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ON DEVIATIONS AND SPREADS OF MEROMORPHIC MINIMAL SURFACES

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Abstract

In this paper we consider the influence that the number of separated maximum points of the norm of a meromorphic minimal surface (m.m.s) has on the magnitudes of growth and value distribution. We present sharp estimations of spread of m.m.s in terms of Nevanlinna's defect, magnitude of deviation and the number of separated points of the norm of m.m.s. We also give examples showing that the estimates are sharp.

1. Introduction

In the years 1960 - 1970 Beckenbach and collaborators generalized the original Nevanlinna's theory by introducing theory of meromorphic minimal surfaces [5, 6]. A surface S is called a *minimal* if the mean curvature of S vanishes on all point on the surface ([7]). We remind the main definitions and results of Beckenbach's theory. We say that the surface

$$S = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i = x_i(u, v), i = 1, 2, 3, \ (u, v) \in D \subset \mathbb{R}^2\}$$

is given in terms of *isothermal parameters* u, v ([5]) if $E = G, \ F = 0$, where $x_i(u, v), \ i = 1, 2, 3$, are a twice continuously differentiable real-valued functions for $(u, v) \in D \subset \mathbb{R}^2$. Here E, F, G are the coefficients of the first fundamental form for the surface S:

$$E = \| \mathbf{x}_u \|^2 = \sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial u}\right)^2, \quad F = (\mathbf{x}_u, \mathbf{x}_v) = \sum_{j=1}^{3} \frac{\partial x_j}{\partial u}\frac{\partial x_j}{\partial v},$$
$$G = \| \mathbf{x}_v \|^2 = \sum_{j=1}^{3} \left(\frac{\partial x_j}{\partial v}\right)^2,$$

where $\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$. A necessary and sufficient condition for a regular surface S, given in terms of isothermal parameters, to be minimal is this that the coordinate functions $x_i(u, v), \ i = 1, 2, 3$, are harmonic ([7], [21]).

We recall now some facts from the theory of harmonic functions. The point $z_0 \in \mathbb{C}$ is an isolated singular point of a function $x(z) = x(u, v)$ $(z = u + iv)$, if in a neighborhood of a point z_0 the function $x(z)$ is harmonic. If the point $z_0 \in \mathbb{C}$ is an isolated singular point of the harmonic function $x(z)$, then in neighborhood of a point z_0 the function $x(z)$ can be presented by a series of the form

$$x(z) = c \log r + \sum_{k=-\infty}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta), \quad (b_0 = 0), \tag{1}$$

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where $z - z_0 = re^{i\theta}$. Expansion (1) is an analog of Laurent's series of a harmonic function. This expansion allows us to define poles, logarithmic poles and essential singular points [18].

Point $z_0 \in D$ is called *regular* for the function $x(z)$, if for representation (1) in the neighborhood of the point z_0 we have $c = 0$ and $\min\limits_{a_k^2+b_k^2\neq 0}\{k\} \geq 0$. We shall assume that $z_0 \neq \infty$.

If $\min\limits_{a_k^2+b_k^2\neq 0}\{k\} = t \geq 1$ and $x(z_0) = a_0$, then the point z_0 is called an *a_0-point of order t* of the harmonic function. In particulary if $a_0 = 0$ then the point z_0 is called *zero of order t* of the harmonic function.

We say that a point $z_0 \in D$ is *a pole of order $t = |l|$* of a function $x(z)$, if in the representation (1) we have $\min\limits_{a_k^2+b_k^2\neq 0}\{k\} = l < 0$. On the other hand if in (1) we have $c \neq 0$ and $\min\limits_{a_k^2+b_k^2\neq 0}\{k\} \geq 0$ thet the point z_0 is called a *logarithmic pole*.

If in (1) there are infinitely many coefficients with negative indices, such that $a_k^2 + b_k^2 \neq 0$ then we say that z_0 is an essential singular point of the function $x(z)$. We say that a harmonic function $x(z)$ is a *meromorphic harmonic* function in the domain D if, except for the poles, there are no more singular points of the function $x(z)$ in D.

Definition. [5] The surface $S = \{x_1(u, v), x_2(u, v), x_3(u, v)\}$ is called a meromorphic minimal surface (m.m.s, for short) in a domain D if the parameters u, v are isothermal (i.e. $E = G$ and $F = 0$ for each $(u, v) \in D$) and the coordinate functions $x_1(u, v), x_2(u, v), x_3(u, v)$ are single valued and harmonic in D, except for the poles.

In this paper we shall consider meromorphic minimal surfaces defined on the whole complex plane $\mathbb{C}$. We say that a surface S is an *entire minimal surface* if the coordinate functions are harmonic in the whole plane $\mathbb{C}$. A point $z_0 \in D$ is called a *pole* of m.m.s. in a domain D, if at least one of the coordinate functions $x_1(z), x_2(z), x_3(z)$ has a pole at z_0. Moreover if l_1, l_2, l_3 are the orders of the poles of functions $x_1(z), x_2(z), x_3(z)$ accordingly, then $l = \max\{l_1, l_2, l_3\}$ is called the *order of the pole of a m.m.s.* at z_0. A meromorphic minimal surface S cannot have a logarithmic poles[5].

A point $z_0 \in D$ is called an $\mathbf{a} = (a_1, a_2, a_3)$-point of a surface S, if z_0 is an a_i-point of the harmonic function $x_i(z)$, $i = 1, 2, 3$. Let l_i be the order of an a_i-point of the function $x_i(z)$. Then $l = \min\{l_1, l_2, l_3\}$ is the order of an $\mathbf{a}$-point of a surface S. The $\mathbf{a}$-points and the poles of a m.m.s. are isolated [5].

For m.m.s. S Beckenbach and Hutchison defined three functions: $m(r, \mathbf{a}, S)$- a *proximity* function of S, $N(r, \mathbf{a}, S)$- an ***a**-points counting* function of S and $H(r, \mathbf{a}, S)$- a *visibility* function, which are defined in the following way:

$$m(r, \mathbf{a}, S) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \|\mathbf{x}(re^{i\theta})\|\, d\theta & \text{for } \mathbf{a} = \infty\ , \\ \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\|\mathbf{x}(re^{i\theta})-\mathbf{a}\|}\, d\theta & \text{for } \mathbf{a} \neq \infty\ , \end{cases}$$

where $\log^+ x = \max(\log x, 0)$ for $x \geq 0$ and $\|\mathbf{x}(z)\| = \sqrt{x_1^2(z) + x_2^2(z) + x_3^2(z)}$ $(z = re^{i\theta})$,

