bar{q}_2) = \mathfrak{R}(\bar{q}_2 \bar{q}_1).$$

A pure quaternion is of the form

$$\mathrm{Pu}(q) = \frac{1}{2}(q - \bar{q}) = ix + jy + kz$$

and a unit quaternion is of the form

$$\mathrm{Un}(q) = \frac{q}{|q|} \in S,$$

where S is the unit sphere in $\mathbb{H}$. Let $\mathfrak{sp}(1)$ be the Lie algebra of Lie group $Sp(1)$. Then

$$\mathfrak{sp}(1) = Pu(\mathbb{H}) \cap S$$

is the set of all unit pure quaternions. Any quaternion is expressible as a power of a pure quaternion and it can be also written in the form

$$q = |q|e^{\mathfrak{u}\theta} := |q|(\cos\theta + \mathfrak{u}\sin\theta),$$

where $\mathfrak{u} \in \mathfrak{sp}(1)$ and $\theta \in [0,\pi]$.

2.2. Moore's determinant for quaternionic matrices. We shall introduce the Moore's determinant of a Hermitian quaternionic $n \times n$ matrix M. Such a matrix has quaternionic entries and satisfies $M^* = M$, where M^* is the conjugate transpose of M. There are several kinds of determinants associated to a quaternionic matrix. Moore's determinant is the one which is convenient for our purposes. For more information about Moore's determinant, see [22].

DEFINITION 2.1. For any Hermitian $n \times n$ quaternionic matrix M, $M = (m_{ij})$, $1 \le i \le j \le n$ the *Moore's determinant* of M is given by

$$\det_{\mathbb{M}}(M) = \sum_{\sigma \in S_n} \varepsilon(\sigma) m_{n_{11}n_{12}} \cdots m_{n_{1l_1}n_{11}} m_{n_{21}n_{22}} \cdots m_{n_{rl_r}n_{r1}},$$

where S_n is group of permutations and $\varepsilon(\sigma) = (-1)^{n-r}$.

Note that any $\sigma \in S_n$ can be written as a product of disjoint cycles. We permute each cycle until the leading index is the smallest and then sort the cycles in increasing order according to the first number of each cycle. That is,

$$\sigma = (n_{11} \cdots n_{1l_1})(n_{21} \cdots n_{2l_2}) \cdots (n_{r1} \cdots n_{rl_r}),$$

where for each i, we let $n_{i1} < n_{ij}$ for all $j > 1$, $n_{11} < n_{21} \cdots < n_{r1}$.

When we deal with the calculations of quaternionic matrices, the following two known results can be used.

Lemma 2.2 (Corollary 6.2 in [28]). *A quaternionic matrix A is normal if and only if there exists a unitary matrix U such that*

$$U^*AU = \mathrm{diag}\{h_1 + ik_1, \ldots, h_n + ik_n\}$$

and A is Hermitian if and only if $k_1 = \cdots = k_n = 0$.

Lemma 2.3 (Theorem 1.1.9. in [2]).
(i) *The Moore determinant of any complex Hermitian matrix considered as quaternionic Hermitian quaternionic matrix is equal to its usual determinant.*
(ii) *For any Hermitian quaternionic matrix A and any quaternionic matrix C*

$$\det_{\mathbb{M}}(C^*AC) = \det_{\mathbb{M}}(A)\det_{\mathbb{M}}(C^*C).$$

Let $I_n^s = \{(i_1, \ldots, i_s) \mid 1 \le i_1 < i_2 \cdots < i_s \le n\}$ and $A_{I_n^s}$ denote the sub-matrix of an $n \times n$-matrix A, formed by choosing the elements of the original matrix from the rows whose indices are in $i_1, \ldots, i_s$ and the columns whose indexes are in $i_1, \ldots, i_s$.

Lemma 2.4 (Proposition 1.1.11 of [2]). *For any quaternionic Hermitian $(n \times n)$-matrix A and any diagonal real matrix*

$$T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

we have

$$\operatorname{det}_{\mathbb{M}}(A + \lambda I) = \sum_{s=1}^{n} \lambda^s \cdot \operatorname{det}_{\mathbb{M}}(A_{I_n^s}).$$

S. Alesker proved the standard Sylvester criterion for positive definite quaternionic matrices in Theorem 1.1.13 of [2, p.10]. For positive semi-definite Hermitian quaternionic matrices, we have the following analogous theorem.

Theorem 2.5. *A Hermitian quaternionic matrix M is positive semi-definite if and only if its principal minors are all nonnegative.*

Proof. We first prove necessity. Suppose that the Hermitian quaternionic matrix A is positive semi-definite matrix and let the Hermitian quaternionic matrix A_k be the matrix associated with the principal minors of order k, given by

$$A_k = \begin{pmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{pmatrix}.$$

Let $Y^* A Y$ and $X^* A_k X$ be the quadratic forms with the Hermitian quaternionic matrices A and A_k, respectively. For any

$$X_0 = (b_{i_1}, b_{i_2}, \ldots, b_{i_k})^* \neq \mathbf{0},$$

denote by Y_0 the vector $Y_0 = (c_1, \ldots, c_n)^*$, where $c_j = b_j$ if $j = i_1, i_2, \ldots, i_k$, else $c_j = 0$.

Note that if A is positive semi-definite, then $Y_0^* A Y_0 \geq 0$ implies $X_0^* A_k X_0 \geq 0$. By Lemma 2.2 and Lemma 2.3, there exists a unitary matrix T_k, satisfying

$$\operatorname{det}_{\mathbb{M}}(T_k^* A_k T_k) = \operatorname{det}_{\mathbb{M}}(A_k)\operatorname{det}_{\mathbb{M}}(T_k^* T_k) = \operatorname{det}_{\mathbb{M}}(A_k)\operatorname{det}_{\mathbb{M}}^2(T_k) \geq 0.$$

Next, we prove sufficiency. Suppose that the principal minors of A are all non-negative. Choose the k-th leading principal minor and let A_k be its corresponding matrix:

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}.$$

According to Lemma 2.4, we have that

$$\operatorname{det}_{\mathbb{M}}(A_k + \lambda I_k) = \lambda^k + p_1 \lambda^{k-1} + \cdots + p_{k-1}\lambda + p_k,$$

where p_i is the sum of all principal minors of A_k. By our assumptions, all $p_i \geq 0$ and therefore

$$\operatorname{det}_{\mathbb{M}}(A_k + \lambda I_k) > 0$$

when $\lambda > 0$. By Theorem 1.1.13 of [2, p.10], the quaternionic version standard Sylvester

criterion, $A_k + \lambda I_k$ is a positive-definite matrix when $\lambda > 0$.

