

# A CLASSIFICATION PROBLEM ON MAPPING CLASSES ON FIBER SPACES OVER TEICHMÜLLER SPACES

YINGQING XIAO and CHAOHUI ZHANG

(Received October 27, 2016, revised October 12, 2017)

## Abstract

Let  $\tilde{S}$  be an analytically finite Riemann surface which is equipped with a hyperbolic metric. Let  $S = \tilde{S} \setminus \{\text{one point } x\}$ . There exists a natural projection  $\Pi$  of the  $x$ -pointed mapping class group  $\text{Mod}_S^x$  onto the mapping class group  $\text{Mod}(\tilde{S})$ . In this paper, we classify elements in the fiber  $\Pi^{-1}(\chi)$  for an elliptic element  $\chi \in \text{Mod}(\tilde{S})$ , and give a geometric interpretation for each element in  $\Pi^{-1}(\chi)$ . We also prove that  $\Pi^{-1}(t_a^n \circ \chi)$  or  $\Pi^{-1}(t_a^n \circ \chi^{-1})$  consists of hyperbolic mapping classes provided that  $t_a^n \circ \chi$  and  $t_a^n \circ \chi^{-1}$  are hyperbolic, where  $a$  is a simple closed geodesic on  $\tilde{S}$  and  $t_a$  is the positive Dehn twist along  $a$ .

## 1. Introduction

Let  $S$  be an analytically finite Riemann surface of genus  $p$  which contains  $n + 1$  punctures  $\{x, x_1, \dots, x_n\}$ . Assume that  $2p + n > 4$ . Let  $\text{Mod}_S^x$  denote the subgroup of the mapping class group  $\text{Mod}(S)$  that consists of isotopy classes of self-maps of  $S$  that fix  $x$ , which implies that  $\text{Mod}_S^x$  is a subgroup of  $\text{Mod}(S)$  with a finite index and each element of  $\text{Mod}_S^x$  can be projected, under the natural projection

$$\Pi : \text{Mod}_S^x \rightarrow \text{Mod}(\tilde{S}),$$

to an element of  $\text{Mod}(\tilde{S})$ , where  $\tilde{S} = S \cup \{x\}$ .

Fix an element  $\chi \in \text{Mod}(\tilde{S})$ . Consider

$$\Pi^{-1}(\chi) = \{\theta \in \text{Mod}_S^x \mid \Pi(\theta) = \chi\}.$$

In the case where  $\chi = \text{id}$ ,  $\Pi^{-1}(\chi)$  is a normal subgroup of  $\text{Mod}_S^x$  which can be identified with the fundamental group  $\pi_1(\tilde{S}, x)$ , and thus it is isomorphic to a covering group of  $\tilde{S}$ . In [8] Kra classified all elements in  $\Pi^{-1}(\text{id})$  by using the terms introduced in [3] and [11]. He also investigated on elements in  $\Pi^{-1}(\chi)$  for a non-trivial element  $\chi$  and showed that if  $\chi \in \text{Mod}(\tilde{S})$  is hyperbolic, then elements in  $\Pi^{-1}(\chi)$  are either hyperbolic or pseudo-hyperbolic (see the definitions in Section 2). The problem of whether or not  $\Pi^{-1}(\chi)$  contains hyperbolic mapping classes were also studied in [13, 15].

In this paper, we study the problem of classifying elements in  $\Pi^{-1}(\chi)$  when  $\chi \in \text{Mod}(\tilde{S})$  is elliptic with prime order  $m \geq 3$ . We also study compositions of elliptic mapping classes and mapping classes induced by simple Dehn twists.

This paper is organized as follows. Section 2 is an overview of Teichmüller spaces and Bers fiber spaces. Main results of this paper are stated in the section also. In Section 3 we

prove some lemmas. Section 4 is devoted to the proof of Theorem 2.6 and Theorem 2.7. In Section 5 we prove Theorem 2.8.

**2. Main Results**

We begin with an overview of Teichmüller spaces and some basic properties. More details can be found in Bers [3, 4].

**§2.1.** Let  $S$  be an analytically finite Riemann surface which is endowed with a hyperbolic metric. The *Teichmüller space*  $T(S)$  is the space of equivalence classes  $[\mu]$  of conformal structures  $(w : S \rightarrow \mu(S))$ , where  $(w : S \rightarrow \mu(S))$  and  $(w' : S \rightarrow \mu'(S))$  are in the same equivalence class  $[\mu]$  if there is a conformal map  $h : \mu(S) \rightarrow \mu'(S)$  such that  $h \circ w$  is isotopic to  $w'$ .

Every self-map  $f$  of  $S$  induces an element in  $\text{Mod}(S)$ . It is well known that  $\text{Mod}(S)$  acts effectively and discontinuously on  $T(S)$  as a group of biholomorphisms when  $2p + n > 4$ . Following Thurston [11], a non-periodic self-map  $f$  of  $S$  is called reducible if there is a non-contractible homotopically independent curve system  $\{\gamma_1, \dots, \gamma_s\}$ ,  $s \geq 1$ , on  $S$  so that  $\gamma_i$  is not homotopic to  $\gamma_j$  whenever  $i \neq j$  and  $1 \leq i, j \leq s$ , and for each  $\gamma_i$ ,  $f(\gamma_i)$  is homotopic to a  $\gamma_j$ . A self-map  $f$  of  $S$  is called irreducible if no such system can be found. By [11],  $f$  is irreducible if and only if it is isotopic to a pseudo-Anosov map (see [11] for the definition of a pseudo-Anosov map).

The mapping class group  $\text{Mod}(S)$  can be viewed as a group of biholomorphisms of  $T(S)$ . Let  $\langle \cdot, \cdot \rangle$  denote the Teichmüller distance. By introducing the index

$$a(\sigma) = \inf\{\langle y, \sigma(y) \rangle \text{ for } y \in T(S)\},$$

Bers [4] classified elements  $\sigma$  of  $\text{Mod}(S)$  as follows. A mapping class  $\sigma$  is elliptic if  $a(\sigma) = 0$  and the value is achievable; parabolic if  $a(\sigma) = 0$  and the value is not achievable; hyperbolic if  $a(\sigma) > 0$  and the value is achievable; or pseudo-hyperbolic if  $a(\sigma) > 0$  and the value is not achievable. Remarkable theorems in [4] state that (i)  $\sigma$  is elliptic if and only if it is induced by a periodic map, (ii)  $\sigma$  is hyperbolic if and only if  $\sigma$  is induced by an irreducible (or pseudo-Anosov) map, and (iii)  $\sigma$  is parabolic or pseudo-hyperbolic if and only if  $\sigma$  is induced by a reducible map.

**§2.2.** Let  $\tilde{S}, S$  be as in the introduction. Associated to each  $[\mu] \in T(\tilde{S})$  is a Jordan domain  $\mathcal{D}_\mu$  depending holomorphically on the equivalence class  $[\mu]$ . The *Bers fiber space*  $F(\tilde{S})$  is the set of pairs

$$\{([\mu], z) \mid [\mu] \in T(\tilde{S}), z \in \mathcal{D}_\mu\}.$$

There is a holomorphic projection  $\pi : F(\tilde{S}) \rightarrow T(\tilde{S})$ .

By Theorem 10 of [3], there is a holomorphic bijection (Bers isomorphism, see [3])  $\varphi$  of  $F(\tilde{S})$  onto  $T(S)$  that makes the diagram

$$\begin{array}{ccc} F(\tilde{S}) & \xrightarrow{\varphi} & T(S) \\ \pi \downarrow & & \downarrow \iota \\ T(\tilde{S}) & \xrightarrow{\text{id}} & T(\tilde{S}) \end{array}$$

commutative, where  $\iota : T(S) \rightarrow T(\tilde{S})$  is the forgetful map. Notice that  $S$  is of type  $(p, n + 1)$ .

**§2.3.** The mapping class group  $\text{Mod}(\tilde{S})$  can be lifted to a group  $\text{mod}(\tilde{S})$  that acts biholomorphically and effectively on  $F(\tilde{S})$ . To do this, we let  $\mathbf{D}$  be a unit disk endowed with a hyperbolic metric. Let  $\chi \in \text{Mod}(\tilde{S})$  be represented by  $f$ . Fix a lift  $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$  under a universal covering map

$$(2.1) \quad \varrho : \mathbf{D} \rightarrow \tilde{S}.$$

Let  $G$  be the covering group. Then every lift of  $f$  is of the form  $g_1 \circ \hat{f} \circ g_2$ , where  $g_1$  and  $g_2 \in G$ . Let  $\hat{f}$  and  $\hat{f}'$  be two lifts of self-maps of  $\tilde{S}$ . We say  $\hat{f}$  and  $\hat{f}'$  are equivalent if  $\hat{f}|_{S^1} = \hat{f}'|_{S^1}$ . The equivalence class of  $\hat{f}$  is denoted by  $[\hat{f}]$ . The group  $\text{mod}(\tilde{S})$  consists of all such  $[\hat{f}]$ . It is known that the group  $G$  is regarded as a normal subgroup of  $\text{mod}(\tilde{S})$  by conjugation, which keeps each fiber of  $F(\tilde{S})$  invariant.