$$N(r, \mathbf{a}, S) = \begin{cases} \int_0^r \frac{n(\rho,\infty,S)-n(0,\infty,S)}{\rho} d\rho + n(0, \infty, S)\log r & \text{for } \mathbf{a} = \infty\ , \\ \int_0^r \frac{n(\rho,\mathbf{a},S)-n(0,\mathbf{a},S)}{\rho} d\rho + n(0, \mathbf{a}, S)\log r & \text{for } \mathbf{a} \neq \infty\ , \end{cases}$$

where $n(r, \mathbf{a}, S)$ and $n(r, \infty, S)$ denote, respectively, the number of $\mathbf{a}$-points ($\mathbf{a} \in \mathbb{R}^3$) and poles of meromorphic minimal surface S in the circle $\{z\colon |z| \leq r\}$, counted according to multiplicity,

$$H(r, \mathbf{a}, S) = \begin{cases} 0 & \text{for } \mathbf{a} = \infty \ , \\ \int_0^r \frac{h(\rho, \mathbf{a}; S)}{\rho} d\rho & \text{for } \mathbf{a} \neq \infty \ , \end{cases}$$

where $h(\rho, \mathbf{a}, S) = \frac{1}{2\pi} \iint_{A_\rho(0)} \triangle \log \|\mathbf{x}(u, v) - \mathbf{a}\| du dv$, $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace's operator and $A_\rho(0) = \{z \in \mathbb{C}\colon |z| \leq \rho\}$. The function $T(r, S) = m(r, \infty, S) + N(r, \infty, S)$ is called the *characteristic* of a meromorphic minimal surface S. The number

$$\lambda = \liminf_{r \to \infty} \frac{\log T(r, S)}{\log r}$$

is called the lower order of S and the quantity

$$\delta(\mathbf{a}, S) = \liminf_{r \to \infty} \frac{m(r, \mathbf{a}, S)}{T(r, S)}$$

is called a defect of S. In [5] Beckenbach and Hutchison get an analog of Nevanlinna's first fundamental theorem for minimal surfaces. The theorem states that if S is a meromorphic minimal surface then for each $\mathbf{a} \in \mathbb{R}^3$

$$m(r, \mathbf{a}, S) + N(r, \mathbf{a}, S) + H(r, \mathbf{a}, S) = T(r, S) + O(1) \ \ (r \to \infty).$$

Beckenbach and Cootz in [6] generalize Nevanlinna's second fundamental theorem to minimal surfaces. The theorem says that for a meromorphic minimal surface S and points $\mathbf{a}_k \in \mathbb{R}^3$ $(k = 1, \ldots, q)$ we have the following inequality

$$\sum_{k=1}^{q} m(r, \mathbf{a}_k, S) \ \leq \ 2T(r, S) + O(\log(rT(r, S))), \ r \notin E, \ r \to \infty,$$

where E is a set of finite measure. Notice that $N(r, \mathbf{a}, S)$ vanishes almost everywhere in $\mathbb{R}^3$ so the most important function in Beckenbach's theory is $H(r, \mathbf{a}, S)$. In 1979 Marchenko applied Petrenko's theory of growth of meromorphic functions ([19]) to the theory of meromorphic minimal surfaces. In [14] were defined the quantities

$$\mathcal{L}(r, \mathbf{a}, S) = \begin{cases} \max\limits_{|z|=r} \log^+ \frac{1}{\|\mathbf{x}(z) - \mathbf{a}\|} & \text{for } \mathbf{a} \neq \infty \ , \\ \max\limits_{|z|=r} \log^+ \|\mathbf{x}(z)\| & \text{for } \mathbf{a} = \infty \ , \end{cases}$$

and

$$\beta(\mathbf{a}, S) = \liminf_{r \to \infty} \frac{\mathcal{L}(r, \mathbf{a}, S)}{T(r, S)}.$$

$\beta(\mathbf{a}, S)$ is called the *magnitude of deviation of the meromorphic minimal surface* S at the point $\mathbf{a}$. In [14] a sharp upper estimate of $\beta(\mathbf{a}, S)$ for surfaces of the finite lower order was also obtained.

Theorem A. [14] *If S is a meromorphic minimal surface of the finite lower order λ, then for each $\mathbf{a} \in \mathbb{R}^3 \cup \{\infty\}$*

$$\beta(\boldsymbol{a}, S) \leq B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & for\ \ \lambda \leq \frac{1}{2}\ , \\ \pi\lambda & for\ \ \lambda > \frac{1}{2}\ . \end{cases}$$

In 2004 Ciechanowicz and Marchenko applied a quantity measuring the number of separated maximum modulus points of a meromorphic function to obtain an upper estimate of deviation for meromorphic functions ([8], see also [16]). We defined in [13] a similar quantity for meromorphic minimal surfaces.

Let $\phi(r)$ be a positive, nondecreasing convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, S))$ and $p_\phi(r, \infty, S)$ be the number of component intervals of the set

$$\{\,\theta\colon\ \log \|\mathbf{x}(re^{i\theta})\| > \phi(r)\,\}$$

possessing at lest one maximum modulus point of the function $\|\mathbf{x}(re^{i\theta})\|$. Moreover, let us denote $p_\phi(\infty, S) = \liminf\limits_{r\to\infty} p_\phi(r, \infty, S)$. We set

$$p(\infty, S) = \sup_{\{\phi\}} p_\phi(\infty, S).$$

In [13] we get an upper estimate of the magnitude of deviation for a meromorphic minimal surface of the finite lower order.

Theorem B. *For a meromorphic minimal surface S of the finite lower order λ, we have*

$$\beta(\infty, S) \leq \begin{cases} \frac{\pi\lambda}{p(\infty,S)} & if\ \ \frac{\lambda}{p(\infty,S)} \geq \frac{1}{2}\ , \\ \frac{\pi\lambda}{\sin \pi\lambda} & if\ \ p(\infty, S) = 1\ \ and\ \ \lambda < \frac{1}{2}\ , \\ \frac{\pi\lambda}{p(\infty,S)} \sin \frac{\pi\lambda}{p(\infty,S)} & if\ \ p(\infty, S) > 1\ \ and\ \ \frac{\lambda}{p(\infty,S)} < \frac{1}{2}\ . \end{cases}$$

Corollary. *For a meromorphic minimal surface S of the finite lower order λ, we have*

$$p(\infty, S) \leq \max\left(1, \left[\frac{\pi\lambda}{\beta(\infty, S)}\right]\right).$$

Moreover if $\beta(\infty, S) > 0$ then

$$1 \leq p(\infty, S) < +\infty.$$

2. Main Results

In 1967, Edrei [10] defined the spread of a meromorphic function. Petrenko defined in [20] a similar quantity for m.m.s. Let $\Lambda(r)$ be a positive, nondecreasing continuous function, such that $\Lambda(r) = o(T(r, S))$ $(r \to \infty)$. Let us denote

$$\sigma_\Lambda(\infty, S) := \limsup_{r\to\infty} mes\{\theta\colon\ \log \|\mathbf{x}(re^{i\theta})\| > \Lambda(r)\},$$

where $mesA$ means Lebesgue measure of the set A. We set

$$\sigma(\infty, S) := \inf_{\Lambda} \sigma_\Lambda(\infty, S).$$

The quantity $\sigma(\infty, S)$ is called *the spread of a meromorphic minimal surface*.