Now, suppose that A is not positive semi-definite matrix, then there exists a non-zero vector

$$X_0 = (b_1, b_2, \ldots, b_n)^*$$

such that

$$X_0{}^* A X_0 = c < 0.$$

Let

$$\lambda = \frac{-c}{X_0{}^* X_0} = \frac{-c}{b_1{}^2 + \cdots + b_n{}^2} > 0.$$

Then

$$X_0{}^*(A + \lambda I) X_0 = X_0{}^* A X_0 + X_0{}^* \lambda I X_0 = \lambda X_0{}^* X_0 + X_0{}^* A X_0 = c - c = 0.$$

Since $\lambda > 0$, we have a contradiction. $\square$

2.3. Rank of quaternionic matrix. The rank of a quaternionic matrix was studied in [23, pp.64-70] and [28, p.43]. We follow the former for our definition.

DEFINITION 2.6. Let A be a finite but non-zero quaternionic matrix, then the *rank* of A, denoted by rank(A), is defined to be the order of its maximal non-singular minor. If $A = \mathbf{0}$ then rank$(A) = 0$.

REMARK 2.7. For each finite quaternionic matrix A the rank is uniquely determined. It is equal to the number of columns (or rows) of A in a maximal set of left linear independent columns (or rows) of A. Consequently, a finite matrix is non-singular if and only if the matrix is square and its rank is equal to its order.

It can be checked directly that the quaternionic matrices A, A^*, AA^* and A^*A have the same rank.

2.4. Quaternionic hyperbolic space. In this section, we briefly sketch the quaternionic hyperbolic space. For more information, see [19, 20, 26].

First, we set

$$H_1 = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & I_{n-1} & \mathbf{0} \\ 0 & \mathbf{0} & -1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & I_{n-1} \end{pmatrix}.$$

Let $\mathbb{H}^{n,1}$ be the $(n+1)$-dimensional $\mathbb{H}$-vector space equipped with the Hermitian form of signature $(n, 1)$ given by

$$\Phi_i(\mathbf{z}, \mathbf{w}) = \langle \mathbf{z}, \mathbf{w} \rangle_i = \mathbf{w}^* H_i \mathbf{z},$$

where $\mathbf{z}$, $\mathbf{w}$ are column vectors in $\mathbb{H}^{n,1}$ and $i = 1, 2$.

REMARK 2.8. Throughout this paper, the upper or lower index i implies that it is in terms of the Hermitian form Φ_i, where $i = 1$ or 2.

Let $\mathbb{P}$ be the natural right projection from $\mathbb{H}^{n,1} \setminus \{0\}$ onto the quaternionic projective space $\mathbb{HP}^n$. We define subsets V^i_-, V^i_0, V^i_+ of $V = \mathbb{H}^{n,1} \setminus \{0\}$ by

$$V^i_- = \{V \mid \langle \mathbf{z}, \mathbf{z} \rangle_i < 0\},$$
$$V^i_0 = \{V \mid \langle \mathbf{z}, \mathbf{z} \rangle_i = 0\},$$
$$V^i_+ = \{V \mid \langle \mathbf{z}, \mathbf{z} \rangle_i > 0\}.$$

We say that $\mathbf{z} \in \mathbb{H}^{n,1}$ is negative, isotropic or positive if $\mathbf{z}$ is in V_-, V_0 or V_+, respectively. Motivated by relativity, these are sometimes called time-like, light-like and space-like. Their projections to $\mathbb{HP}^n$ are called negative, isotropic and positive points, respectively.

The quaternionic hyperbolic is then defined by $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V^i_-)$ and its boundary is $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V^i_0)$. For a given finite point $p = (q_1, q_2, \ldots, q_n)^T$ in $\partial\mathbf{H}^n_{\mathbb{H}}$, denote its lift to be $\mathbb{H}^{n,1}$ is $\mathbf{p}$. Then

$$\mathbf{p} = (q_1\lambda, q_2\lambda, \ldots, q_n\lambda, \lambda)^T, \text{ where } \lambda \neq 0.$$

The lift of ∞ is $(\lambda, 0, \ldots, 0, 0)^T$. Moreover, we call $\mathbf{p}$ the standard lift when $\lambda = 1$.

$\mathbb{P}(V^1_-)$ is a ball model and another one is a Siegel domain model. We may pass between them by a corresponding Cayley transform.

The metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z} \rangle^2_i} = \left(\begin{array}{cc} \langle \mathbf{z}, \mathbf{z} \rangle_i & \langle d\mathbf{z}, \mathbf{z} \rangle_i \\ \langle \mathbf{z}, d\mathbf{z} \rangle_i & \langle d\mathbf{z}, d\mathbf{z} \rangle_i \end{array} \right).$$

Equivalently,

$$\cosh^2\left(\frac{\rho(\mathbb{P}\mathbf{z}, \mathbb{P}\mathbf{w})}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle_i \langle \mathbf{w}, \mathbf{z} \rangle_i}{\langle \mathbf{z}, \mathbf{z} \rangle_i \langle \mathbf{w}, \mathbf{w} \rangle_i},$$

where ρ is the distance function.

Consider the Lie group $\mathrm{Sp}(n,1) = \{M \in \mathrm{GL}(n+1, \mathbb{H}) \mid M^*H_iM = H_i\}$. Then the isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\pm I_{n+1}$. It is a non-compact real semi-simple Lie group.

We follow [20] to give the classification for non-trivial elements of $\mathrm{PSp}(n,1)$.

DEFINITION 2.9. A non-trivial element of $\mathrm{PSp}(n,1)$ is
(1) *loxodromic* if they fix exactly two points of $\partial\mathbf{H}^n_{\mathbb{H}}$;
(2) *parabolic* if they fix exactly one point of $\partial\mathbf{H}^n_{\mathbb{H}}$;
(3) *elliptic* if they fix exactly at least one point of $\mathbf{H}^n_{\mathbb{H}}$.

Lemma 2.10 (Proposition 2.1.3. of [11])**.** $\mathrm{Sp}(n,1)$ *is doubly transitive on quaternionic lines in* V^i_0.

The following proposition follows from Lemma 2.10.

Proposition 2.11. *If* $\mathbf{z}_1, \mathbf{z}_2$ *are two distinct isotropic vectors in* $\mathbb{H}^{n,1}$, *then* $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_i \neq 0$.