It is important to note that the Bers isomorphism  $\varphi : F(\tilde{S}) \rightarrow T(S)$  induces an isomorphism  $\varphi^*$  of  $\text{mod}(\tilde{S})$  onto  $\text{Mod}_S^x$  by carrying each element  $[\hat{f}]$  to  $\varphi^*([\hat{f}]) = \varphi \circ [\hat{f}] \circ \varphi^{-1}$ , where we recall that  $\text{Mod}_S^x$  is a subgroup of  $\text{Mod}(S)$  with index  $n + 1$ . Observe also that every element  $\text{Mod}_S^x$  can be described as a mapping class that fixes  $x$ . So an element  $\theta \in \text{Mod}_S^x$  naturally projects to an element  $\chi$  in  $\text{Mod}(\tilde{S})$ .

**§2.4.** Interestingly, there are highly non-trivial mapping classes in  $\text{Mod}_S^x$  that project to the trivial mapping class. To describe them, let  $[\alpha] \in \pi_1(\tilde{S}, x)$ . Then  $\alpha$  is considered a trace of an isotopy of  $x$ . Extend the isotopy to an isotopy  $\{f_t : \tilde{S} \rightarrow \tilde{S}\}$  so that  $f_0 = \text{id}$  and  $f_1(x) = x$ . Thus  $f_1|_S$  defines a mapping class in  $\text{Mod}_S^x$ . Define

$$(2.2) \quad j : \pi_1(\tilde{S}, x) \rightarrow \text{Mod}_S^x.$$

by sending  $[\alpha]$  to the mapping class of  $f_1|_S$ . A closed curve  $c \subset \tilde{S}$  is called to fill  $\tilde{S}$  if its complement in  $\tilde{S}$  is a disjoint union of topological disks and (possibly) punctured disks. Likewise,  $c$  is called semi-filling with respect to a curve system  $\{\gamma_1, \dots, \gamma_u\}$ ,  $u \geq 1$ , if  $c$  fills the component of  $\tilde{S} \setminus \{\gamma_1, \dots, \gamma_u\}$  where  $c$  resides. It was shown in [8] that  $j([c])$  is hyperbolic if and only if  $c$  fills  $\tilde{S}$ , and  $j([c])$  is pseudo-hyperbolic if and only if  $c$  is a semi-filling geodesic.

**§2.5.** To state our main result, we return to an elliptic mapping class  $\chi \in \text{Mod}(\tilde{S})$  with prime order  $m \geq 3$ . Let  $\theta \in \Pi^{-1}(\chi)$  be induced by a self-map  $f_\theta$  of  $S$ . In the case where  $f_\theta$  is non-periodic reducible self-map of  $S$ ,  $\theta$  can be further classified as a type (I) or type (II) mapping class, where  $\theta$  is called a type (I) mapping class if  $f_\theta$  is reduced by the boundary  $\partial\Delta$  of a twice punctured disk  $\Delta \subset S$  enclosing  $x$  and another puncture of  $\tilde{S}$  such that the restriction  $f_\theta|_{S \setminus \Delta}$  is isotopic to a periodic self-map with prime order  $m$ .  $\theta$  is called to be of type (II) if there is a curve system  $\mathcal{A} = \{\alpha_1, \dots, \alpha_v\}$ ,  $v \geq 1$ , where each  $\alpha_i$  is non-contractible on  $\tilde{S}$ , such that the following three conditions hold:

- $f_\theta$  leaves invariant the component  $\mathcal{R}$  of  $S \setminus \mathcal{A}$  that contains  $x$ ,
- $f_\theta|_{\mathcal{R}}$  is irreducible, and permutes other components of  $S \setminus \mathcal{A}$ , and
- $f_\theta^m$  can be expressed as  $j([c])$  for a semi-filling loop  $c$  with respect to  $\mathcal{A}$ .

It is obvious that each type (I) or type (II) mapping class projects to a periodic mapping class of order  $m$ . Hence  $f$  has some fixed points on the compactification of  $\tilde{S}$  some of which may be punctures of  $\tilde{S}$ . As mentioned before, for each  $\chi \in \text{Mod}(\tilde{S})$ ,  $\Pi^{-1}(\chi)$  consists of elliptic, hyperbolic, and non-hyperbolic elements, where non-hyperbolic elements are either

parabolic or pseudo-hyperbolic. From the definition, we know that any type (I) mapping class is parabolic, while any type (II) mapping class is pseudo-hyperbolic.

**§2.6.** Under our circumstances,  $\chi$  is elliptic with prime order  $m \geq 3$ . Our main theorems below state that for any element  $\theta \in \Pi^{-1}(\chi)$ , if  $\theta$  is not elliptic, then  $\theta$  is either of type (I), or of type (II), or hyperbolic. More precisely, from Nielsen's theorem (see Ivanov [7] for example),  $\chi$  has a fixed point in  $T(\tilde{S})$ . We assume without loss of generality that the fixed point is represented by  $\tilde{S}$ . Thus there is a representative  $f$  of  $\chi$  that can be realized as a conformal automorphism of  $\tilde{S}$  with order  $m$ .

**Theorem.** *For each non-trivial elliptic mapping class  $\chi \in \text{Mod}(\tilde{S})$  with prime order  $m \geq 3$ , we have:*

- (1)  $\Pi^{-1}(\chi)$  contains (infinitely many) elliptic elements if and only if  $f$  fixes at least one point of  $\tilde{S}$ ,
- (2)  $\Pi^{-1}(\chi)$  contains (infinitely many) (I) parabolic elements if and only if  $f$  fixes at least one puncture of  $\tilde{S}$ ,
- (3)  $\Pi^{-1}(\chi)$  always contains (infinitely many) type (II) or hyperbolic mapping classes, and
- (4) if in addition  $\tilde{S} \setminus \langle f \rangle$  has genus  $\tilde{p} = p/m > 1$ , then  $\Pi^{-1}(\chi)$  contains (infinitely many) hyperbolic elements.

**§2.7.** Let  $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$  be a lift of  $f$ . Then  $\hat{f}$  is a conformal automorphism of  $\mathbf{D}$ . Thus  $\hat{f} \in \text{PSL}(2, \mathbf{R})$  but  $\hat{f}$  is not an element of  $G$ . Note that any element in  $\Pi^{-1}(\chi)$  can be written in the form  $\varphi^*([\hat{f}])$  for a conformal automorphism  $\hat{f}$  of  $\mathbf{D}$ .

More information on  $\Pi^{-1}(\chi)$  is contained in the following result.

**Theorem.** *Let  $\chi \in \text{Mod}(\tilde{S})$  be a non-trivial elliptic element with prime order  $m \geq 3$ , and let  $\theta \in \Pi^{-1}(\chi)$  be non-elliptic which can be expressed as  $\theta = \varphi^*([\hat{f}])$  for some conformal automorphism  $\hat{f}$  of  $\mathbf{D}$ . Then  $\theta$  is either hyperbolic, or of type (I) or of type (II). More precisely, we have*

- (1)  $\theta$  is of type (I) if and only if  $\hat{f}$  fixes a fixed point of a parabolic element of  $G$ ,
- (2)  $\theta$  is of type (II) if and only if  $\hat{f}$  fixes a geodesic  $\lambda_c$  that can be projected to a semi-filling closed geodesic  $c \subset \tilde{S}$  that is invariant under  $f$ ,
- (3)  $\theta$  is hyperbolic if and only if  $\hat{f}$  keeps invariant a geodesic  $\lambda_c \in \mathbf{D}$  that can be projected to a filling closed geodesic  $c \subset \tilde{S}$  that is invariant under  $f$ .

**§2.8.** Finally, we consider some compositions of elliptic mapping classes  $\chi$  and Dehn twists  $t_\alpha$  along a simple closed curve  $\alpha \subset \tilde{S}$ , and study the corresponding fiber in  $\text{Mod}_S^x$ . Our last result states:

**Theorem.** *Let  $\chi \in \text{Mod}(\tilde{S})$  be elliptic with prime order  $m \geq 3$ . There exist simple closed geodesics  $\alpha \subset \tilde{S}$  such that  $t_\alpha^n \circ \chi$  and  $t_\alpha^n \circ \chi^{-1}$  are both hyperbolic for all integers  $n$  with a finite number of exclusions. In the case where both  $t_\alpha^n \circ \chi$  and  $t_\alpha^n \circ \chi^{-1}$  are hyperbolic, either  $\Pi^{-1}(t_\alpha^n \circ \chi)$  or  $\Pi^{-1}(t_\alpha^n \circ \chi^{-1})$  consists of hyperbolic mapping classes.*

**§2.9. Remark:** When  $n = 0$ , that is,  $\tilde{S}$  is closed, it was shown in [13] that for any hyperbolic mapping class  $\chi \in \text{Mod}(\tilde{S})$ ,  $\Pi^{-1}(\chi)$  consists of hyperbolic mapping classes. It is not known, however, whether  $\Pi^{-1}(\chi')$  contains hyperbolic mapping class for a general mapping class  $\chi'$  of  $\tilde{S}$  if  $n > 0$ . Theorem 2.8 above provides an example that the single fiber  $\Pi^{-1}(\chi')$  may contain infinitely many hyperbolic mapping classes. Another example is given in Theorem 2

of [13].