In this paper we get a sharp lower estimation for spread of m.m.s. in terms of $\delta(\infty, S)$, $\beta(\infty, S)$ and $p(\infty, S)$. Our main results are as follows.

Theorem 1. *If S is a meromorphic minimal surface of the finite lower order λ and $\delta(\infty, S) > 0$, then*

$$\sigma(\infty, S) \geq \min\left\{2\pi, \frac{4p(\infty, S)}{\lambda} \arcsin\sqrt{\frac{\delta(\infty, S)}{2}}\right\}.$$

It is easy to see that if $\delta(\infty, S) > 0$ then $p(\infty, S) \geq 1$. So we have

Corollary 1. *If S is a meromorphic minimal surface of the finite lower order λ, then*

$$\sigma(\infty, S) \geq \min\left\{2\pi, \frac{4}{\lambda} \arcsin\sqrt{\frac{\delta(\infty, S)}{2}}\right\}.$$

Corollary 1 was obtained by Petrenko in 1981 [20]. In section 7 of this paper we present an example of m.m.s. for which equality holds in Corollary 1. In the case of meromorphic functions the sharp estimation of spread in terms of Nevanlinna defect $\delta(\infty, f)$ was obtained by Baernstein in 1973 [1]. It was a solution of Edrei's conjecture [10].

Theorem 2. *If S is a meromorphic minimal surface of the finite lower order λ and $\beta(\infty, S) > 0$, then*

$$\sigma(\infty, S) \geq \min\left\{2\pi, \frac{2p(\infty, S)}{\lambda} \arcsin\frac{\beta(\infty, S)p(\infty, S)}{\pi\lambda}\right\}.$$

If $\beta(\infty, S) > 0$ then $p(\infty, S) \geq 1$. Hence we have

Corollary 2. *If S is a meromorphic minimal surface of the finite lower order λ, then*

$$\sigma(\infty, S) \geq \min\left\{2\pi, \frac{2}{\lambda} \arcsin\frac{\beta(\infty, S)}{\pi\lambda}\right\}.$$

The sharp estimation of spread for meromorphic functions in terms of magnitude of Petrenko's deviation $\beta(\infty, f)$ was proved by Marchenko in 1982 [15].

If $E \subset (0, \infty)$ is a measurable set then the quantities

$$\overline{logdens} E = \limsup_{R\to\infty} \frac{1}{\log R} \int\limits_{E\cap[1,R]} \frac{dt}{t},$$

$$\underline{logdens} E = \liminf_{R\to\infty} \frac{1}{\log R} \int\limits_{E\cap[1,R]} \frac{dt}{t}$$

are called, respectively, the *upper* and *lower logarithmic density* of E.

Theorem 3. *Let S be a meromorphic minimal surface of the finite lower order λ and order ρ. For $0 < \gamma < \infty$ put*

$$B(\gamma) := \begin{cases} \frac{\pi\gamma}{\sin\pi\gamma} & \text{for } \gamma \leq \frac{1}{2}, \\ \pi\gamma & \text{for } \gamma > \frac{1}{2}, \end{cases} \qquad E(\gamma) := \{r > 0 : \mathcal{L}(r, \infty, S) \leq B(\gamma)T(r, S)\},$$

Then

$$\overline{logdens}\ E(\lambda) \geq 1 - \frac{\lambda}{\gamma} \ \ and \ \ \underline{logdens}\ E(\gamma) \geq 1 - \frac{\rho}{\gamma}.$$

In the case of meromorphic functions the estimation of logarithmic density of a set $E(\gamma) := \{r > 0 : \log^+ \max |f(z)| \leq B(\gamma)T(r, f)\}$ was obtained by Marchenko in 1998 [17].

3. Auxiliary results

Let $S = \{\mathbf{x}(z) = (x_1(z), x_2(z), x_3(z)) \colon z \in \mathbb{C}\}$ be a meromorphic minimal surface and let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ such that $\phi(r) = o(T(r, S))$. We consider the function given by

$$u_\phi(z) = \max\{ \log \|\mathbf{x}(z)\|, \phi(|z|) \}.$$

In [13] we proved the following lemma.

Lemma 1. *The function $u_\phi(z)$ is a $\delta-$subharmonic function in $\mathbb{C}$, i.e.*

$$u_\phi(z) = u_1(z) - u_2(z),$$

where $u_1(z), u_2(z)$ are subharmonic functions in $\mathbb{C}$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u_2(re^{i\theta})d\theta = N(r, \infty, S).$$

Set

$$m^*(r, \theta, u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\phi(re^{i\varphi})d\varphi,$$
$$T^*(r, \theta, u_\phi) = m^*(r, \theta, u_\phi) + N(r, \infty, S),$$

where $r \in (0, \infty)$, $\theta \in [0, \pi]$, E is a measurable set and $|E|$ is the Lebesgue measure of E [1, 13]. Now for each $t \in (0, +\infty)$, consider the set

$$G_t = \{re^{i\varphi} \colon u_\phi(re^{i\varphi}) > t\},$$

and let

$$u^*_\phi(re^{i\varphi}) = \sup\{t \colon re^{i\varphi} \in G^*_t\},$$

where G^*_t is the symmetric rearrangement of the set G_t [12]. The function $u^*_\phi(re^{i\varphi})$ is non-negative and non-increasing in the interval $[0, \pi]$, even with respect to ϕ and for each fixed r equimeasurable with $u_\phi(re^{i\varphi})$. Moreover, it satisfies the equalities:

$$u^*_\phi(r) = \max\{\log \max_{|z|=r} \|\mathbf{x}(z)\|, \phi(r)\},$$
$$u^*_\phi(re^{i\pi}) = \max\{ \log \min_{|z|=r} \|\mathbf{x}(z)\|, \phi(r) \},$$
$$m^*(r, \theta, u_\phi) = \frac{1}{\pi} \int_0^\theta u^*_\phi(re^{i\varphi})\, d\varphi.$$

From Baernstein's theorem ([2]), the function $T^*(r, \theta, u_\phi)$ is subharmonic in $D = \{re^{i\theta} \colon 0 < r < \infty,\ 0 < \theta < \pi\}$, continuous in $D \cup (-\infty, 0) \cup (0, \infty)$ and logarithmically convex in $r > 0$ for each fixed $\theta \in [0, \pi]$. Moreover,

$$T^*(r,0,u_\phi) = N(r,\infty,S),$$
$$T^*(r,\pi,u_\phi) = T(r,S) + o(T(r,S)) \quad (r\to\infty),$$
$$\frac{\partial}{\partial\theta}T^*(r,\theta,u_\phi) = \frac{u^*_\phi(re^{i\theta})}{\pi} \quad \text{for } 0<\theta<\pi.$$

Let $\alpha(r)$ be a real-valued function of a real variable r and define

$$L\alpha(r) = \liminf_{h\to 0}\frac{\alpha(re^h)+\alpha(re^{-h})-2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable in r, then $L\alpha(r) = r\frac{d}{dr}r\frac{d}{dr}\alpha(r)$.