3. A characterization of special-Gram quaternionic matrices

In this section, our model for the quaternionic hyperbolic space will be the ball model with the Hermitian form Φ_1. The method used to drive our results is similar to the one

used in [13]. Here, we also have to take under consideration the non-commutativity of quaternions.

3.1. Special-Gram quaternionic matrix. Let $\mathfrak{p} = (p_1, \ldots, p_m)$ be an ordered m-tuple of pairwise distinct isotropic points in $\partial \mathbf{H}_{\mathbb{H}}^n$ and $\mathbf{p}_i \in \mathbb{H}^{n,1}$ be any lift of p_i, then we call $\tilde{\mathfrak{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)$ a lift of $\mathfrak{p}$.

First, we consider the special-Gram quaternionic matrix.

DEFINITION 3.1. The *special-Gram quaternionic matrix* associated to $\mathfrak{p}$ is $G = G(\tilde{\mathfrak{p}}) = \big(\langle \mathbf{p}_j, \mathbf{p}_i \rangle_1 \big)$, where $\mathbf{p}_1$ is the standard lift of p_1.

Note that G depends on the choice of lifts $\mathbf{p}_i$ for $i = 2, \ldots, m$. By Proposition 2.11, we know that g_{ij} is zero for $i = j$ and non-zero for $i \neq j$. It is straightforward to check that the special-Gram quaternionic matrices are invariant under the action of the isometry transformations, i.e.,

$$G(\tilde{\mathfrak{p}}) = G(T(\tilde{\mathfrak{p}})) = (T(\mathbf{p}_1), \ldots, T(\mathbf{p}_m)), \ T \in \mathrm{PSp}(n, 1).$$

Let $D = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_m)$ be the diagonal quaternionic $m \times m$ matrix with quaternionic entries $\lambda_i \neq 0$ and choose a lift $(\mathbf{p}_1, \mathbf{p}_2 \lambda_2, \ldots, \mathbf{p}_m \lambda_m)$ of $\mathfrak{p}$, then

$$\widehat{G} = G((\mathbf{p}_1, \mathbf{p}_2 \lambda_2, \ldots, \mathbf{p}_m \lambda_m)) = (\langle p_j \lambda_j, p_i \lambda_i \rangle_1) = (\bar{\lambda}_i \langle p_j, p_i \rangle_1 \lambda_j).$$

In other words, $\widehat{G} = D^* G D$.

Proposition 3.2. *Let $D = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_m)$ be a non-singular diagonal matrix. For any two quaternionic matrices $\widehat{M}$ and M, we say $\widehat{M} \sim M$ if $\widehat{M} = D^* M D$. Then the relation $\sim$ is an equivalence relation.*

Proof. The reflexive property and symmetric property are trivial. We now check the transitivity. Suppose that $M_1 \sim M_2$ and $M_2 \sim M_3$. Then we have $M_1 = D_1^* M_2 D_1$ and $M_2 = D_2^* M_3 D_2$, where D_1 and D_2 are the non-singular diagonal matrices as above D. Hence

$$M_1 = (D_2 D_1)^* M_3 D_2 D_1.$$

Since $D_2 D_1$ is again a non-singular diagonal matrix as above D, then we see that $M_1 \sim M_3$.
$\square$

DEFINITION 3.3. Two special-Gram quaternionic matrices M and $\widehat{M}$ are *equivalent* if there exists a non-singular diagonal matrix $D = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_m)$ such that $\widehat{M} = D^* M D$.

We will suppose that $\mathbf{p}_1$ be the standard lift of p_1. To each ordered m-tuple $\mathfrak{p}$ of pairwise distinct points in $\partial \mathbf{H}_{\mathbb{H}}^n$ is assigned an equivalence class of special-Gram quaternionic matrices. Let G and $\widehat{G}$ be two equivalence special-Gram quaternionic matrices associated to an m-tuple $\mathfrak{p}$. By Lemma 2.3, we have

$$\mathrm{det}_{\mathbb{M}}(\widehat{G}) = \lambda \, \mathrm{det}_{\mathbb{M}}(G),$$

where $\lambda > 0$. Clearly, the sign of $\mathrm{det}_{\mathbb{M}}(G)$ is independent of the chosen lifts $\mathbf{p}_i$.

Proposition 3.4. *Let $\mathfrak{p} = (p_1, \ldots, p_m)$ be an ordered m-tuple of pairwise distinct points in $\partial \mathbf{H}_{\mathbb{H}}^n$. Then the equivalence class of a special-Gram quaternionic matrix associated to $\mathfrak{p}$ contains a unique matrix $G(\mathfrak{p}) = (g_{ij}) = (\langle \mathbf{p}_j, \mathbf{p}_i \rangle_1)$ with $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \ldots, m$.*

Proof. Let $\tilde{\mathfrak{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)$ be a lift of $\mathfrak{p} = (p_1, \ldots, p_m)$. We know that $g_{ij} \neq 0$ whence $i \neq j$ due to the fact that p_i is distinct and isotropic. Now we re-scale $\mathbf{p}_j$ appropriately, replacing $\mathbf{p}_j$ by $\mathbf{p}_j \lambda_j$. We obtain that

$$g_{1j} = 1,$$

where $\lambda_1 = 1$ and $\lambda_j = \langle \mathbf{p}_j, \mathbf{p}_1 \rangle_1^{-1}$ with $j = 2, \ldots, m$.

Suppose that

$$(\mathbf{p}_1, \mathbf{p}_2 \mu_2, \ldots, \mathbf{p}_m \mu_m)$$

is another lift of $\mathfrak{p} = (p_1, \ldots, p_m)$. We start from $(\mathbf{p}_1, \mathbf{p}_2 \mu_2, \ldots, \mathbf{p}_m \mu_m)$ to find the matrix (g'_{ij}), where (g'_{ij}) is in the equivalence class of the special-Gram quaternionic matrix associated to $\mathfrak{p}$ with $g'_{ii} = 0$ and $g'_{1j} = 1$ for $j = 2, \ldots, m$. Then we find

$$(g_{ij}) = (g'_{ij}).$$

It is clear that the special-Gram quaternionic matrix G associated to $\mathfrak{p}$ is unique. □

REMARK 3.5. We stress here that if we choose an arbitrary lift for p_1 in Proposition 3.4, then the result does not hold. One can show that by choosing two different but arbitrary lifts for $(\mathbf{p}_1, \ldots, \mathbf{p}_m)$ and $(\mathbf{p}_1 \mu_1, \mathbf{p}_2 \mu_2, \ldots, \mathbf{p}_m \mu_m)$ of $\mathfrak{p}$, this leads to the corresponding matrices (g_{ij}) and (g'_{ij}) with $g'_{1j} = g_{1j}/|\mu_1|^2$. Hence $(g'_{ij}) = (g_{ij})$ only if $|\mu_1| = 1$.