### 3. Some preliminary results

**§3.1.** Let  $\theta, \theta' \in \text{Mod}(S)$  be non-trivial. We call  $\theta$  and  $\theta'$  commuting mapping classes of  $S$  if  $\theta \circ \theta'(\tau) = \theta' \circ \theta(\tau)$  for every  $\tau \in T(S)$ . We have

**Lemma.** *Suppose that  $\theta$  and  $\theta'$  are infinite order commuting mapping classes of  $S$ . Then  $\theta$  is hyperbolic if and only if  $\theta'$  is hyperbolic.*

**REMARK.** The authors are grateful to the referee for pointing out that this result is essentially known, whose proof was given in Ivanov [7]. Here we provide with an alternate approach.

*Proof.* Let  $f_\theta$  and  $f_{\theta'}$  denote a self-maps of  $S$  that induce  $\theta$  and  $\theta'$ , respectively. Obviously, the condition is equivalent to that  $f_\theta \circ f_{\theta'}$  is isotopic to  $f_{\theta'} \circ f_\theta$  on  $S$ . Suppose that  $f_{\theta'}$  is reduced by a loop system  $E = \{e_1, \dots, e_k\}$ . By taking a suitable power we may assume that  $f_{\theta'}$  is a component map. In particular,  $f_{\theta'}(e_i) = e_i, i = 1, \dots, k$ . Let  $\mathcal{P} = \{P_1, \dots, P_{s_0}\}$  denote all components of  $S \setminus E$  on which  $f_{\theta'}$  is isotopic to a pseudo-Anosov map. Let  $\{Q_1, \dots, Q_s\}$  denote all components of  $S \setminus E$  on which  $f_{\theta'}$  is the identity. Let  $E_0 = \{e_1, \dots, e_t\}$  be the subset of  $E$  consisting of boundary components of  $P_i, i = 1, \dots, s_0$ .  $E \setminus E_0$  consists of loops on each of which  $f_{\theta'}$  is either the identity or a power of a non-trivial Dehn twist.

Consider the self-map  $\xi = f_\theta \circ f_{\theta'} \circ f_\theta^{-1}$ . Then  $\xi(f_\theta(\alpha)) = f_\theta(\alpha)$ , which says that  $\xi$  restricts to the identity or non-trivial Dehn twist on the loop  $f_\theta(\alpha)$  for each  $\alpha \in E \setminus E_0$ . By hypothesis,  $\xi$  is isotopic to  $f_{\theta'}$ . It turns out that  $f_{\theta'}$  restricts to the identity, or a non-trivial Dehn twist on  $f_\theta(\alpha)$ . As such,  $f_\theta(\alpha)$  is also in  $E \setminus E_0$ . It follows that  $f_\theta$  leaves invariant the set  $E \setminus E_0$ . If this set is not empty, we are done. Otherwise,  $\mathcal{P}$  is not empty. Let  $P_1 \in \mathcal{P}$ . The map  $\xi$  is isotopic to pseudo-Anosov on  $f_\theta(P_1)$ . Since  $\xi$  is isotopic to  $f_{\theta'}$ ,  $f_\theta(P_1)$  is in  $\mathcal{P}$  as well. This means that  $f_\theta$  permutes  $P_i$  in  $\mathcal{P}$ . Thus  $f_\theta$  is reduced by the boundary loops of  $P_i, P_i \in \mathcal{P}$ . So  $f_\theta$  is reducible. Since  $\theta$  and  $\theta'$  are symmetric, the lemma is proved.  $\square$

**§3.2.** By assumption,  $\tilde{S}$  is of type  $(p, n)$  with  $2p + n > 4$ . In particular,  $\tilde{S}$  is not of type  $(0, 3), (0, 4), (1, 1), (1, 2)$ , or  $(2, 0)$ . The following lemma is merely a special case (torsion free) of Lemma 3.8 of [12].

**Lemma.** *Let  $\theta$  be a holomorphic automorphism of the Bers fiber space  $F(\tilde{S})$  that leaves each fiber invariant. Then  $\theta$  coincides with an element of  $G$ .*

**§3.3.** Let  $c$  be a primitive, closed filling geodesic on  $\tilde{S}$ , which means that  $[c]$  is not a power of any element of  $\pi_1(\tilde{S}, x)$ . Let  $j : \pi_1(\tilde{S}, x) \rightarrow \text{Mod}_x^x$  be defined in (2.2). By Theorem 2 of [8],  $j([c])$  is hyperbolic. Hence by Lemma 3.1, any infinite order mapping class commuting with  $j([c])$  must also be hyperbolic. More precisely, we have

**Lemma.** *Let  $\theta \in \text{Mod}_x^x$  be of infinite order and  $\theta \neq (j([c]))^p$  for any  $p \in \mathbf{Z}$ . Then  $\theta$  commutes with  $j([c])$  if and only if  $\Pi(\theta)$  is a non-trivial elliptic element and is induced by a conformal automorphism  $f : \tilde{S} \rightarrow \tilde{S}$  that keeps the filling closed geodesic  $c$  invariant (as a set).*

Proof. Suppose that an infinite order element  $\theta \in \text{Mod}_S^x$  satisfies the condition

$$(3.1) \quad \theta \circ j([c]) = j([c]) \circ \theta.$$

Since  $c$  is a filling geodesic, from Theorem 2 of [8],  $j([c])$  is hyperbolic, and thus it is induced by a pseudo-Anosov map of  $S$ . By Lemma 3.1,  $\theta$  is also hyperbolic. Hence by Theorem 15.7 of [7], both  $\theta$  and  $j([c])$  are powers of the same hyperbolic mapping class  $\delta$  of  $\text{Mod}_S^x$ . Write

$$\theta = \delta^s \text{ and } j([c]) = \delta^t.$$

If  $\delta = j([c_0])$  for a  $[c_0] \in \pi_1(\tilde{S}, x)$ , then  $\theta = j([c_0])^s = j([c_0]^s)$ . In this case,  $\Pi(\theta)$  is trivial. Thus  $\theta \in G$ . Since  $G$  is centerless, either  $\theta = (j([c]))^p$  or  $\theta^p = j([c])$  for some  $p \in \mathbf{Z}$ . By assumption, the first case does not occur. The second case says that  $c$  is not primitive. This again contradicts the hypothesis. We conclude that  $\delta \neq j([c_0])$ .

Let  $G_0 = \langle \delta, \varphi^*(G) \rangle$ . As a subgroup of  $\text{Mod}_S^x$ ,  $G_0$  acts on  $T(S)$  discontinuously. Note also that  $G$  is a normal subgroup of  $\text{mod}(\tilde{S})$  as biholomorphisms on  $F(\tilde{S})$ . The group  $\text{mod}(\tilde{S})$  is isomorphic to  $\text{Mod}_S^x$  under  $\varphi^*$ . So  $\varphi^*(G)$  is a normal subgroup of  $\text{Mod}_S^x$  and thus  $\varphi^*(G)$  is a normal subgroup of  $G_0$ .

From Nielsen's theorem, there is a point  $\sigma \in T(\tilde{S})$  such that  $\Pi(\delta)(\sigma) = \sigma$ . We assume that  $\sigma = [0]$  is represented by  $\tilde{S}$ . Consider the fiber  $\mathbf{D} = \pi^{-1}([0]) \subset F(\tilde{S})$ . Note that  $\varphi^{-1}(\delta)|_{\mathbf{D}}$  acts as a conformal automorphism.  $\varphi^{-1}(\delta)|_{\mathbf{D}}$  is an element of  $\mathbf{PSL}(2, \mathbf{R})$ . Denote  $\hat{f} = \varphi^{-1}(\delta)|_{\mathbf{D}}$ , and let  $G'_0 = (\varphi^*)^{-1}(G_0)|_{\mathbf{D}}$ . We see that  $G'_0 = \langle \hat{f}, G \rangle$  acts on  $\mathbf{D}$  discontinuously,  $\hat{f}$  does not belong to  $G$ , and  $G$  is a normal subgroup of  $G'_0$ . In particular, we have

$$\hat{f} \circ G \circ \hat{f}^{-1} = G.$$

It follows that  $\hat{f}$  can be projected to a conformal automorphism  $f$  of  $\tilde{S}$  under the projection  $\varrho : \mathbf{D} \rightarrow \tilde{S}$ . It is also easy to see that  $\mathbf{D}/G'_0 = \tilde{S}/\langle f \rangle$ .

By construction,  $\hat{f}^r = j([c])$ . As elements of  $\mathbf{PSL}(2, \mathbf{R})$ , both  $\hat{f}$  and  $j([c])$  keep a geodesic  $\lambda_c$  invariant. Since  $\varrho(\lambda_c) = c$ ,  $f(c) = c$ , as desired.