Lemma 2. *Let $S = \{\boldsymbol{x}(z) = (x_1(z), x_2(z), x_3(z))\colon z\in\mathbb{C}\}$ be a meromorphic minimal surface. For almost all $\theta\in[0,\pi]$ and for all $r>0$ such that the function $\|\boldsymbol{x}(z)\|$ has neither zeros nor poles in $\{z\colon |z|=r\}$, we have*

$$LT^*(r,\theta,u_\phi)\geq -\frac{1}{\pi}\frac{\partial u^*_\phi(r,\theta)}{\partial\theta}.$$

In the case of twice differentiable function $T^*(r,\theta,u_\phi)$ Lemma 2 follows by subharmonicity of this function because

$$\begin{aligned} LT^*(r,\theta,u_\phi) = r\tfrac{d}{dr}r\tfrac{d}{dr}T^*(r,\theta,u_\phi) &= r^2\Delta T^*(r,\theta,u_\phi) - \tfrac{\partial^2}{\partial\theta^2}T^*(r,\theta,u_\phi)\\ &\geq -\tfrac{\partial^2}{\partial\theta^2}T^*(r,\theta,u_\phi) = -\tfrac{1}{\pi}\tfrac{\partial u^*_\phi(r,\theta)}{\partial\theta}, \end{aligned}$$

where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is Laplace's operator, $z = re^{i\theta} = x+iy$. The proof of Lemma 2 is analogous to proof of lemma 2 in [13].

Let now $\{R_n\}$ be a sequence of positive numbers such that

$$\lambda = \liminf_{r\to\infty}\frac{\log T(r,S)}{\log r} = \lim_{n\to\infty}\frac{\log T(2R_n,S)}{\log R_n} \tag{2}$$

and for $n\geq n_0$

$$T(2R_n,S) < R_n^{\lambda+1}.$$

Let ϕ be nondecreasing positive convex function of $\log r$ such that $\phi(r) = o(T(r,S))$ and $0<p_\phi(\infty,S)<\infty$. For $\tau>0$ we choose numbers α and ψ satisfying following conditions

$$0<\alpha\leq\min\{\pi,\frac{\pi p_\phi(\infty,S)}{2\tau}\},\quad -\frac{\pi p_\phi(\infty,S)}{2\tau}\leq\psi\leq\frac{\pi p_\phi(\infty,S)}{2\tau}-\alpha.$$

We denote

$$\begin{aligned} h_{\phi,\tau}(r)\colon = \frac{p^2_\phi(\infty,S)}{\pi}&\left(\cos\frac{\tau\psi}{p_\phi(\infty,S)}u^*_\phi(r) - \cos\frac{\tau(\alpha+\psi)}{p_\phi(\infty,S)}u^*_\phi(re^{i\alpha})\right)\\ &-\tau p_\phi(\infty,S)\left(\sin\frac{\tau(\alpha+\psi)}{p_\phi(\infty,S)}T^*(r,\alpha,u_\phi) - \sin\frac{\tau\psi}{p_\phi(\infty,S)}N(r,\infty,S)\right). \end{aligned} \tag{3}$$

In [13, p. 148] we get an equality which we present as the following lemma.

Lemma 3. *Let S be a meromorphic minimal surface of the finite lower order λ. Then for every $\epsilon>0$ there is a sequence $\{r_k\}\to\infty$ $(k\to\infty)$ such that for $k>k_0(\epsilon)$*

$$h_{\phi,\lambda}(r_k) < \epsilon T(r_k, S). \tag{4}$$

When the lower order λ is zero, (4) *holds if λ is any positive number.*

For $\tau > 0$, $0 \le \alpha \le \min\{\pi, \frac{\pi}{2\tau}\}$ and $\frac{\pi}{2\tau} \le \psi \le \frac{\pi}{2\tau}$ denote

$$\begin{aligned}\widetilde{h}_{\phi,\tau}(r) :&= \frac{1}{\pi}\cos(\tau\psi)u^*_\phi(r) - \frac{1}{\pi}\cos(\tau(\alpha+\psi))u^*_\phi(re^{i\alpha}) \\ &\quad - \tau\sin(\tau(\alpha+\psi))T^*(r,\alpha,u_\phi) + \tau\sin(\tau\psi)N(r,\infty,S).\end{aligned} \tag{5}$$

Lemma 4. *Let S be a meromorphic minimal surface of the finite lower order λ and $A = \{r \in (0,\infty) : \widetilde{h}_{\phi,\tau}(r) > 0\}$. Then*

$$\tau \int\limits_{A\cap[1,R_n]} \frac{dt}{t} \le (1+o(1))\log T(2R_n, S) \quad (n \to \infty),$$

where R_n is defined in (2)*.*

Proof. In accordance with [16], we put

$$\sigma(r) = \int\limits_0^\alpha T^*(r,\theta,u_\phi)\cos\tau(\theta+\psi)\,d\theta.$$

Since $T^*(r,\theta,u_\phi)$ is a convex function of $\log r$, we have $LT^*(r,\theta,u_\phi) \ge 0$. By applying Fatou's lemma we get

$$\begin{aligned}L\sigma(r) &= L\int\limits_0^\alpha T^*(r,\theta,u_\phi)\cos\tau(\theta+\psi)\,d\theta \\ &\ge \int\limits_0^\alpha LT^*(r,\theta,u_\phi)\cos\tau(\theta+\psi)\,d\theta \ge 0\end{aligned} \tag{6}$$

Hence $\sigma(r)$ is a convex function of $\log r$, so $r\sigma'_-(r)$ is an increasing function on $(0,\infty)$. Then for almost all $r > 0$

$$L\sigma(r) = r\frac{d}{dr}(r\sigma'_-(r)).$$

By Lemma 2 and inequality (6) we get that for almost all $r > 0$,

$$L\sigma(r) = r\frac{d}{dr}(r\sigma'_-(r)) \ge -\int\limits_0^\alpha \frac{1}{\pi}\frac{\partial u^*_\phi(re^{i\theta})}{\partial\theta}\cos\tau(\theta+\psi)\,d\theta.$$