DEFINITION 3.6. The unique matrix defined in Proposition 3.4 is called a *normalized special-Gram quaternionic matrix* and denoted by $\underline{G}(\mathfrak{p})$ or simply by $\underline{G}$ if there is no danger of confusion.

It follows that there is a correspondence between the space of the ordered m-tuple of pairwise distinct isotropic points and the space of normalized special-Gram quaternionic matrices.

3.2. Characterization of special-Gram quaternionic matrices. In this section, we discuss some properties of special-Gram quaternionic matrices associated with m-tuple of isotropic pairwise distinct points, where $m > 1$.

The Hermitian form $\Phi_1(\mathbf{z}, \mathbf{w})$ restricted on a subspace $W \subset \mathbb{H}^{n,1}$ is called *degenerate* if there exists a non-zero vector $w \in W$ such that $\langle w, v \rangle_1 = 0$ for all $v \in W$. We call the subspace W *degenerate* if the Hermitian form is degenerate, otherwise *non-degenerate*. Clearly, w must lie in V_0.

Proposition 3.7. *Let W be a degenerate subspace of $\mathbb{H}^{n,1}$ and*

$$R(W) = \{\mathbf{w} \in W \mid \langle \mathbf{w}, \mathbf{u} \rangle_1 = 0, \forall \mathbf{u} \in W\}.$$

Then $R(W)$ is an original point or a quaternionic line in V_0.

Proof. Let $\dim(W) = k+1$. If $k = 0$ then the proof is completed. Now suppose $1 \le k \le n$. If there exist two linearly independent isotropic vectors $\mathbf{w}_1, \mathbf{w}_2 \in R(W)$, we have $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_1 = 0$ by definition. On the other hand, since $\mathbf{w}_1, \mathbf{w}_2 \in V_0$, we obtain $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_1 \ne 0$ by Proposition 2.11. Therefore, we get a contradiction.   □

Corollary 3.8. *Let $W \subset \mathbb{H}^{n,1}$. For the Hermitian form Φ_1 (but as well as for Φ_2) acting on $\mathbb{H}^{n,1}$, the restriction $\Phi_1 \mid W$ loses at most one dimension and the Witt index of a totally isotropic subspace is 1.*

The following definition comes from [11, p.52].

Definition 3.9. A subspace $W \subset \mathbb{H}^{n,1}$ is called *hyperbolic* if the restriction $\Phi_1 \mid W$ is non-degenerate and indefinite; it is *elliptic* if $\Phi_1 \mid W$ is positive definite; and it is *parabolic* if $\Phi_1 \mid W$ is degenerate.

Let W be a $(k+1)$-dimensional subspace of $\mathbb{H}^{n,1}, 1 \le k \le n$ and denote its signature by (n_+, n_-, n_0), where n_+ (resp. n_-, n_0) is the number of positive (resp. negative, zero) eigenvalues of the Hermitian matrix of Φ_1. Then Corollary 3.8 implies

Corollary 3.10. *The restriction of the Hermitian form Φ_1 on $\mathbb{H}^{n,1}$ to W can only have signature $(k,1,0)$, $(k+1,0,0)$, $(k,0,1)$ or $(k-1,1,1)$.*

Suppose that $\mathfrak{p} = (p_1, \ldots, p_m)$ is an ordered m-tuple of pairwise distinct points in quaternionic projection space $\mathbb{HP}^n$ and $L(\mathbf{p}_i, \ldots, \mathbf{p}_m) \subset \mathbb{H}^{n,1}$ is the subspace spanned by the distinct vectors $\mathbf{p}_i, \ldots, \mathbf{p}_m$, where $\mathbf{p}_i \in \mathbb{H}^{n,1}$ is the lift of the isotropic point p_i, $i = 1, \ldots, m$. Moreover assume that $\dim(L(\mathbf{p}_i, \ldots, \mathbf{p}_m)) = k+1$. Then only the following three cases can occur:

(1) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is hyperbolic, if it has signature $(k,1,0)$, where $1 \le k \le n$;
(2) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is elliptic, if it has signature $(k+1,0,0)$, where $1 \le k \le n$;
(3) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is parabolic, if it has signature $(k,0,1)$ or $(k-1,1,1)$, where $1 \le k \le n$.

Let $G(\tilde{\mathfrak{p}}) = (g_{ij}) = (\langle \mathbf{p}_j, \mathbf{p}_i \rangle_1)$ be the special-Gram matrix associated to an m-tuple $\mathfrak{p} = (p_1, \ldots, p_m)$ of pairwise distinct points in the quaternionic projective space $\mathbb{HP}^n$, where $\mathbf{p}_i$ is the lift of $p_i \in \mathbb{HP}^n$, $i = 1, \ldots, m$. By Lemma 2.2, we have

$$G(\tilde{\mathfrak{p}}) = \begin{pmatrix} \langle \mathbf{p}_1, \mathbf{p}_1 \rangle_1 & \langle \mathbf{p}_2, \mathbf{p}_1 \rangle_1 & \cdots & \langle \mathbf{p}_m, \mathbf{p}_1 \rangle_1 \\ \langle \mathbf{p}_1, \mathbf{p}_2 \rangle_1 & \langle \mathbf{p}_2, \mathbf{p}_2 \rangle_1 & \cdots & \langle \mathbf{p}_m, \mathbf{p}_2 \rangle_1 \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{p}_1, \mathbf{p}_m \rangle_1 & \langle \mathbf{p}_2, \mathbf{p}_m \rangle_1 & \cdots & \langle \mathbf{p}_m, \mathbf{p}_m \rangle_1 \end{pmatrix} = C^* J C,$$

where $C \in \mathrm{GL}(m, \mathbb{H})$. By restricting Φ_1 on $\mathbb{H}^{n,1}$ to the subspace $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$, we obtain the Hermitian matrix $J = \mathrm{diag}(\alpha_1, \ldots, \alpha_m)$ where $\alpha_i = \pm 1$ or 0.