Conversely, we assume that  $c$  is a filling geodesic on  $\tilde{S}$ , and  $f : \tilde{S} \rightarrow \tilde{S}$  satisfies  $f(c) = c$ . We lift the map  $f$  to a conformal automorphism  $\hat{f}$  of  $\mathbf{D}$  so that  $\hat{f}(\lambda_c) = \lambda_c$ , where  $\lambda_c$  is a geodesic in  $\mathbf{D}$  such that  $\varrho(\lambda_c) = c$ .  $\hat{f}$  is a hyperbolic element of  $\mathbf{PSL}(2, \mathbf{R})$ . Since  $\chi^m = \text{id}$ , by Lemma 3.2,  $\hat{f}^m$  is an element  $g_c$  of  $G$ , where  $g_c$  corresponds to  $c$  under the isomorphism  $\pi_1(\tilde{S}, x) \xrightarrow{\cong} G$ .  $[\hat{f}]$  commutes with  $g_c$  if both are considered elements of  $\text{mod}(\tilde{S})$ . It follows that  $\theta$  commutes with  $j([c])$ . □

**§3.4.** Now let  $\chi \in \text{Mod}(\tilde{S})$  be elliptic with prime order  $m \geq 3$ . Let  $f : \tilde{S} \rightarrow \tilde{S}$  be a representative of  $\chi$  and let  $\theta \in \text{Mod}_S^x$  be such that  $\Pi(\theta) = \chi$ . Let  $f_\theta : S \rightarrow S$  be a representative of  $\theta$ . Suppose that there is a subsurface  $\mathcal{R}$  of  $S$  satisfying the properties:

- $x \in \mathcal{R}$ ,  $f_\theta$  keeps  $\mathcal{R}$  invariant, and
- $\partial\mathcal{R} = \{d_1, \dots, d_u\}$ , where  $u \geq 1$  and  $d_i$  are also non-contractible loops on  $\tilde{S}$ .

Under these circumstances, we have:

**Lemma.**  $f_\theta|_{\mathcal{R}}$  is periodic if and only if  $f_\theta$  is periodic.

Proof. Suppose that  $f_\theta|_{\mathcal{R}}$  is periodic. Then the restriction  $f_\theta^n|_{\mathcal{R}}$  is the identity for some  $n \in \mathbf{Z}$ . On the other hand, by assumption, we know that  $\Pi(\theta^m) = \chi^m$  is the identity, by

Lemma 3.2,  $f_\theta^m = \varphi \circ \gamma \circ \varphi^{-1}$  for some element  $\gamma \in G$ . This tells us that  $f_\theta^m$  leaves the identity on any component of  $S \setminus \{d_1, \dots, d_u\}$  other than  $\mathcal{R}$ .

If  $f_\theta^m|_{\mathcal{R}}$  is not the identity, then  $\{f_\theta^m|_{\mathcal{R}}\}$  is infinitely cyclic, which says that  $(f_\theta^m|_{\mathcal{R}})^q \neq \text{id}$  for any integer  $q \neq 0$ . In particular,  $(f_\theta^m|_{\mathcal{R}})^n \neq \text{id}$ . But  $(f_\theta^m|_{\mathcal{R}})^n = (f_\theta^n|_{\mathcal{R}})^m = \text{id}$ . This is a contradiction, showing that  $f_\theta^m|_{\mathcal{R}}$  is the identity. This also implies that  $m = n$ .

We conclude that  $f_\theta^m$  restricts to the identity on any components of  $S \setminus \{d_1, \dots, d_u\}$ . It remains to exclude the case where  $f_\theta^m$  is a multi-twists along some loops  $d_i$ .

Notice that  $d_i$  is non-contractible on  $\tilde{S}$ . Assume that  $f_\theta^m|_{N_1}$  is non-trivial, where  $N_1$  is a thin annular neighborhood of  $d_1$ .  $N_1$  avoids the puncture  $x$  and is disjoint from any other loops  $d_j$  for  $j \neq 1$ . Since  $d_1$  is non-contractible on  $\tilde{S}$  and since  $m \geq 3$  is a prime integer,  $\mathcal{R}$  is not an  $x$ -punctured cylinder, which means that  $\Pi(\theta^m)$  is non-trivial on the image of  $N_1$  under the forgetful map. Thus  $\chi^m$  is not the identity. But this contradicts that  $\chi$  is periodic with order  $m$ . It follows that  $\theta^m$  is the identity and hence  $\theta$  is an elliptic mapping class of order  $m$ .

The converse is trivial. □

**§3.5.** A similar argument yields the following result:

**Lemma.** *Under the same notation and hypothesis of Lemma 3.4, if  $f_\theta^m|_{\mathcal{R}}$  is a non-trivial Dehn twist along  $\partial\Delta$  where  $\Delta \subset \mathcal{R} \subset S$  is a twice punctured disk enclosing  $x$  and another puncture, then  $f_\theta$  represents a type (I) mapping class.*

*Proof.* By definition,  $f_\theta|_{\mathcal{R}}$  is a type (I) reducible on  $\mathcal{R}$ . By the same argument of Lemma 3.4,  $f_\theta$  is itself a type (I) reducible map on  $S$ . □

**§3.6.** Let  $\chi \in \text{Mod}(\tilde{S})$  be elliptic which is represented by a conformal automorphism  $f$  of  $\tilde{S}$ . Suppose that  $\chi$  has a prime order  $m \geq 3$ .

**Lemma.** *Let  $c$  be a simple non-contractible loop on  $\tilde{S}$  such that  $c$  is not homotopic to  $f(c)$ . If  $f^q(c)$  is homotopic to  $c$  for an integer  $q$  with  $1 < q \leq m$ , then  $q = m$ .*

*Proof.* Without loss of generality we may assume that  $c$  is a simple geodesic. Since  $f$  is conformal,  $f(c)$  is also a geodesic and is not homotopic to  $c$ .

Suppose that  $f^q(c) = c$ . Then either  $f^q|_c$  is the identity or cyclic. If  $f^q|_c$  is the identity,  $f^q$  is the identity on  $\tilde{S}$ . So  $q = m$ . If  $f^q|_c$  is cyclic, since  $m$  is a prime number,  $f^q|_c$  must be of order  $m$ , which means that  $f$  is of order  $qm$ . But  $q > 1$ . This is impossible. □

**§3.7.** Recall that for any  $[c] \in \pi_1(\tilde{S}, x)$ ,  $j([c]) \in \text{Mod}_S^x$  is the mapping class defined in (2.2). Let  $f_c : S \rightarrow S$  be a suitable representative of  $j([c])$  and  $c$  a representative of  $[c]$ .

If  $c$  is a loop around a puncture of  $\tilde{S}$ ,  $f_c$  is a Dehn twist along a twice punctured disc  $\Delta$  enclosing  $x$ . So  $f_c$  restricts to the identity on any component of  $S \setminus \partial\Delta$ .

If  $c$  is a simple non-contracting loop of  $\tilde{S}$ ,  $f_c$  is a spin map which is reducible. Denote by  $\mathcal{C}$  the corresponding cylinder containing  $x$ . Then  $f_c$  restricts to the identity on any component of  $S \setminus \mathcal{C}$ .

Consider the universal covering map (2.1). Fix a point  $\hat{x}$ . Let  $g_c \in G$  be the element corresponding to  $c$ . Assume that  $\hat{x}$  is in the axis  $\lambda_c$  of  $g_c$ . Construct a quasiconformal automorphism  $w$  of  $\mathbf{D}$  that is supported on a thin neighborhood of  $\lambda_c$  and has the properties

that

- $w(\hat{x}) = g_c(\hat{x})$  and
- $w$  commutes with each element of  $G$ .

Let  $W : \tilde{S} \rightarrow \tilde{S}$  be the projection of  $w$ . There is a homotopy  $\omega_t$  (which is called the Ahlfors homotopy in the literature) between  $w$  and the identity so that for any  $t \in [0, 1]$ ,  $\omega_t$  commutes with each element of  $G$ . Hence  $\omega_t$  can be projected to a homotopy  $\Omega_t$  on  $\tilde{S}$  so that  $\Omega_0 = \text{id}$  and  $\Omega_1 = W$ . Since  $W$  fixes  $x$ ,  $j([c])$  is the mapping class of  $\Omega_1|_S$ . From this construction, we see that  $W$  is the identity outside a neighborhood of  $c$ . Let  $\{e_1, \dots, e_{k_0}\}$  be the curve system on  $S$  so that one component of  $S \setminus \{e_1, \dots, e_{k_0}\}$  is a minimum surface containing  $c$ . This means that  $f_c$  restricts to the identity on each component of  $S \setminus \{e_1, \dots, e_{k_0}\}$  that avoids  $c$ .

Putting all the information together, we summarize:

**Lemma.** *For any  $[c] \in \pi_1(\tilde{S}, x)$ ,  $j([c])$  is represented by a map  $f_c$  which restricts to the identity on any subsurface of  $S$  that avoids  $c$ .*

**§3.8.** We continue to assume that  $\chi \in \text{Mod}(\tilde{S})$  is elliptic which is represented by a conformal automorphism  $f$  of  $\tilde{S}$  fixing a point or a puncture of  $\tilde{S}$ . Let  $f^* : \mathbf{D} \rightarrow \mathbf{D}$  be a (conformal) lift of  $f$  that fixes a pre-image  $y^*$  of a fixed point  $y$  of  $f$ . The point  $y^* \in \partial\mathbf{D}$  if and only if  $y$  is a puncture of  $\tilde{S}$ .