If for $r > 0$ there are no zeros and poles of $\|\mathbf{x}(z)\|$ on the circle $\{z \in \mathbb{C} : |z| = r\}$, the function $u_\phi(re^{i\theta})$ satisfies the Lipschitz condition in θ. Hence $u^*_\phi(re^{i\theta})$ also satisfies the Lipschitz condition on $[0,\pi]$. It follows from [12] that $u^*_\phi(re^{i\theta})$ is absolutely continuous on $[0,\pi]$. Integrating by parts twice we get

$$\int\limits_0^\alpha \frac{1}{\pi}\frac{\partial u^*_\phi(re^{i\theta})}{\partial\theta}\cos\tau(\theta+\psi)\,d\theta$$

$$= -\frac{1}{\pi}\Big(\cos(\tau\psi)u_\phi^*(r) - \cos(\tau(\alpha+\psi))u_\phi^*(re^{i\alpha})\Big)$$
$$+ \tau\Big(\sin(\tau(\alpha+\psi))T^*(r,\alpha,u_\phi) - \sin(\tau\psi)N(r,\infty,S)\Big) - \tau^2\sigma(r)$$
$$= -\widetilde{h}_{\phi,\tau}(r) - \tau^2\sigma(r).$$

Since $u_\phi^*(re^{i\theta})$ is decreasing in θ, then for almost all $r > 0$

$$\widetilde{h}_{\phi,\tau}(r) + \tau^2\sigma(r) \geq 0. \tag{7}$$

Thus for almost all $r > 0$ we get $r\frac{d}{dr}(r\sigma_-'(r)) \geq \widetilde{h}_{\phi,\tau}(r) + \tau^2\sigma(r) \geq 0$. From [16], dividing both sides of this inequality by $r^{\tau+1}$ and integrating by parts over the interval $[r, R_n]$ we obtain

$$\int_r^{R_n} \frac{\widetilde{h}_{\phi,\tau}(t)}{t^{\tau+1}}dt \leq \int_r^{R_n} \frac{1}{t^\tau}\frac{d}{dt}(t\sigma_-'(t))dt - \tau^2\int_r^{R_n}\frac{\sigma(t)}{t^{\tau+1}}dt \tag{8}$$
$$\leq \left(\frac{t\sigma_-'(t)}{t^\tau} + \tau\frac{\sigma\tau}{t^\tau}\right)\Bigg|_r^{R_n}, \quad 0 < r \leq R_n.$$

Now we apply Barry's method [3, 4]. We consider the function

$$\Phi(r) = -\int_r^{R_n}\frac{\widetilde{h}_{\phi,\tau}(t)}{t^{\tau+1}}dt, \quad 0 < r \leq R_n.$$

From the inequality (7) we have

$$\Phi(r) \geq -\frac{\sigma_-'(R_n)}{R_n^{\tau-1}} - \tau\frac{\sigma(R_n)}{R_n^\tau} + \frac{\sigma_-'(r)}{r^{\tau-1}} + \tau\frac{\sigma(r)}{r^\tau}.$$

We put

$$\psi(r) = r^\tau\left(\Phi(r) + \frac{\sigma_-'(R_n)}{R_n^{\tau-1}} + \tau\frac{\sigma(R_n)}{R_n^\tau}\right). \tag{9}$$

Then we get

$$\psi(r) \geq r\sigma_-'(r) + \tau\sigma(r), \quad 0 < r \leq R_n.$$

From (7) we get, for almost all $r > 0$

$$r\psi'(r) = \tau\psi(r) + \widetilde{h}_{\phi,\tau}(r) \geq \tau r\sigma_-'(r) + \tau^2\sigma(r) + \widetilde{h}_{\phi,\tau}(r) \geq \tau r\sigma_-'(r) \geq 0.$$

The function $T^*(r,\alpha,u_\phi)$ is increasing for $r \geq r_0$ ([9]), then $\sigma(r)$ is increasing on $[r_0, R_n]$. Therefore $r\sigma_-'(r) \geq 0$ for all $r \geq r_0$. Moreover $\sigma(r) > 0$ for all $r \geq r_0$. Then for all $r \geq r_0$ we have

$$\psi(r) \geq r\sigma_-'(r) + \tau\sigma(r) > 0.$$

Let $r \in A = \{r \in (0,\infty) : \widetilde{h}_{\phi,\tau}(r) > 0\}$. Then $r\psi'(r) = \tau\psi(r) + \widetilde{h}_{\phi,\tau}(r) > \tau\psi(r) > 0$ for all $r_0 \leq r \leq R_n$. Hence $\frac{\psi'(r)}{\psi(r)} > \frac{\tau}{r}$, so for $r \geq r_0$ we have

$$\tau\int_{A\cap[1,R_n]}\frac{dr}{r} \leq \int_{A\cap[r_0,R_n]}\frac{\psi'(r)}{\psi(r)}dr + \tau\log r_0 \tag{10}$$

$$\leq \int_{r_0}^{R_n} \frac{\psi'(r)}{\psi(r)} dr + \tau \log r_0$$
$$= \log \frac{\psi(R_n)}{\psi(r_0)} + \tau \log r_0 \quad (n \to \infty).$$

On the other hand, $\psi(R_n) = R_n \sigma'_-(R_n) + \tau\sigma(R_n)$. From the definition of $\sigma(r)$ it follows that

$$\sigma(r) = \int_0^\alpha T^*(r, \theta, u_\phi) \cos \tau(\theta + \psi) d\theta \leq \int_0^\alpha T^*(r, \theta, u_\phi) d\theta$$
$$\leq \int_0^\alpha (T(r, S) + o(T(r, S))) d\theta \leq \pi T(r, S) + o(T(r, S)) \quad (r \to \infty).$$

It follows from the monotonicity of $r\sigma'_-(r)$ that

$$r\sigma'_-(r) \leq \int_r^{2r} \sigma'_-(t) dt \leq \sigma(2r) \leq \pi T(2r, S) + o(T(r, S)) \quad (r \to \infty).$$

Then from this inequality and (9) and (10) we obtain

$$\tau \int_{A \cap [1, R_n]} \frac{dr}{r} \leq \log \frac{\psi(R_n)}{\psi(r_0)} + O(1) \leq \log \psi(R_n) + O(1)$$
$$= \log [R_n \sigma'_-(R_n) + \tau\sigma(R_n)] + O(1)$$
$$\leq \log T(2R_n, S) + o(T(R_n, S)) \quad (n \to \infty),$$

which completes the proof of Lemma 4. □

4. Proof of Theorem 1

Consider the case $\lambda > 0$. Let $\Lambda = \Lambda(r)$ be a positive, nondecreasing continuous function which satisfies $\Lambda(r) = o(T(r, S))$ $(r \to \infty)$ and $\phi = \phi(r)$ a positive, nondecreasing convex function of $\log r$ for $r > 0$ which satisfies $\phi(r) = o(T(r, S))$. Since $\delta(\infty, S) > 0$ we have $\beta(\infty, S) > 0$. Hence $1 \leq p_\phi(\infty, S) < +\infty$. We will prove that

$$\sigma_\Lambda(\infty, S) \geq \min \left\{ 2\pi, \frac{4p_\phi(\infty, S)}{\lambda} \arcsin \sqrt{\frac{\delta(\infty, S)}{2}} \right\}. \tag{11}$$