Note that $G(\tilde{\mathfrak{p}})$ has the same signature as J. Therefore the type of $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ can be given according to the signature of the special-Gram qutaternionic matrix $G(\tilde{\mathfrak{p}})$:

(1) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is hyperbolic, if $G(\tilde{\mathfrak{p}})$ has signature (n_+, n_-, n_0) with $1 \le n_+ \le n$, $n_- = 1$, and $n_+ + 1 + n_0 = m$.
(2) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is elliptic, if $G(\tilde{\mathfrak{p}})$ has signature (n_+, n_-, n_0) with $1 \le n_+ \le n$, $n_- = 0$, and $n_+ + n_0 = m$.
(3) $L(\mathbf{p}_i, \ldots, \mathbf{p}_m)$ is parabolic, if $G(\tilde{\mathfrak{p}})$ has signature (n_+, n_-, n_0) with $1 \le n_+ \le n$, $1 \le n_0 \le$

n, $n_- = 0$ or 1, and $n_+ + n_- + n_0 = m$.

The above conditions are called the *signature conditions*.

We now focus on the case that $\mathfrak{p} = (p_1, \ldots, p_m)$ is an ordered m-tuple of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$, with the lift $\tilde{\mathfrak{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_m)$. Since the Moore's determinant of the upper left hand 2×2-block of $G(\tilde{\mathfrak{p}})$ is negative, we see that $L(\mathbf{p}_1, \ldots, \mathbf{p}_m)$ always has signature with $n_- = 1$. Thus any special-Gram quaternionic matrix associated to an m-tuple of pairwise distinct isotropic points has exactly signature (n_+, n_-, n_0) with $n_- = 1$, $1 \le n_+ \le n$, and $1 + n_+ + n_0 = m$. Then the following holds.

Theorem 3.11. *Let $G = (g_{ij})$ be a Hermitian quaternionic matrix with $g_{ii} = 0$, $g_{ij} \ne 0$ for $i \ne j$, $m > 1$. Then G is a special-Gram quaternionic matrix associated with some ordered m-tuple $\mathfrak{p} = (p_1, \ldots, p_m)$ of distinct isotropic points in $\partial\mathbf{H}^n_{\mathbb{H}}$ if and only if $\mathrm{rank}(G) \le n+1$ and G has the signature (n_+, n_-, n_0) with $n_- = 1$, $1 \le n_+ \le n$, and $1 + n_+ + n_0 = m$.*

Proof. The proof is on the lines of the proof of [13, Proposition 2.2]. Let $G = (g_{ij})$ be a Hermitian quaternionic matrix with $g_{ii} = 0$, $g_{ij} \ne 0$ for $i \ne j$, $m > 1$. According to Lemma 2.2, there exists a matrix $U \in \mathrm{GL}(m, \mathbb{H})$ such that $U^* G U = B$, where $B = (b_{ij})$ is the diagonal $m \times m$-matrix such that $b_{ii} = 1$ for $1 \le i \le n_+$, $b_{ii} = -1$ for $i = n_+ + 1$ and $b_{ij} = 0$ for all others. Now let $A = (a_{ij})$ be the $(n+1) \times m$-matrix such that $a_{ii} = 1$ for $1 \le i \le n_+$, $a_{ii} = -1$ for $i = n_+ + 1$ and $a_{ij} = 0$ for all others. Then we easily obtain that $B = A^* H_1 A$. Thus the i-th column vector of the matrix AU^{-1} can be defined to be $\mathbf{p}_i$. We see that $\langle \mathbf{p}_j, \mathbf{p}_i \rangle_1 = g_{ij}$. Finally, we find the isotropic points p_i corresponding to $\mathbf{p}_i$. □

Assume that W is a subspace of $\mathbb{H}^{n,1}$. According to Proposition 2.1.1. of [11], each linear isometry of W can be extended to an element of $\mathrm{Sp}(n, 1)$. Then we obtain the following.

Proposition 3.12. *Suppose that $\mathfrak{p} = (p_1, \ldots, p_m)$ and $\mathfrak{p}' = (p'_1, \ldots, p'_m)$ are two m-tuple of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$. Then $\mathfrak{p}$ and $\mathfrak{p}'$ are congruent in $\mathrm{PSp}(n, 1)$ if and only if their associated special-Gram quaternionic matrices are equivalent.*

Corollary 3.13. *Suppose that $\mathfrak{p} = (p_1, \ldots, p_m)$ and $\mathfrak{p}' = (p'_1, \ldots, p'_m)$ are two m-tuple of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$, and let $\underline{G}(\mathfrak{p})$ and $\underline{G}(\mathfrak{p}')$ be their normalized special-Gram quaternionic matrices. Then $\mathfrak{p}$ and $\mathfrak{p}'$ are congruent in $\mathrm{PSp}(n, 1)$ if and only if $\underline{G}(\mathfrak{p}) = \underline{G}(\mathfrak{p}')$.*

Let $\mathcal{F}_1(n, m)$ be the $\mathrm{PSp}(n, 1)$-configuration space of the ordered m-tuple of pairwise distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$, i.e., the space of m-tuple cut by the action of $\mathrm{PSp}(n, 1)$. Then we have

Corollary 3.14. *The configuration space $\mathcal{F}_1(n, m)$ is in bijection with the space of normalized special-Gram quaternionic matrices.*

By Proposition 3.4, the normalized special-Gram quaternionic matrix associated with an ordered m-tuple of pairwise distinct isotropic points is necessarily of the following form:

$$\underline{G}(\mathfrak{p}) = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & g_{23} & g_{24} & \cdots & g_{2m} \\ 1 & \overline{g}_{23} & 0 & g_{34} & \cdots & g_{3m} \\ 1 & \overline{g}_{24} & \overline{g}_{34} & 0 & \cdots & g_{4m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{g}_{2m} & \overline{g}_{3m} & \overline{g}_{4m} & \cdots & 0 \end{pmatrix},$$

where $g_{ij} \neq 0$ for $i \neq j$.