**Lemma.** *With the above notation and terminology, there is a hyperbolic element  $g \in G$  and an integer  $N$  such that for all  $k \geq N$ ,  $g^k \circ f^*$  are hyperbolic Möbius transformations.*

*Proof.* There are two cases to discuss.

Case 1.  $f$  fixes a point  $y \in \tilde{S}$ . In this case, for any two non-antipodal points  $\alpha, \beta \in \mathbf{S}^1$ , we use  $[\alpha, \beta]$  (resp.  $(\alpha, \beta)$ ) to denote the minor closed (resp. open) arc on  $\mathbf{S}^1$ . Also, the Euclidean length of the segment is denoted by  $\text{len}(\alpha, \beta)$ . The lift  $f^*$  of  $f$  that fixes  $y^*$ , where  $y^*$  is a point in the orbit  $\{\rho^{-1}(y)\} \subset \mathbf{D}$ . In this setting  $f^*$  is a Möbius transformation keeping  $\mathbf{S}^1$  invariant. One may assume that  $y^* = 0$  and  $f^*$  is of the form

$$z \rightarrow [\exp(2\pi i/m)]z, \quad z \in \mathbf{D}.$$

That is,  $f^*$  is a rotation. It sends any point  $\alpha \in \mathbf{S}^1$  to a point  $\beta = f^*(\alpha)$ . The length  $\lambda := \text{len}(\alpha, \beta)$  does not depend on  $\alpha$ .

Let  $g \in G$  be a hyperbolic element so that its axis  $A_g$  has a relatively large Euclidean length, in the sense that  $A_g$  and  $f^*(A_g)$  intersect. This is achievable by Theorem 5.3.8 of Beardon [1]. Let  $A_g \cap \mathbf{S}^1 = \{\text{Fix}^+(g), \text{Fix}^-(g)\}$ , where  $\text{Fix}^+(g)$  and  $\text{Fix}^-(g)$  are repelling and attracting fixed point of  $g$ . Orient  $A_g$  so that it points from  $\text{Fix}^+(g)$  to  $\text{Fix}^-(g)$ . It is clear that  $A_g$  divides  $\mathbf{D}$  into two half-plane  $U$  and  $U'$ , where  $U$  and  $U'$  are the half planes lying in the right and left side of  $A_g$ , respectively. Assume that  $U$  contains some diameter of  $\mathbf{D}$ .

We see that  $\text{len}(z, g(z))$ ,  $z \in U \cap \mathbf{S}^1$ , attains its maximum value at some point  $z_0 \in \mathbf{S}^1$  that is away from  $\text{Fix}^+(g)$  and  $\text{Fix}^-(g)$ . As a point  $z \in U \cap \mathbf{S}^1$  tends to either  $\text{Fix}^+(g)$  or  $\text{Fix}^-(g)$ ,  $\text{len}(z, g(z))$  tends to zero.

Choose a sufficiently large integer  $N$  so that when  $k \geq N$ , the maximum value of  $\text{len}(z, g^k(z))$ ,  $z \in U \cap \mathbf{S}^1$ , is  $\lambda_0 > \lambda$ . Let  $z_0 \in U \cap \mathbf{S}^1$  be the point so that  $\text{len}(z_0, g^N(z_0)) = \lambda_0$ . Then it still holds that for a fixed  $k > N$ ,  $\text{len}(z, g^k(z))$  tends to zero, whenever  $z \in U \cap \mathbf{S}^1$  tends to either  $\text{Fix}^+(g^k)$  or  $\text{Fix}^-(g^k)$ ,

Now from the intermediate-value theorem in Calculus, there is a point  $z'$  in the arc  $(\text{Fix}^+(g), z_0)$  and point  $z''$  in  $(z_0, \text{Fix}^-(g))$  so that  $\text{len}(z', g^k(z')) = \lambda$  and  $\text{len}(z'', g^k(z'')) = \lambda$ . Since  $\lambda = \text{len}(z', f^*(z')) = \text{len}(z'', f^*(z''))$ , we conclude that

$$g^k(z') = f^*(z') \text{ and } g^k(z'') = f^*(z''), \quad z', z'' \in U \cap \mathbf{S}^1.$$

It follows that  $(g^k)^{-1} \circ f^*(z') = g^{-k} \circ f^*(z') = z'$  and  $(g^k)^{-1} \circ f^*(z'') = g^{-k} \circ f^*(z'') = z''$ , which says that  $g^{-k} \circ f^*$  has two distinct fixed points in  $U \cap \mathbf{S}^1$ . Notice that  $g^{-k}$  and  $f^*$  are Möbius transformations. We assert that  $g^{-k} \circ f^*$  must be a hyperbolic Möbius transformation. Since  $G_0 = \langle G, f^* \rangle$  is discrete, as a hyperbolic element of  $G_0$ ,  $g^{-k} \circ f^*$  cannot fix any fixed points of parabolic elements of  $G_0$ . Similarly,  $g^k \circ f^*$  has two distinct fixed points in  $U' \cap \mathbf{S}^1$ . So  $g^k \circ f^*$  is also a hyperbolic element.

Case 2.  $f$  fixes a puncture  $y$  of  $\tilde{S}$ . In this case, we take the upper half space model for the hyperbolic space  $\mathbf{H}$ . The boundary is  $\partial\mathbf{H} = \mathbf{R}$ . We assume that  $\infty$  lies above  $y$  under (2.1), and the parabolic element  $\gamma$  that fixes  $\infty$  is  $z \rightarrow z + 1$ . The lift  $f^*$  fixes  $\infty$ . Under the assumption, since  $f^m$  is the identity,  $f^*$  sends each point  $z$  in  $\mathbf{R} \cup \mathbf{H}$  to  $z + k/m$  for some integer  $k$  that is prime to  $m$ .

We then use the same argument in Case 1. Let  $g \in G$  be a hyperbolic element so that  $\text{len}(\text{Fix}^+(g), \text{Fix}^-(g)) > 1$  (by Theorem 5.3.8 of Beardon [1]). Its axis  $A_g$  divides  $\mathbf{R} \cup \mathbf{H}$  into two regions. Let  $U$  be the bounded region. For a large but fixed  $k$ , there are two distinguished points  $z', z'' \in U \cap \mathbf{R}$  so that  $\text{len}(z', g^k(z')) = 1$  and  $\text{len}(z'', g^k(z'')) = 1$ . It follows that  $g^{-k} \circ f^* \in G_0$  is hyperbolic. Similarly,  $g^k \circ f^* \in G_0$  is also hyperbolic with two distinct fixed points in  $U' \cap \mathbf{R}$ . Once again, both  $g^{-k} \circ f^*$  and  $g^k \circ f^*$  cannot fix any fixed points of parabolic elements of  $G$ ; otherwise,  $G_0$  would not be discrete. Lemma 3.8 is proved. □

#### 4. Proof of Theorem 2.6 and Theorem 2.7

**§4.1.** We first prove Theorem 2.7. Theorem 2.6 will be derived easily. As usual, we let  $\theta$  be induced by a self-map  $f_\theta$  of  $S$ . First we assume that  $f_\theta$  is completely reduced by

$$(4.1) \quad \mathcal{L} = \{c_1, \dots, c_u\}, \quad u \geq 1,$$

(see Bers [3] for the definition of completely reducible map. From Lemma 5.9 of [3], each reducible mapping class is completely reducible). Let  $\sigma_t$  be a smooth flow in  $T(S)$  that is obtained from pinching all the loops in (4.1) to cusps. Let  $\partial T(S)$  denote the Bers boundary of  $T(S)$  (see [2]). Let  $\{x_i\} \in \sigma_t \subset T(S)$  be any discrete instances represented by  $S_i$  so that  $x_i \rightarrow \partial T(S)$ . Let  $\tilde{x}_i = \pi \circ \varphi^{-1}(x_i) \in T(\tilde{S})$  be represented by  $\tilde{S}_i$ . Then  $\tilde{S}_i$  is obtained from  $S_i$  by filling in the puncture  $x$ . Let

$$\mathcal{R}(\tilde{S}) = T(\tilde{S}) / \text{Mod}(\tilde{S})$$

denote the Riemann moduli space of  $\tilde{S}$ , and let  $\varpi : T(\tilde{S}) \rightarrow \mathcal{R}(\tilde{S})$  be the natural projection.

Let  $\Lambda \subset \mathcal{R}(\tilde{S})$  and  $\Lambda' \subset T(\tilde{S})$  denote the sequences  $\{\varpi(\tilde{x}_i)\}$  and  $\{\tilde{x}_i\}$ , respectively. There are two cases to consider.

Case 1.  $\Lambda$  lies in a compact subset of  $\mathcal{R}(\tilde{S})$ . In this case, there is a sub-sequence  $\{\tilde{x}_i\} \subset \Lambda'$ , which may tend to the boundary  $\partial T(\tilde{S})$ , yet  $\tilde{S}_i$  does not possess short closed geodesics. By Lemma 2 of [13], all loops in (4.1) bound disks  $D_j \subset S$  that encloses  $x$  and another puncture

$x_j$ . But all loops in (4.1) are disjoint and homotopically independent. It follows that  $u = 1$ , which means that (4.1) consists of only one single loop, say  $c_1$ , that is the boundary of a twice punctured disk  $\Delta$  enclosing  $x$  and a puncture  $x_1$  of  $\tilde{S}$ .