If $\sigma_\Lambda(\infty, S) \geq \min\left\{2\pi, \frac{\pi p_\phi(\infty, S)}{\lambda}\right\}$ then (11) is fulfilled. Let $\sigma_\Lambda(\infty, S) < \min\left\{2\pi, \frac{\pi p_\phi(\infty, S)}{\lambda}\right\}$. Choose α such that

$$\frac{\sigma_\Lambda(\infty, S)}{2} < \alpha < \min\left\{\pi, \frac{\pi p_\phi(\infty, S)}{2\lambda}\right\}.$$

Hence, for $r \geq r_0$ we have $u^*_\phi(r, \alpha) \leq \max\{\Lambda(r), \phi(r)\} = o(T(r, S))$ and

$$T^*(r, \alpha, u_\phi) = T(r, S) + o(T(r, S)) \quad (r \to \infty).$$

From Lemma 3 there exists a sequence $\{r_k\}$ such that for $k \geq k_0(\epsilon)$

$$h_{\phi,\tau}(r_k) < \varepsilon T(r_k, S),$$

So we have

$$(12)\quad -u^*_\phi(r_k,\alpha)\cos\frac{\lambda(\alpha+\psi)}{p_\phi(\infty,S)} + \log\max_{|z|=r_k}\|\mathbf{x}(z)\|\cos\frac{\lambda\psi}{p_\phi(\infty,S)} + \frac{\lambda\pi}{p_\phi(\infty,S)}N(r_k,\infty,S)\sin\frac{\lambda\psi}{p_\phi(\infty,S)} - \frac{\lambda\pi}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)\sin\frac{\lambda\psi}{p_\phi(\infty,S)} < \varepsilon T(r_k,S)\ (k\to\infty).$$

Let us put $\psi = -\frac{\pi p_\phi(\infty,S)}{2\lambda}$. Then, for all $k \ge k_0(\epsilon)$

$$-u^*_\phi(r_k,\alpha)\sin\frac{\lambda\alpha}{p_\phi(\infty,S)} - \frac{\pi\lambda}{p_\phi(\infty,S)}N(r_k,\infty,S) + \frac{\pi\lambda}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} < \epsilon T(r_k,S)\quad(k\to\infty).$$

Since $u^*_\phi(r,\alpha) \le \max\{\Lambda(r),\phi(r)\}$ we have

$$-\max\{\Lambda(r_k),\phi(r_k)\}\sin\frac{\lambda\alpha}{p_\phi(\infty,S)} - \frac{\pi\lambda}{p_\phi(\infty,S)}N(r_k,\infty,S) + \frac{\pi\lambda}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} < \epsilon T(r_k,S)\quad(k\to\infty).$$

By the definition of defect we have $1-\delta(\infty,S) = \limsup_{r\to\infty}\frac{N(r,\infty,S)}{T(r,S)}$. Hence, for $r \ge r_0(\epsilon)$

$$N(r,\infty,S) < (1-\delta(\infty,S)+\epsilon)T(r,S).$$

Since $\max\{\Lambda(r),\phi(r)\} = o(T(r,S))$ there is a sequence $\{r_k\}$ tending to infinity such that

$$o(T(r_k,S))\sin\frac{\lambda\alpha}{p_\phi(\infty,S)} - \frac{\pi\lambda}{p_\phi(\infty,S)}(1-\delta(\infty,S)+\epsilon)T(r_k,S) + \frac{\pi\lambda}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} < \epsilon T(r_k,S)\quad(k\to\infty).$$

Dividing both sides of this inequality by $T(r_k,S)$ and taking the limit as $k\to\infty$ we obtain

$$-1+\delta(\infty,S)-\epsilon+\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} < \frac{\epsilon p_\phi(\infty,S)}{\pi\lambda}.$$

By taking the limit as $\epsilon\to 0$ in this inequality we have

$$-1+\delta(\infty,S)+\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} \le 0,$$

so

$$1-\cos\frac{\lambda\alpha}{p_\phi(\infty,S)} \le \delta(\infty,S)$$

for each α such that $\frac{\sigma_\Lambda(\infty,S)}{2} < \alpha < \min\left\{\pi, \frac{\pi p_\phi(\infty,S)}{2\lambda}\right\}$. Then by taking the limit as $\alpha \to \frac{\sigma_\Lambda(\infty,S)}{2}$ we have

$$1 - \cos \frac{\lambda \sigma_\Lambda(\infty, S)}{2p_\phi(\infty, S)} \geq \delta(\infty, S).$$

It follows that

$$\sin^2 \frac{\lambda \sigma_\Lambda(\infty, S)}{4p_\phi(\infty, S)} \geq \frac{\delta(\infty, S)}{2}.$$

Hence

$$\sigma_\Lambda(\infty, S) \geq \frac{4p_\phi(\infty, S)}{\lambda} \arcsin \sqrt{\frac{\delta(\infty, S)}{2}}.$$

Then for any Λ and ϕ we have

$$\sigma_\Lambda(\infty, S) \geq \min \left\{ 2\pi, \frac{4p_\phi(\infty, S)}{\lambda} \arcsin \sqrt{\frac{\delta(\infty, S)}{2}} \right\}.$$

Hence

$$\sigma(\infty, S) \geq \min \left\{ 2\pi, \frac{4p(\infty, S)}{\lambda} \arcsin \sqrt{\frac{\delta(\infty, S)}{2}} \right\}.$$

Then Theorem 1 is proved for $\lambda > 0$. If the lower order of S is zero then from Lemma 3 the inequality

$$h_{\phi,\lambda}(r_k) < \epsilon T(r_k, S)$$

holds if λ is any positive number. Then by repeating all steps in proof of first case we get

$$\sigma_\Lambda(\infty, S) \geq \min \left\{ 2\pi, \frac{4p_\phi(\infty, S)}{\lambda} \arcsin \sqrt{\frac{\delta(\infty, S)}{2}} \right\},$$

for any positive number λ. We have in that case $\sigma_\Lambda(\infty, S) = 2\pi$ and hence $\sigma(\infty, S) = 2\pi$. This completes the proof of Theorem 1.