Proposition 3.15. *Suppose that* $\underline{G} = (g_{ij})$ *is a Hermitian quaternionic* $m \times m$-*matrix,* $m > 2$, *with* $g_{ii} = 0$, $g_{1j} = 1$ *for* $j = 2, \ldots, m$ *and* $g_{ij} \neq 0$ *for* $i \neq j$. *Then there exists a matrix in* $GL(m, \mathbb{H})$ *which transforms* $\underline{G}$ *into the matrix*

$$
\left(
\begin{array}{cc|c}
0 & 1 & \mathbf{0} \\
1 & 0 & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & G^{\star}
\end{array}
\right),
$$

where $G^{\star}$ *is Hermitian quaternionic* $(m-2) \times (m-2)$-*matrix given by*

$$
G^{\star} =
\begin{pmatrix}
-(g_{23} + \overline{g}_{23}) & -\overline{g}_{23} - g_{24} + g_{34} & \cdots & -\overline{g}_{23} - g_{2m} + g_{3m} \\
-g_{23} - \overline{g}_{24} + \overline{g}_{34} & -(g_{24} + \overline{g}_{24}) & \cdots & -\overline{g}_{24} - g_{2m} + g_{4m} \\
\vdots & \vdots & \ddots & \vdots \\
-g_{23} - \overline{g}_{2m} + \overline{g}_{3m} & -g_{24} - \overline{g}_{2m} + \overline{g}_{4m} & \cdots & -(g_{2m} + \overline{g}_{2m})
\end{pmatrix}.
$$

Proof. Let

$$
T =
\begin{pmatrix}
1 & 0 & -g_{23} & -g_{24} & \cdots & -g_{2m} \\
0 & 1 & -1 & -1 & \cdots & -1 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
$$

Then

$$
T^{*} \underline{G} T =
\left(
\begin{array}{cc|c}
0 & 1 & \mathbf{0} \\
1 & 0 & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & G^{\star}
\end{array}
\right).
$$

$\square$

The matrix $G^{\star}$ is called the *associated* matrix with $\underline{G}$.

According to Lemma 2.3, we have

$$
\det_{\mathbb{M}}(\underline{G}) \det_{\mathbb{M}}(T^{*}T) = -\det_{\mathbb{M}}(G^{\star}).
$$

Corollary 3.16. *Let* $\underline{G} = (g_{ij})$ *be a Hermitian quaternionic* $m \times m$-*matrix satisfying the conditions of Proposition* 3.15 *and* $G^{\star}$ *be the associated matrix with* $\underline{G}$. *Then*

$$
\det_{\mathbb{M}}(\underline{G}) = \frac{-\det_{\mathbb{M}}(G^{\star})}{\det_{\mathbb{M}}(T^{*}T)}, \text{ and } \operatorname{rank}(\underline{G}) = \operatorname{rank}(G^{\star}) + 2.
$$

Using Theorem 2.5 and Proposition 3.11, we conclude the following theorem.

Theorem 3.17. *Let* $\underline{G} = (g_{ij})$ *be a Hermitian quaternionic* $m \times m$-*matrix,* $m > 2$, *such that* $g_{ii} = 0$, $g_{1j} = 1$ *for* $j = 2, \ldots, m$, *and* $g_{ij} \neq 0$ *for* $i \neq j$. *Let* $G^{\star}$ *be the associated matrix to* G. *Then* $\underline{G}$ *is a normalized special-Gram quaternionic matrix associated with some ordered* m-*tuple* $\mathfrak{p} = (p_1, \ldots, p_m)$ *of pairwise distinct points in* $\partial \mathbf{H}_{\mathbb{H}}^{n}$ *if and only if* $\operatorname{rank}(G^{\star}) \leq n - 1$ *and all principal minors of* $G^{\star}$ *are non-negative.*

4. The moduli space

4.1. Invariants. We recall Cartan's angular invariant and quaternionic cross-ratio. One may find more information in [3, 8, 27].

Let $\mathfrak{p} = (p_1, p_2, p_3)$ be an ordered triple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$. The quaternionic Cartan's angular invariant $\mathbb{A}_{\mathbb{H}}(\mathfrak{p})$ of $\mathfrak{p}$ is defined by

$$\mathbb{A}_{\mathbb{H}}(\mathfrak{p}) = \arccos \frac{\mathfrak{R}(-\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_1)}{|\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_1|},$$

where $\mathbf{p}_i \in \mathbb{H}^{n,1}$ are the lifts of p_i, $\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_1 = \langle \mathbf{p}_1, \mathbf{p}_2 \rangle_1 \langle \mathbf{p}_2, \mathbf{p}_3 \rangle_1 \langle \mathbf{p}_3, \mathbf{p}_1 \rangle_1$ and $\mathfrak{R}(-\langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \rangle_1) > 0$.

One can prove that $0 \le \mathbb{A}_{\mathbb{H}}(\mathfrak{p}) \le \pi/2$ and $\mathbb{A}_{\mathbb{H}}(\mathfrak{p})$ is independent of the chosen lifts and the order of three points.

We give the definition of quaternionic cross-ratio, following [8]. Suppose that $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ is a quadruple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$. Their cross-ratio is defined by

$$\mathbb{X}(p_1, p_2, p_3, p_4) = \langle \mathbf{p}_3, \mathbf{p}_1 \rangle_1 \langle \mathbf{p}_3, \mathbf{p}_2 \rangle_1^{-1} \langle \mathbf{p}_4, \mathbf{p}_2 \rangle_1 \langle \mathbf{p}_4, \mathbf{p}_1 \rangle_1^{-1},$$

where $\mathbf{p}_i$ are the lifts of p_i. Observe that

$$\mathbb{X}(p_1 \lambda_1, p_2 \lambda_2, p_3 \lambda_3, p_4 \lambda_4) = \overline{\lambda}_1 \langle \mathbf{p}_3, \mathbf{p}_1 \rangle_1 \langle \mathbf{p}_3, \mathbf{p}_2 \rangle_1^{-1} \langle \mathbf{p}_4, \mathbf{p}_2 \rangle_1 \langle \mathbf{p}_4, \mathbf{p}_1 \rangle_1^{-1} \overline{\lambda}_1^{-1}.$$

Let now $\mathfrak{p} = (p_1, \ldots, p_m)$ be an m-tuple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$. For short, let

$$\mathbb{X}_{2j} = \mathbb{X}(p_1, p_2, p_3, p_j), \ \mathbb{X}_{3j} = \mathbb{X}(p_1, p_3, p_2, p_j), \ \mathbb{X}_{kj} = \mathbb{X}(p_1, p_k, p_3, p_j),$$

where $m \ge 4$, $4 \le k \le m-1$, $k < j$. It is clear that the number of the above cross-ratios is equal to $m(m-3)/2$.

REMARK 4.1. Suppose that the lift of p_1 is always the standard lift, then we have $\lambda_1 = 1$. Hence we see that the above cross-ratios independent of chosen the lifts of p_i, where $i = 2, \ldots, m$. This combine with Remark 3.5 is the reason why that we let the lift of p_1 be the standard lift when we define the Special-Gram Quaternionic matrix. For the treatment of the cross-ratios as the invariants up to the action of $\mathrm{Sp}(1)$, see [17].

The proof of the following proposition follows after direct computations.