Since  $\Pi(\theta) = \chi \in \text{Mod}(\tilde{S})$  is elliptic with prime order  $m$ ,  $f_\theta|_{S \setminus \Delta}$  is isotopic to a periodic self-map of prime order  $m$ . Hence  $\theta = \varphi^*([\hat{f}])$  is a type (I) mapping class.

Case 2.  $\Lambda$  is not compact in  $\mathcal{R}(\tilde{S})$ . In this case, we claim that  $\theta$  is a type (II) mapping class as defined in §2.5. Indeed, we pick an arbitrary loop, say  $c_i$ , in (4.1) that is non-contractible on  $\tilde{S}$  (the image of  $c_i$  on  $\tilde{S}_i$  is denoted by  $\tilde{c}_i$ ). By assumption,  $m \geq 3$  is a prime number. Consider the image loops  $\mathcal{L}^i = \{f_\theta^j(c_i), j \geq 0\}$ . There is a possibility that  $\mathcal{L}^i$  consists of a simple loop  $c_i$  only. That is,  $f_\theta$  keeps  $c_i$  invariant. But if  $c_i$  is not homotopic to  $f_\theta(c_i)$ , by Lemma 3.6,  $c_i$  is not homotopic to  $f_\theta^q(c_i)$  for any  $q$  with  $1 < q < m$ .

We claim that  $c_i$  is homotopic to  $f_\theta^m(c_i)$ . Indeed,  $\tilde{c}_i$  is homotopic to  $f^m(\tilde{c}_i)$ . So if  $c_i$  is not homotopic to  $f_\theta^m(c_i)$ , then since  $c_i$  and  $f_\theta^m(c_i)$  lie in  $\mathcal{L}$ , they are disjoint. Hence  $c_i$  and  $f_\theta^m(c_i)$  must bound a cylinder  $\mathcal{P}_i$  that encloses the puncture  $x$ . It follows that  $f_\theta(\mathcal{P}_i)$  is also a cylinder that encloses the puncture  $x$  as well.

This means that either  $\mathcal{P}_i = f_\theta(\mathcal{P}_i)$ , or  $\mathcal{P}_i$  and  $f_\theta(\mathcal{P}_i)$  intersect. In the later case, the boundary  $\partial\mathcal{P}_i$  of  $\mathcal{P}_i$  is  $\{c_i, f_\theta^m(c_i)\}$  and the boundary  $\partial f_\theta(\mathcal{P}_i)$  of  $f_\theta(\mathcal{P}_i)$  is  $\{f_\theta(c_i), f_\theta(f_\theta^m(c_i))\}$ . Since  $c_i$  and  $f_\theta(c_i)$  is disjoint,  $c_i$  must intersect with  $f_\theta(f_\theta^m(c_i))$ . This is impossible. In the former case, if  $c_i$  is homotopic to  $f_\theta(c_i)$ , then this contradicts Lemma 3.6; if  $c_i$  is homotopic to  $f_\theta(f_\theta(c_i))$ , then  $f_\theta(c_i)$  is homotopic to  $f_\theta^m(c_i)$ , which says that  $c_i$  is homotopic to  $f_\theta^{m-1}(c_i)$ , which again contradicts Lemma 3.6.

We conclude that  $c_i$  is homotopic to  $f_\theta^m(c_i)$ , and that (4.1) is a disjoint union

$$(4.2) \quad \bigcup_{i=0}^N \mathcal{L}^i, \quad N < \infty,$$

where  $\mathcal{L}^i$  is either an  $m$ -cycle of loops, or consists of a simple loop only.

Note that every  $\mathcal{L}^i$  in (4.2) consists of either dividing loops or non-dividing loops. Let  $\mathcal{R}^*$  be the component of  $S - \bigcup \mathcal{L}^i$  containing the puncture  $x$ . We claim that  $\mathcal{R}^*$  is invariant under  $f_\theta$ . To see this, we construct  $\mathcal{R}^*$  in the following steps:

(A) If  $\mathcal{L}^1$  consists of non-dividing loops and  $S \setminus \{\mathcal{L}^1\}$  is a single component, we define  $\mathcal{R}_1$  to be  $S \setminus \{\mathcal{L}^1\}$ . If  $\mathcal{L}^1$  consists of dividing loops, or non-dividing loops but  $S \setminus \{\mathcal{L}^1\}$  has more than one components, we let  $\mathcal{R}_1$  be the component of  $S \setminus \{\mathcal{L}^1\}$  that the puncture  $x$  resides. It is easy to see that  $\mathcal{R}_1$  is invariant under  $f_\theta$ .

(B) If  $\mathcal{R}_1$  contains a loop in  $\mathcal{L}^2$  then since  $\mathcal{R}_1$  is invariant under  $f_\theta$ ,  $\mathcal{R}_1$  contains every element in  $\mathcal{L}^2$  and we follow the same procedure as in (A) to obtain  $\mathcal{R}_2$ . If  $\mathcal{R}_1$  does not contain any loop in  $\mathcal{L}^2$ , then we simply ignore  $\mathcal{L}^2$  and examine  $\mathcal{L}^3$ , and so on.

Since  $N$  is finite, the process terminates after finite many steps. The resulting subsurface is  $\mathcal{R}^*$ . From the construction we see that  $\mathcal{R}^*$  is invariant under  $f_\theta$  and encloses the puncture  $x$ .

Our next claim is that  $f_\theta|_{\mathcal{R}^*}$  is irreducible (or pseudo-Anosov) self-map of  $\mathcal{R}^*$ . Suppose that  $f_\theta|_{\mathcal{R}^*}$  is periodic, by Lemma 3.4,  $f_\theta$  is periodic. Thus  $\theta$  is elliptic. This is a contradiction. Next we assume that  $f_\theta|_{\mathcal{R}^*}$  is reducible. The only possibility for this to occur is that  $f_\theta|_{\mathcal{R}^*}$  is reduced by the single loop  $c_1$ , where  $c_1$  is the boundary of a twice punctured disk  $\Delta$  enclosing  $x$  and another puncture (otherwise, we would continue the above procedure (A)). This leads

to three possibilities:

- $f_\theta^m|_{\mathcal{R}^*}$  is the Dehn twist along  $c_1$ ,
- $f_\theta^m|_{\mathcal{R}^*}$  is irreducible (or pseudo-Anosov) on  $S \setminus \Delta$ , or
- $f_\theta^m|_{\mathcal{R}^*}$  is irreducible (or pseudo-Anosov) on  $\Delta$  but the identity on  $S \setminus \Delta$ .

If the first situation occurs, by Lemma 3.5,  $f_\theta$  is of type (I). We claim that the second situation does not occur. Suppose for the contrary. By filling in the puncture  $x$ ,  $c_1$  shrinks to a puncture, and all other loops in (4.1) is non-contractible. So  $f_\theta^m|_{\mathcal{R}^*}$  descends to a pseudo-Anosov self-map on a component of  $\tilde{S} \setminus \{\tilde{c}_2, \dots, \tilde{c}_u\}$ . This contradicts that  $\chi^m$  is the identity.

Otherwise, we conclude that the third case must occur. That is,  $f_\theta|_{\mathcal{R}^*}$  is irreducible (or pseudo-Anosov). Hence  $f_\theta^m|_{\mathcal{R}^*}$  is also irreducible (or pseudo-Anosov) and  $f_\theta^m$  restricts to the identity on any other component of  $S \setminus \mathcal{L}$ . Since  $f_\theta^m$  projects to the identity, by Theorem 2 of [8],  $(f_\theta)^m = j([c])$ , where  $c$  is a semi-filling loop on  $\tilde{S}$  but a filling loop on  $\mathcal{R}^*$ . This proves that  $\theta$  must be of type (II).

**§4.2.** To prove (1) of Theorem 2.7, suppose that  $\hat{f}$  fixes a point  $y^* \in \partial \mathbf{D} = \mathbf{S}^1$ , where  $y^*$  is the fixed point of a parabolic element  $T$  of  $G$ . There is an integer  $n$  such that

$$(4.3) \quad [\hat{f}] \circ T \circ [\hat{f}]^{-1} = T^n.$$

From Theorem 2 of [8, 9],  $\varphi^*(T^n) = (\varphi^*(T))^n$  is a power of the Dehn twist along the boundary  $\partial \Delta$  of a twice punctured disc  $\Delta$  enclosing  $x$  and another puncture  $x_1$ . Thus  $\varphi^*([\hat{f}]) \circ \varphi^*(T) \circ \varphi^*([\hat{f}]^{-1})$  keeps  $\varphi^*([\hat{f}])(\partial \Delta)$  invariant. From (4.3),  $\varphi^*([\hat{f}])(\partial \Delta) = \partial \Delta$ . Therefore,  $n = 1$ . Since  $\Pi \circ \varphi^*([\hat{f}]) = \chi$  is of order  $m$ ,  $f_\theta|_{S \setminus \Delta}$  is isotopic to a periodic self-map of order  $m$ . It follows that  $\varphi^*([\hat{f}])$  is of type (I) and is reduced by  $\partial \Delta$ .