5. Proof of Theorem 2

Let $\Lambda = \Lambda(r)$ be a positive, nondecreasing continuous function which satisfies $\Lambda(r) = o(T(r, S))$ $(r \to \infty)$ and $\phi = \phi(r)$ a positive, nondecreasing convex function of $\log r$ for $r > 0$ which satisfies $\phi(r) = o(T(r, S))$. Since $\beta(\infty, S) > 0$ we have $1 \leq p_\phi(\infty, S) < +\infty$. We will prove that

$$\sigma_\Lambda(\infty, S) \geq \min \left\{ 2\pi, \frac{2p_\phi(\infty, S)}{\lambda} \arcsin \frac{p_\phi(\infty, S)\beta(\infty, S)}{\pi\lambda} \right\}.$$

Let $\sigma_\Lambda(\infty, S) < \min \left\{ 2\pi, \frac{\pi p_\phi(\infty, S)}{\lambda} \right\}$. Choose α such that

$$\frac{\sigma_\Lambda(\infty, S)}{2} < \alpha < \min \left\{ \pi, \frac{\pi p_\phi(\infty, S)}{2\lambda} \right\}.$$

Since $\sigma_\Lambda(\infty, S) < 2\alpha$, then for $r \geq r_0$ we have $u^*_\phi(r, \alpha) \leq \max\{\Lambda(r), \phi(r)\}$. Put $\psi = 0$. By (12), we have

$$-u_\phi^*(r_k,\alpha)\cos\frac{\lambda\alpha}{p_\phi(\infty,S)}+\log\max_{|z|=r_k}\|\mathbf{x}(z)\|-\frac{\pi\lambda}{p_\phi(\infty,S)}\sin\frac{\lambda\alpha}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)$$
$$<\epsilon T(r_k,S)\quad(k\to\infty).$$

Since $u_\phi^*(r,\alpha)\le\max\{\Lambda(r),\phi(r)\}$ we have

$$-\max\{\Lambda(r_k),\phi(r_k)\}\cos\frac{\lambda\alpha}{p_\phi(\infty,S)}+\log\max_{|z|=r_k}\|\mathbf{x}(z)\|$$
$$-\frac{\pi\lambda}{p_\phi(\infty,S)}\sin\frac{\lambda\alpha}{p_\phi(\infty,S)}T^*(r_k,\alpha,u_\phi)<\epsilon T(r_k,S)\quad(k\to\infty).$$

Since $T^*(r_k,\alpha,u_\phi)=T(r_k,S)(1+o(1))\quad(k\to\infty)$ then

$$\beta(\infty,S)<\tfrac{\pi\lambda}{p_\phi(\infty,S)}\sin\tfrac{\lambda\alpha}{p_\phi(\infty,S)}+\tfrac{\max\{\Lambda(r_k),\phi(r_k)\}}{T(r_k,S)}\cos\tfrac{\lambda\alpha}{p_\phi(\infty,S)}+2\epsilon\ \ (k\to\infty).$$

But $\max\{\Lambda(r_k),\phi(r_k)\}=o(T(r_k,S))$, so by passing to the limit in this inequality as $k\to\infty$, $\epsilon\to 0$ and $\alpha\to\frac{\sigma_\Lambda(\infty,S)}{2}$ we have

$$\beta(\infty,S)\le\frac{\pi\lambda}{p_\phi(\infty,S)}\sin\frac{\lambda\sigma_\Lambda(\infty,S)}{2p_\phi(\infty,S)}.$$

Hence

$$\sigma_\Lambda(\infty,S)\ge\frac{2p_\phi(\infty,S)}{\lambda}\arcsin\frac{p_\phi(\infty,S)\beta(\infty,S)}{\pi\lambda}.$$

Therefore for any Λ and ϕ we have

$$\sigma_\Lambda(\infty,S)\ge\min\left\{2\pi,\frac{2p_\phi(\infty,S)}{\lambda}\arcsin\frac{p_\phi(\infty,S)\beta(\infty,S)}{\pi\lambda}\right\}.$$

Hence

$$\sigma(\infty,S)\ge\min\left\{2\pi,\frac{2p(\infty,S)}{\lambda}\arcsin\frac{p(\infty,S)\beta(\infty,S)}{\pi\lambda}\right\}.\tag{13}$$

Theorem 2 is proved for the case $\lambda>0$. Let now the lower order of S be zero. Then inequality (13) holds for any $\lambda>0$. Therefore $\sigma(\infty,S)=2\pi$. This completes proof of Theorem 2.

6. Proof of Theorem 3

If $\gamma\le\lambda$ the theorem is straightforward. Assume that $\gamma>\lambda$. Let us also take a number τ such that $\lambda<\tau<\gamma$. In Lemma 4 we put $\alpha=\min\left(\pi,\frac{\pi}{2\tau}\right)$, $\psi=\frac{\pi}{2\tau}-\alpha$. Then

$$\begin{aligned}\widetilde{h}_{\phi,\tau}(r)&=\frac{1}{\pi}u_\phi^*(r)\sin(\tau\alpha)-\tau T^*(re^{i\alpha},S)+\tau\cos(\tau\alpha)N(r,\infty,S)\\&\ge\frac{1}{\pi}u_\phi^*(r)\sin(\tau\alpha)-\tau(1+o(1))T(r,S)\\&\ge\frac{\sin(\tau\alpha)}{\pi}(\mathcal{L}(r,\infty,S)-B(\gamma)T(r,S))\quad(r>r_0),\end{aligned}\tag{14}$$

where $\widetilde{h}_{\phi,\tau}(r)$ is defined in (5). From Lemma 4 we have

$$\tau \int\limits_{A\cap[1,R_n]} \frac{dt}{t} \le (1+o(1))\log T(2R_n, S) \quad (n\to\infty),$$

where $A = \{r \in (0,\infty) \colon \widetilde{h}_{\phi,\tau}(r) > 0\}$. Then by (2) we have $\tau \underline{logdens} A \le \lambda$. Let

$$A_1 = \{r \in (r_0,\infty) : \mathcal{L}(r,\infty,S) - B(\gamma)T(r,S) > 0\}.$$

From (14) it follows that $A_1 \subset A$, so

$$\underline{logdens} A_1 \le \underline{logdens} A \le \frac{\lambda}{\tau}.$$

Then

$$\overline{logdens}\{r \in (0,\infty) : \mathcal{L}(r,\infty,S) \le B(\gamma)T(r,S)\} \ge 1 - \frac{\lambda}{\tau}.$$

Hence for each τ, such that $\lambda < \tau < \gamma$ we have

$$\overline{logdens} E(\gamma) \ge 1 - \frac{\lambda}{\tau},$$

and taking the limit with $\tau \to \gamma$ we obtain

$$\overline{logdens} E(\gamma) \ge 1 - \frac{\lambda}{\gamma}.$$

The proof for the lower logarithmic density can be done in a similar way with taking for R_n any positive numbers. This completes the proof of Theorem 3.