Proposition 4.2. *Let $\mathfrak{p} = (p_1, \ldots, p_m)$ be an m-tuple of pairwise distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$ and $\underline{G}(\mathfrak{p}) = (g_{ij})$ be the normalized special-Gram quaternionic matrix of $\mathfrak{p}$. Then the following relations hold:*

$$\mathbb{A}_{\mathbb{H}} = \mathbb{A}_{\mathbb{H}}(p_1, p_2, p_3) = \arg(-\overline{g}_{23}), \quad g_{23} = -re^{-\mathfrak{u}\mathbb{A}_{\mathbb{H}}},$$
$$\mathbb{X}_{2j} = \mathbb{X}(p_1, p_2, p_3, p_j) = g_{23}^{-1} g_{2j}, \quad g_{2j} = -re^{-\mathfrak{u}\mathbb{A}_{\mathbb{H}}} \mathbb{X}_{2j},$$
$$\mathbb{X}_{3j} = \mathbb{X}(p_1, p_3, p_2, p_j) = \overline{g}_{23}^{-1} g_{3j}, \quad g_{3j} = -re^{\mathfrak{u}\mathbb{A}_{\mathbb{H}}} \mathbb{X}_{3j},$$
$$\mathbb{X}_{kj} = \mathbb{X}(p_1, p_k, p_2, p_j) = \overline{g}_{2k}^{-1} g_{kj}, \quad g_{kj} = -r\overline{\mathbb{X}}_{2k} e^{\mathfrak{u}\mathbb{A}_{\mathbb{H}}} \mathbb{X}_{kj},$$

where $r = |g_{23}|$ and $\mathfrak{u} \in \mathfrak{sp}(1)$ and all indices are in accordance with the indices of the above-defined cross-ratios.

Suppose that $\{\mathbb{X}_1, \ldots, \mathbb{X}_d\}$ is the set of the above cross-ratios associated with an ordered

m-tuple $\mathfrak{p}$ of distinct points in $\partial\mathbf{H}_{\mathbb{H}}^n$, where $d = m(m-3)/2$. We observe that

$$\{\mathbb{X}_1, \ldots, \mathbb{X}_d, \mathbb{A}_{\mathbb{H}}\}$$

is the parameter system which is invariant under the action of $\mathrm{PSp}(n, 1)$ and independent of the chosen lifts of p_i, where $i = 2, \ldots, m$. According to Corollary 3.13, this system with a unit pure quaternion $\mathfrak{u}$ and a positive number r forms the minimal parameter system that uniquely determines the $\mathrm{PSp}(n, 1)$-configuration space $\mathcal{F}_1(n, m)$. Then we have the following result which is useful for us to study the moduli space.

Theorem 4.3. *$\mathcal{F}_1(n, m)$ is uniquely determined by a positive number r and a unit pure quaternion $\mathfrak{u} \in \mathfrak{sp}(1)$ and the invariants given the above $\mathbb{X}_{2j}, \mathbb{X}_{3j}, \mathbb{X}_{kj}, \mathbb{A}_{\mathbb{H}}$.*

4.2. Moduli space and proof of the Theorem 1.3. Assume that $\mathfrak{p} = (p_1, \ldots, p_m)$ is an m-tuple of pairwise distinct points in $\partial\mathbf{H}_{\mathbb{H}}^n$ and $\underline{G}(\mathfrak{p}) = (g_{ij})$ is the normalized special-Gram quaternionic matrix of $\mathfrak{p}$. Let $G^\star$ be the associated $(m-2) \times (m-2)$-matrix to $\underline{G}(\mathfrak{p})$. The principal minors of $G^\star$ are $\det_{\mathbb{M}}(G^\star_{I^s_{m-2}})$.

By Proposition 4.2, we may treat $\det_{\mathbb{M}}(G^\star_{I^s_{m-2}})$ as functions of

$$(\mathbb{X}_{2j}, \mathbb{X}_{3j}, \ldots, \mathbb{X}_{kj}, \mathfrak{u}, \mathbb{A}_{\mathbb{H}}, r).$$

Identify $w = (q_1, \ldots, q_d, \mathfrak{u}, t_1, t_2)$ with $(\mathbb{X}_{2j}, \mathbb{X}_{3j}, \ldots, \mathbb{X}_{kj}, \mathfrak{u}, \mathbb{A}_{\mathbb{H}}, r)$, where $q_i \in \mathbb{H}$ is nonzero, $i = 1, \ldots, d$ and $d = m(m-3)/2$.

We define the map

$$D^\star_{I^s_{m-2}} : \mathbb{H}^{m(m-3)/2} \times \mathfrak{sp}(1) \times \mathbb{R}^2 \to \mathbb{R},$$

given by

$$w \mapsto D^\star_{I^s_{m-2}}(w) = \det_{\mathbb{M}}(G^\star_{I^s_{m-2}}).$$

Suppose that $[\mathfrak{p}] \in \mathcal{F}_1(n, m)$. Then according to Theorem 3.17 and Proposition 4.2, we define the map

$$\tau_1 : \mathcal{F}_1(n, m) \to \mathbb{H}^{m(m-3)/2} \times \mathfrak{sp}(1) \times \mathbb{R}^2,$$

given by

$$[\mathfrak{p}] \mapsto w.$$

Let

$$\mathbb{M}_1(n, m) = \{w \in \mathbb{H}^d \times \mathfrak{sp}(1) \times \mathbb{R}^2 \mid D^\star_{I^s_{m-2}}(w) \geq 0, s \leq n-1; D^\star_{I^s_{m-2}}(w) = 0, s > n-1\},$$

where $w = (q_1, \ldots, q_d, \mathfrak{u}, t_1, t_2), 0 \neq q_i \in \mathbb{H}$, and $\mathfrak{u} \in \mathfrak{sp}(1), t_1 \in [0, \pi/2], t_2 > 0$ for $i = 1, \ldots, d = m(m-3)/2$.

Proof of the Theorem 1.3. We shall show that the above defined map τ_1 is a homeomorphism between $\mathcal{F}_1(n, m)$ and $\mathbb{M}_1(n, m)$. Let $\mathbb{M}_1(n, m)$ be equipped with the topology inherited from $\mathbb{H}^{m(m-3)/2} \times \mathfrak{sp}(1) \times \mathbb{R}^2$. Hence we only need to prove that τ_1 is bijective.

Injectivity follows straightforwardly by Theorem 4.3. It is only necessary to show that the map τ_1 is surjective.