Conversely, if  $\theta$  is of type (I), we are in Case 1 of §4.1. In this case, from Lemma 1 of [13], there is a parabolic element  $T \in G$  such that  $\varphi^*(T)$  is the Dehn twist along  $\partial \Delta$ . In particular,  $\varphi^*(T)$  commutes with  $\theta$ . So  $T$  commutes with  $[\hat{f}]$ . But  $[\hat{f}]|_{\mathbf{D}} = \hat{f}$ . From the same argument of Corollary 1 of [13],  $\hat{f}$  and  $T$  share a common fixed point if both are viewed as elements of real transformations.

**§4.3.** To prove (2) of Theorem 2.7, we suppose that  $\theta$  is a type (II) mapping class, we are in Case 2 of §4.1. Then  $f_\theta$  keeps invariant some subsurface  $\mathcal{R}^*$  of  $S$ , where  $\mathcal{R}^*$  is a subsurface of  $S$  described in steps (A) and (B) right after (4.2). By hypothesis,  $(f_\theta|_{\mathcal{R}^*})^m$  is irreducible, which means that there is a filling closed geodesic  $c$  on  $\mathcal{R}^*$  such that  $(f_\theta|_{\mathcal{R}^*})^m$  is isotopic to  $j([c])|_{\mathcal{R}^*}$ .

By Lemma 3.7,  $j([c])$  is the identity outside  $\mathcal{R}^*$ . We see that  $f_\theta$  commutes with  $j([c])$  as self-maps of  $S$ , which implies that the element  $g_c \in G$  that corresponds to  $c$  under (2.1) commutes with  $[\hat{f}]$ . The curve  $c$ , as a loop on  $\tilde{S}$ , is semi-filling. So  $g_c$  is semi-essential hyperbolic. Since  $[\hat{f}]|_{\mathbf{D}} = \hat{f}$ ,  $g_c$  and  $\hat{f}$  share a common axis  $\lambda_c$  if both are considered real Möbius transformations.  $\lambda_c$  projects to a semi-filling geodesic  $c$  on  $\tilde{S}$ . It is immediate that  $f$  keeps the geodesic  $\varrho(\lambda_c)$  invariant.

Finally, we assume that  $\hat{f}$  fixes a geodesic  $\lambda \subset \mathbf{D}$  with  $\varrho(\lambda) \subset \tilde{S}$  being a semi-filling geodesic. Then  $\hat{f}$  commutes with an element  $g_c$  of  $G$ , where  $g_c$  corresponds to the semi-filling geodesic  $c = \varrho(\lambda_c)$ . This implies that  $\theta = \varphi^*([\hat{f}])$  commutes with  $\varphi^*(g_c) = j([c])$ . Since  $j([c])$  is a reducible mapping class. By Lemma 3.1,  $\theta$  is reducible as well. From the same discussion as above,  $\theta$  is not of type (I); otherwise, there is a parabolic element  $T$  of  $G$  such that  $[\hat{f}]$  commutes with  $T$ . As real Möbius transformations,  $T$  and  $\hat{f}$  share a common

fixed point in  $\mathbf{S}^1$ . By assumption,  $\hat{f}$  and  $g_c$  share the two fixed points. It follows that the fixed point of  $T$  is also a fixed point of  $g_c$ . As a consequence, the group generated by  $T$  and  $g_c$ , which is a subgroup of  $G$ , is not discrete. This is absurd. It follows from §4.1 that  $\theta$  must be a type (II) mapping class.

**§4.4.** To prove (3) of Theorem 2.7, we note that  $\theta^m = \theta'$  is also a hyperbolic mapping class. Since  $\Pi(\theta') = \text{id}$ , by Lemma 3.2,  $\varphi^{*-1}(\theta') \in G$ . From Theorem 2 of [8],  $\theta' = \varphi^*(g_c) = j([c])$  for an essential hyperbolic element  $g_c$  of  $G$  that corresponds to a closed filling geodesic  $c$  on  $\tilde{S}$ . Obviously,  $\theta$  commutes with  $\theta'$ . From the proof of Lemma 3.3,  $\hat{f}$  and  $g_c$  share the same axis  $\lambda_c \subset \mathbf{D}$ . It is clear that  $\varrho(\lambda_c)$  is a filling geodesic that is invariant under  $f$ .

Conversely, suppose that  $\hat{f}(\lambda_c) = \lambda_c$ , where  $\lambda_c$  is the axis of an essential hyperbolic element  $g_c$  of  $G$ . As real Möbius transformations,  $\hat{f}$  and  $g_c$  share the same fixed points. So  $\hat{f}$  commutes with  $g_c$ . It follows that  $\theta = \varphi([\hat{f}])$  commutes with  $\varphi^*(g_c)$ . From Theorem 2 of [8] again,  $\varphi^*(g_c)$  is hyperbolic. Hence from Lemma 3.1,  $\theta$  is hyperbolic as well. This completes the proof of Theorem 2.7.

Now we proceed to prove Theorem 2.6.

**§4.5.** If  $\theta$  is elliptic,  $\theta$  has a fixed point  $\tau \in T(S)$ . So  $[\hat{f}](\varphi^{-1}(\tau)) = \varphi^{-1}(\tau)$ . This means that  $[\hat{f}]$  keeps the fiber determined by  $\varphi^{-1}(\tau)$  invariant. We may assume that the fiber is  $\mathbf{D}$ . In this case  $\varphi^{-1}(\tau) \in \mathbf{D}$ . Hence  $f$  fixes the point  $\varrho(\varphi^{-1}(\tau)) \in \tilde{S}$ .

Conversely, if  $f$  fixes a point  $\tilde{y} \in \tilde{S}$ , then we may choose a lift  $\hat{f} \in \mathbf{PSL}(2, \mathbf{R})$  so that  $\hat{f}$  fixes a point  $y \in \{\varrho^{-1}(\tilde{y})\}$  in  $\mathbf{D}$ . Notice that in our case,  $\hat{f} = [\hat{f}]_{\mathbf{D}}$ . We see that as an element of  $\text{mod}(\tilde{S})$ ,  $[\hat{f}]$  has a fixed point in  $F(\tilde{S})$ . Hence  $\varphi^*([\hat{f}])$  has a fixed point in  $T(S)$ , which says  $\varphi^*([\hat{f}])$  is elliptic. This proves (1) of Theorem 2.6. (2) of Theorem 2.6 can be similarly handled.

**§4.6.** Since  $\chi$  is elliptic,  $f$  has fixed points on the compactification of  $\tilde{S}$ . In the case where  $\tilde{S}$  is closed,  $f$  only fixes some points of  $\tilde{S}$ . However, if  $\tilde{S}$  is not closed,  $f$  could fix some punctures as well as some points of  $\tilde{S}$ . In particular, if  $f$  does not fix any punctures of  $\tilde{S}$ ,  $f$  must fix some points of  $\tilde{S}$ . In this situation, let  $\xi_1, \dots, \xi_s$  denote all the points of  $\tilde{S}$  fixed by  $f$ . From §4.5, we know that  $i^{-1}(\chi)$  contains infinitely many elliptic elements. By Lemma 3.8, among the lifts of  $f$  there exist some lifts  $\hat{f}$  (that can be written as  $g^k \circ f^*$ ) that are hyperbolic Möbius transformations. Hence  $\hat{f}$  does not fix any pre-images of  $\xi_1, \dots, \xi_s$ . Those  $\hat{f}$  induce  $[\hat{f}] \in \text{mod}(\tilde{S})$ . It is easy to see that  $\varphi^*([\hat{f}])$  is not elliptic. Since  $f$  does not fix any puncture,  $i^{-1}(\chi)$  does not contain any type (I) mapping classes, from §4.1 it follows that either  $\varphi^*([\hat{f}])$  is of type (II), or  $\varphi^*([\hat{f}])$  is hyperbolic.

Now we assume that  $f$  does fix some punctures, which are denoted by  $y_1, \dots, y_{s_0}$ . If  $f$  fixes a point  $x_1 \in \tilde{S}$ , by Case 1 of Lemma 3.8, there is a lift  $\hat{f}$  that is a hyperbolic element. So it cannot fix any fixed points of parabolic elements of  $G$ . From §4.1, either  $\varphi^*([\hat{f}])$  is of type II, or  $\varphi^*([\hat{f}])$  is hyperbolic.

If  $f$  fixes no points of  $\tilde{S}$ ,  $f$  must fix a puncture  $y$  of  $\tilde{S}$ . By Lemma 3.8 again, there is a lift  $\hat{f}$  that is a hyperbolic element. From §4.1, either  $\varphi^*([\hat{f}])$  is of type II, or it is hyperbolic. This proves (3) of Theorem 2.6.