7. Examples

We consider the surface $S(f)$ given by the relations

$$\begin{cases} x_1(z) &= Re[3f(z) - f^3(z)] \ , \\ x_2(z) &= Re[i(3f(z) + f^3(z))] \ , \\ x_3(z) &= Re[3f^2(z)] \ , \end{cases}$$

where $f(z)$ is a meromorphic function [14]. Then the coordinate functions are harmonic in $\mathbb{C}$. From [5], to prove that $S(f)$ is a m.m.s. it is enough to show that

$$\sum_{i=1}^{3} \left(\frac{dg_i(z)}{dz}\right)^2 \equiv 0,$$

where

$$g_1(z) = 3f(z) - f^3(z),\ g_2(z) = i(3f(z) + f^3(z)),\ g_3(z) = 3f^2(z).$$

By basic computations we see that

$$\begin{aligned} \|\mathbf{x}(z)\|^2 = 9|f(z)|^2 + |f(z)|^6 + 6(Im[f(z)]Im[f^3(z)] \\ - Re[f(z)]Re[f^3(z)]) + 9(Re[f^2(z)])^2. \end{aligned}$$

We consider the set $E(r) = \{\theta \in [0,2\pi] \colon |f(re^{i\theta})| > 4\}$. If $z = re^{i\theta},\ \theta \in E(r)$ then we have

$$\|\mathbf{x}(z)\|^2 \ge |f(z)|^6 - 12|f(z)|^4 \ge \frac{1}{4}|f(z)|^6.$$

Then $\log^+ \|\mathbf{x}(z)\| \geq 3\log^+ |f(z)| + O(1) \quad (r \to \infty)$. On the other hand, for $z = re^{i\theta},\ \theta \in E(r)$ we get

$$\|\mathbf{x}(z)\|^2 \leq 9|f(z)|^2 + |f(z)|^6 + 21|f(z)|^4 \leq 31|f(z)|^6.$$

Then $\log^+ \|\mathbf{x}(z)\| \leq 3\log^+ |f(z)| + O(1) \quad (r \to \infty)$. Thus we obtain

$$\begin{aligned} m(r,\infty,S(f)) &= 3m(r,\infty,f) + O(1), \\ \mathcal{L}(r,\infty,S(f)) &= 3\mathcal{L}(r,\infty,f) + O(1), \quad r \to \infty. \end{aligned} \tag{15}$$

It is easy to see that $N(r,\infty,S(f)) = 3N(r,\infty,f)$, so by (15) we have

$$T(r,S(f)) = 3T(r,f) + O(1), \tag{16}$$

which implies that

$$\delta(\infty,S(f)) = \delta(\infty,f), \quad \beta(\infty,S(f)) = \beta(\infty,f).$$

Let us now consider the Mittag-Leffler function ([11]), i.e.

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(1+\frac{k}{\rho}\right)}, \quad 0 < \rho < \infty.$$

From asymptotic of this function for $\frac{1}{2} < \rho < +\infty$ we have

$$E_\rho(z) = \begin{cases} \rho e^{z^\rho} + O\left(\frac{1}{|z|}\right) & \text{for } |arg(z)| \leq \frac{\pi}{2\rho}\ , \\ O\left(\frac{1}{|z|}\right) & \text{for } \pi \geq |arg(z)| > \frac{\pi}{2\rho}\ . \end{cases}$$

If $\rho = \frac{1}{2}$ then $E_\rho(z) = \cosh\sqrt{z}$ we have $E_{\frac{1}{2}}(z) = \frac{1}{2}e^{z^{\frac{1}{2}}} + O(1)$. If $0 < \rho < \frac{1}{2}$ then $E_\rho(z) = (1+o(1))\rho e^{z^\rho}$, where $|arg(z)| \leq \pi - \delta$ for any positive number δ.
So we have

$$T(r,E_\rho) = \begin{cases} \frac{1}{\pi\rho} r^\rho + o(r^\rho) & \text{for } \frac{1}{2} \leq \rho < \infty\ , \\ \frac{\sin\pi\rho}{\pi\rho} r^\rho + o(r^\rho) & \text{for } 0 < \rho < \frac{1}{2}\ . \end{cases} \tag{17}$$

To get the sharpness of the estimate in Theorem 1 we take any $\lambda > 0$ and $n \in \mathbb{N}$ such that $\frac{\lambda}{n} > \frac{1}{2}$ and consider the function

$$f_1(z) = E_{\frac{\lambda}{n}}(z^n).$$

Then

$$T(r,f_1) = \frac{n}{\pi\lambda} r^\lambda + o(r^\lambda) \quad (r \to \infty)$$

so by (16) we have

$$T(r,S(f_1)) = \frac{3n}{\pi\lambda} r^\lambda + o(r^\lambda) \ \ (r \to \infty).$$

A surface $S(f_1)$ is entire minimal surface, so $\delta(\infty,S(f_1)) = 1$. Moreover, function $f_1(z)$ is entire and has n maximum modulus points so $p(\infty,f_1) = n$. Note that

$$\mathcal{L}(r,\infty,f_1) = r^\lambda + O(1) \quad (r \to \infty),$$

Hence

$$\beta(\infty, f_1) = \frac{\pi\lambda}{n},$$

so

$$\beta(\infty, S(f_1)) = \frac{\pi\lambda}{n},$$

$$p(\infty, S(f_1)) = n,$$

$$\sigma(\infty, S(f_1)) = \frac{n\pi}{\lambda}.$$

Thus the estimates from Theorem 1 and Theorem 2 are attained for $S(f_1)$. To prove the sharpness of the estimate in the second case from Theorem 1 and Theorem 2 consider for each $0 < \lambda \leq \frac{1}{2}$ the function

$$f_2(z) = E_\lambda(z).$$

We have $p(\infty, f_2) = 1$. The function $f_2(z)$ is of the finite lower order λ and from (17) we have

$$T(r, f_2) = \frac{\sin \pi\lambda}{\pi\lambda} r^\lambda + o(r^\lambda) \quad (r \to \infty).$$

Then $T(r, S(f_2)) = \frac{3 \sin \pi\lambda}{\pi\lambda} r^\lambda + o(r^\lambda) \quad (r \to \infty)$, so a surface $S(f_2)$ is of order λ. A surface $S(f_2)$ is a entire minimal surface, so $\delta(\infty, S(f_2)) = 1$. Moreover

$$\mathcal{L}(r, \infty, f_2(z)) = r^\lambda + O(1) \quad (r \to \infty),$$
$$\beta(\infty, S(f_2)) = \beta(\infty, f_2) = \frac{\pi\lambda}{\sin \pi\lambda},$$

and

$$\sigma(\infty, S(f_2)) = 2\pi.$$

Thus the estimations in Theorem 1 and Theorem 2 are also attained for this special case.

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