If $w \in \mathbb{M}_1(n, m)$, we can construct a Hermitian quaternionic $m \times m$-matrix $\underline{G} = (g_{ij})$ with

$g_{ii} = 0$ and $g_{1j} = 1$ for $j = 2, \ldots, m$. We see that $w = (q_1, \ldots, q_d, \mathrm{u}, t_1, t_2)$ identify with $(\mathbb{X}_{2j}, \mathbb{X}_{3j}, \ldots, \mathbb{X}_{kj}, \mathrm{u}, \mathbb{A}_{\mathbb{H}}, r)$. Then we can fix the other entries of $\underline{G}$ by using Proposition 4.2. If this $\underline{G}$ satisfies the conditions in Theorem 3.17, then $\underline{G}$ is the normalized special-Gram quaternionic matrix for some ordered m-tuple of pairwise distinct points $\mathfrak{p}$ in $\partial \mathbf{H}_{\mathbb{H}}^n$. In other words, w uniquely corresponds to a point $[\mathfrak{p}] \in \mathcal{F}_1(n, m)$. This proves that τ_1 is surjective.   □

Corollary 4.4. $\mathbb{M}_1(n, m)$ *is a real* $2m^2 - 6m + 5 - \sum_{i=1}^{m-n-1} \binom{m-2}{n-1+i}$ *dimensional subset of a algebraic variety with the same real dimension when* $m > n + 1$.

4.3. Moduli space of Loxodromic representation family and proof of Theorem 1.6. In this section, we study the moduli space for representation family of discrete, faithful, totally loxodromic and finitely generated groups into $\mathrm{PSp}(2, 1)$. Our target space is the $\mathrm{PSp}(2, 1)$-character variety.

DEFINITION 4.5. A subgroup of $\mathrm{PSp}(n, 1)$ is called *totally loxodromic* if it comprises only loxodromic elements and the identity.

Proposition 4.6 (Corollary 4.5.4. of [11]). *Let G be a totally loxodromic subgroup of* $\mathrm{PSp}(n, 1)$ *and let $M(G)$ be a totally geodesic submanifold with G-invariant such that the quaternionic dimension* $\dim_{\mathbb{H}} M(G)$ *is even. Then G is discrete.*

Let G_0 be a totally loxodromic subgroup of $\mathrm{PSp}(n, 1)$. If $\dim_{\mathbb{H}} M(G_0)$ is even, then by Proposition 4.6, G_0 is discrete. For example, the totally loxodromic subgroups of $\mathrm{PSp}(2, 1)$ that are generated by finite distinct generators without common fixed points are discrete subgroups of $\mathrm{PSp}(2, 1)$.

Suppose that $\Gamma = \langle h_1, h_2, \ldots, h_k \rangle$ is a finitely generated group with a set of generators $h_1, h_2, \ldots, h_k$ and G a topological group. The set of homomorphisms $\mathrm{Hom}(\Gamma, G)$ naturally sits inside G^k via the evaluation map $f : \mathrm{Hom}(\Gamma, G) \to G^k$ given by $\rho \mapsto (\rho(h_1), \ldots, \rho(h_k))$. Hence, $\mathrm{Hom}(\Gamma, G)$ has an induced topology by f.

DEFINITION 4.7. The *representation family* of discrete, faithful, totally loxodromic representations of Γ into G is

$$\mathrm{Lox}(\Gamma, G) = \{\rho \in \mathrm{Hom}(\Gamma, G) \mid \rho \text{ is injective}; \ \rho(\Gamma) \text{ is discrete, loxodromic}\}.$$

We endow $\mathrm{Lox}(\Gamma, G)$ with the topology of pointwise convergence. In this topology, a sequence formed by the homomorphisms $\rho_j : \Gamma \to G$, $j = 1, 2, \ldots$, converges to a homomorphism $\rho : \Gamma \to G$ if and only if for each $h \in \Gamma$ the sequence $\rho_1(h), \rho_2(h), \ldots$ converges to $\rho(h)$ in the topological group G.

In the remainder of this section we assume $G = \mathrm{PSp}(2, 1)$. The points of $\mathrm{Lox}(\Gamma, G)$ are identified with the G-conjugation equivalence classes of $G_0 = \rho(\Gamma)$. Let $G_0 = \langle g_1, \ldots, g_k \rangle$ with $g_i = \rho(h_i)$, then $g_i \neq g_j$ for all $i \neq j$.

Before the proof of Theorem 1.6 we restate the following result.

Lemma 4.8 (Theorem 3.1 (i) of [9]). *Let A be an element of isometry group preserving the Hermitian form H_2 in $\mathbb{H}^{2,1}$. If A is a loxodromic element, then A is conjugate to an element of the form*

$$L = L(\beta, \theta) = \begin{pmatrix} \lambda e^{i\beta} & 0 & 0 \\ 0 & \lambda^{-1} e^{i\beta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \quad \lambda > 0,\ \lambda \neq 1,$$

where $0 \leq \beta,\ \theta \leq \pi$.

Proof of Theorem 1.6. First, we need to verify that the *G*-conjugation equivalence class of each loxodromic element g_i can be uniquely determined by some parameters. Denote by $\mathrm{Sp}(2,1)$ the isometry group preserving the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = -(\overline{z_0} w_1 + \overline{z_1} w_0) + \overline{z_2} w_2$$

which gives the Siegel domain model. Suppose that $A \in \mathrm{Sp}(2,1)$ is a loxodromic element. By Lemma 4.8, we obtain that every loxodromic equivalence class with respect to the conjugation action of $\mathrm{Sp}(2,1)$ can be uniquely determined by parameters λ, β and θ.

According to Lemma 2.10, $\mathrm{Sp}(2,1)$ is doubly transitive on quaternionic lines in V_0^2. Then we associate the quintuple

$$(p_i^+, p_i^-, \lambda, \beta, \theta)$$

to each g_i, where p_i^+ and p_i^- are respectively the attracting and the repelling fixed points of g_i. Since G_0 is discrete, all fixed points $p_1^+, p_1^-, \ldots, p_k^+, p_k^-$ are distinct. Therefore

$$(p_1^+, p_1^-, \ldots, p_k^+, p_k^-;\ \lambda_1 \ldots, \lambda_k;\ \beta_1, \theta_1 \ldots, \beta_k, \theta_k),$$

can be associated with $G_0 = \langle g_1, \ldots, g_k \rangle$.

Now, according to Theorem 1.3, the proof is completed. □

The following corollary follows from Corollary 4.4 and Theorem 1.6.

Corollary 4.9. *The real dimension of the moduli space of* $\mathrm{Lox}(\Gamma, G)$ *is not more than*

$$8k^2 - 9k + 5 - \sum_{i=1}^{2k-3} \binom{2k-2}{1+i}$$

when $k \geq 2$.

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