**§4.7.** We proceed to prove (4) of Theorem 2.6. Let  $\eta : \tilde{S} \rightarrow \tilde{S}/\langle f \rangle$  denote the branched covering. If the orbifold  $\tilde{S}/\langle f \rangle$  has genus  $\tilde{p} = p/m \geq 1$ , we fix a branch point  $\tilde{\xi}$  on  $\tilde{S}/\langle f \rangle$ , let  $\xi = \eta^{-1}(\tilde{\xi}) \in \tilde{S}$ . Also we take a filling loop  $\tilde{c}$  on  $\tilde{S}^*$  that passes through  $\tilde{\xi}$  and avoids any

other branch points, where  $\tilde{S}^*$  is the Riemann surface obtained from forgetting the branch points of  $\tilde{S}/\langle f \rangle$ . Let  $c^* = \eta^{-1}(\tilde{c})$ , and  $c$  the loop obtained from reparameterizing  $c^*$ . Note that  $c$  is a (self-intersecting) closed loop on  $\tilde{S}$  that passes through  $\xi$  and is invariant under  $f$ .

We claim that  $c$  is a filling loop of  $\tilde{S}$ . Otherwise, there is a component  $\Delta$  of  $\tilde{S} \setminus c$  that is neither a disk nor a punctured disk. Then either  $f(\Delta) = \Delta$  or that  $f(\Delta) \cap \Delta$  is empty. In the second case,  $\Delta$  is homeomorphic to a component of  $(\tilde{S}/\langle f \rangle) \setminus \tilde{c}$ , contradicting that  $\tilde{c}$  is a filling loop of  $\tilde{S}^*$ . In the first case,  $\eta|_{\Delta} : \Delta \rightarrow \Delta/\langle f \rangle$  is a finite branched covering. Since  $\Delta$  projects to a disk or punctured disk,  $\eta(\Delta)$  is a disk or punctured disk. This implies that  $\Delta$  must be a disk or punctured disk. This is again a contradiction.

We conclude that  $c$  is a filling loop of  $\tilde{S}$  that is invariant under  $f$ . From (3) of Theorem 2.6, there is a lift  $\theta$  of  $\chi$ , so that  $\theta$  is irreducible. This proves (4) of Theorem 2.6 and hence this completes the proof of Theorem 2.6.

### 5. Proof of Theorem 2.8

**§5.1.** Let  $\chi$  be represented by a conformal automorphism  $f$  on  $\tilde{S}$  which has some fixed points. Let  $\tilde{z}_0$  be one of the fixed points. Let  $z_0 \in \mathbf{D}$  be a point such that  $\pi(z_0) = \tilde{z}_0$ . We may assume without loss of generality that  $z_0 = 0$ .

Let  $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$  be the lift of  $f$  so that  $\hat{f}(0) = 0$ . Then  $\hat{f}$  is a conformal automorphism on  $\mathbf{D}$ , and hence it is a Möbius transformation. It follows that  $\hat{f}$  is a rotation on  $\mathbf{D}$  with rotation angle no larger than  $2\pi/3$ .

**§5.2.** A power of a Dehn twist  $t_{\alpha}^n$  about a simple closed geodesic  $\alpha \subset \tilde{S}$  can also be lifted to map  $\tau^n : \mathbf{D} \rightarrow \mathbf{D}$  so that  $\varrho \circ \tau^n = t_{\alpha}^n \circ \varrho$ . It is known that  $\tau$  determines a disjoint union of half planes  $\Delta_j$  so that  $\tau$  keeps each  $\Delta_j$  invariant. As such, the complement  $\Omega = \mathbf{D} \setminus \cup \Delta_j$  is also an invariant set by  $\tau$  (see [15] for more details). By post composing a suitable element of  $G$ , one may assume that  $0 \in \Omega$ . This implies that for any  $z \in \mathbf{S}^1$  and any  $n$ , the Euclidean distance between  $\tau^n(z)$  and  $z$  is no greater than  $2\pi/2 = \pi$ .

**§5.3.** If  $n > 0$ , the motion direction of  $\tau|_{\mathbf{S}^1}$  is in the clockwise direction. Now either  $\hat{f}$  or  $\hat{f}^{-1}$  is in the clockwise motion direction. Thus we may assume that  $\tau|_{\mathbf{S}^1}$  and  $\hat{f}|_{\mathbf{S}^1}$  are in the same clockwise motion direction. We assert that for any  $z \in \mathbf{S}^1$ , the distance between  $z$  and  $\tau^n \hat{f}$  is no greater than  $2\pi/3 + \pi < 2\pi$ . It follows that  $(\tau^n \hat{f})|_{\mathbf{S}^1}$  has no fixed points. In particular,  $(\tau^n \hat{f})|_{\mathbf{S}^1}$  does not fix any parabolic fixed point of  $G$ . From Lemma 5.1 and Lemma 5.2 of [14],  $\varphi^*([\tau^n \hat{f}])$  cannot fix the boundary of any twice punctured disk enclosing  $x$ . By the same argument of [13, 15],  $\{\varphi^*([\tau^n \hat{f}])\} \subset \Pi^{-1}(t_{\alpha}^n \circ \chi)$  consists of hyperbolic mapping classes if  $t_{\alpha}^n \circ \chi$  itself is a hyperbolic mapping class.

**§5.4.** We need to prove that there exists a simple closed geodesic  $\alpha \subset \tilde{S}$  such that  $t_{\alpha}^n \circ f$  and  $f \circ t_{\alpha}^n$  are pseudo-Anosov for almost all integers  $n$ . To see this, we invoke Theorem III.3 of FLP [5] (see also Ivanov [7]) which asserts that if

$$(5.1) \quad \{f^k(\alpha) \text{ for } k = 0, \dots, m - 1\}$$

fills  $\tilde{S}$  in the sense that  $\tilde{S} \setminus \{f^k(\alpha), k = 0, \dots, m - 1\}$  is a union of disks or punctured disks, then for almost all integers  $n$  (with possibly seven exceptional cases)  $t_{\alpha}^n \circ f$  and  $f \circ t_{\alpha}^n$  represent hyperbolic mapping classes.

It is now easy to obtain a simple curve  $\alpha$  on  $S$  satisfying condition (5.1). Figure 1 demon-

strates a conformal automorphism of order 4 on a compact Riemann surface of genus 4. The curve  $\alpha$  in the figure together with all the images  $f^i(\alpha)$  fill the surface  $S$ .

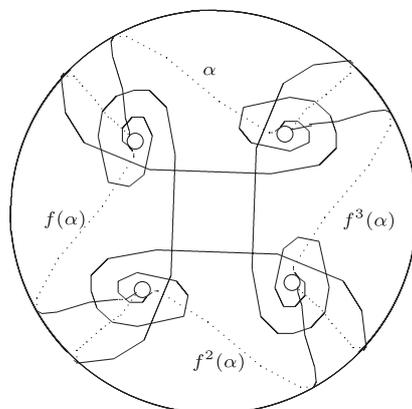


Fig. 1

ACKNOWLEDGEMENTS. This work has been partially supported by NSFC under grants 11371126 and 11571099. The authors thank the referee for carefully reading the manuscript and for his/her valuable suggestions and comments, which much improve the quality of the paper.

---

### References

- [1] A. Beardon: *The geometry of Discrete groups*, Springer-Verlag, NY Heidelberg Berlin, 1983.
- [2] L. Bers: *On boundaries of Teichmüller spaces and on Kleinian groups: I*, Ann. of Math. **91** (1970), 570–600.
- [3] L. Bers: *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
- [4] L. Bers: *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Math. **141** (1978), 73–98.
- [5] A. Faithi, F. Laudenbach and V. Poenaru: *Travaux de Thurston sur les surfaces*, Seminaire Orsay, Asterisque, **66–67**, Soc. Math. de France, 1979.
- [6] N.V. Ivanov: *Mapping class groups*, Manuscript, Dec 21, 1998.
- [7] N.V. Ivanov: *Subgroup of Teichmüller modular groups*, Trans. of Math. Monographs **115**, AMS., Providence, RI, 1992.
- [8] I. Kra: *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.
- [9] S. Nag: *Nongeodesic discs embedded in Teichmüller spaces*, Amer. J. Math. **104** (1982), 339–408.
- [10] H.L. Royden: *Automorphisms and isometries of Teichmüller space*, Ann. of Math. Studies, **66** (1971), 369–383.
- [11] W.P. Thurston: *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431.
- [12] C. Zhang: *On isomorphisms of Bers fiber spaces*, Ann Acad. Sci. Fen. Math. **22** (1997), 255–274.
- [13] C. Zhang: *Hyperbolic mapping classes and their lifts on the Bers fiber space*, preprint (2004).
- [14] C. Zhang: *Singularities of quadratic differentials and extremal Teichmüller mappings defined by Dehn twists*, J. Aust. Math. Soc. **87** (2009), 275–288.

- [15] C. Zhang: *Pseudo-Anosov maps and fixed points of boundary homeomorphisms compatible with a Fuchsian group*, Osaka J. Math. **46** (2009), 783–798.

Yingqing Xiao  
College of Mathematics and Econometrics  
Hunan University  
Changsha, 410082  
China  
e-mail: ouxyq@hnu.edu.cn

Chaohui Zhang  
Zhang, Department of Mathematics  
Morehouse College  
Atlanta, GA 30314  
USA  
e-mail: czhang@morehouse.edu