Nath, A. and Sankaran, P.
Osaka J. Math.
56 (2019), 75–90

ON GENERALIZED DOLD MANIFOLDS

Dedicated to Professor D.S. Nagaraj on the occasion of his sixtieth birthday

Avijit NATH and Parameswaran SANKARAN

(Received August 10, 2017)

Abstract

Let X be a smooth manifold with a (smooth) involution $\sigma : X \to X$ such that $\mathrm{Fix}(\sigma) \neq \emptyset$. We call the space $P(m, X) := \mathbb{S}^m \times X/\sim$ where $(v, x) \sim (-v, \sigma(x))$ a generalized Dold manifold. When X is an almost complex manifold and the differential $T\sigma : TX \to TX$ is conjugate complex linear on each fibre, we obtain a formula for the Stiefel-Whitney polynomial of $P(m, X)$ when $H^1(X; \mathbb{Z}_2) = 0$. We obtain results on stable parallelizability of $P(m, X)$ and a very general criterion for the (non) vanishing of the unoriented cobordism class $[P(m, X)]$ in terms of the corresponding properties for X. These results are applied to the case when X is a complex flag manifold.

1. Introduction

Let $P(m, n)$ denote the space obtained as the quotient by the cyclic group $\mathbb{Z}_2$-action on the product $\mathbb{S}^m \times \mathbb{C}P^n$ generated by the involution $(u, L) \mapsto (-u, \bar{L}), u \in \mathbb{S}^m, L \in \mathbb{C}P^n$ where $\bar{L}$ denotes the complex conjugation. The spaces $P(m, n)$, which seem to have first appeared in the work of Wu, are called Dold manifolds, after it was shown by Dold [6] that, for suitable values of m, n, the cobordism classes of $P(m, n)$ serve as generators in odd degrees for the unoriented cobordism algebra $\mathfrak{N}$. Dold manifolds have been extensively studied and have received renewed attention in recent years; see [9], [15] and also [14], [20], and [4].

The construction of Dold manifolds suggests, among others, the following generalization. Consider an involution on a Hausdorff topological space $\sigma : X \to X$ with non-empty fixed point set and consider the space $P(m, X, \sigma)$ obtained as the quotient of $\mathbb{S}^m \times X$ by the action of $\mathbb{Z}_2$ defined by the fixed point free involution $(v, x) \mapsto (-v, \sigma(x))$. We obtain a locally trivial fibre bundle with projection $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ and fibre space X. If x_0 is a fixed point of σ, then the bundle admits a cross-section $s : \mathbb{R}P^m \to P(m, X, \sigma)$ defined as $s([v]) = [v, x_0]$. If X is a smooth manifold and if σ is smooth, then the above bundle and the cross-section are smooth.

In this paper we study certain manifold-properties of $P(m, X, \sigma)$ (or more briefly $P(m, X)$) where X is a closed connected smooth manifold with an almost complex structure $J : TX \to TX$ and σ is a conjugation, that is, the differential $T\sigma : TX \to TX$ and J anti-commute: $T\sigma \circ J = -J \circ T\sigma$. We give a description of the tangent bundle of $P(m, X)$. Assuming that $\mathrm{Fix}(\sigma) \neq \emptyset$ and $H^1(X; \mathbb{Z}_2) = 0$, we obtain a formula for the Stiefel-Whitney classes of

2010 Mathematics Subject Classification. 57R25, 57R20.

Both authors were partially supported by a XII Plan Project, Department of Atomic Energy, Government of India.

$P(m, X)$ (Theorem 3.1) and a necessary and sufficient condition for $P(m, X)$ to admit a spin structure (Theorem 3.2). We also obtain results on the stable parallelizability of the $P(m, X)$ (Theorem 3.3) and the vanishing of their (unoriented) cobordism class in the cobordism ring $\mathfrak{N}$ (Theorem 3.7).

Recall that a smooth manifold M is said to be parallelizable (resp. stably parallelizable) if its tangent bundle τM (resp. $\epsilon_{\mathbb{R}} \oplus \tau M$) is trivial.

By the celebrated work of Adams [1] on the vector field problem for spheres, one knows that the (additive) order of the element $([\zeta] - 1) \in KO(\mathbb{R}P^m)$ equals $2^{\varphi(m)}$ where ζ is the Hopf line bundle over $\mathbb{R}P^m$ and $\varphi(m)$ is the number of positive integers $j \leq m$ such that $j \equiv 0, 1, 2,$ or $4 \mod 8$.

The complex flag manifold $\mathbb{C}G(n_1, \dots, n_r)$ is the homogeneous space $U(n)/(U(n_1) \times \cdots \times U(n_r))$, where the $n_j \geq 1$ are positive integers and $n = \sum_{1 \leq j \leq r} n_j$. These manifolds are well-known to be complex projective varieties. We denote by $P(m; n_1, \dots, n_r)$ the space $P(m, \mathbb{C}G(n_1, \dots, n_r))$. The complete flag manifold $\mathbb{C}G(1, \dots, 1)$ is denoted $\mathrm{Flag}(\mathbb{C}^n)$. Note that $\mathbb{C}G(n_1, n_2)$ is the complex Grassmann manifold $\mathbb{C}G_{n,n_1}$ of n_1-dimensional vector subspaces of $\mathbb{C}^n$.

We highlight here the results on stable parallelizability and cobordism for a restricted classes of generalized Dold manifolds as in these cases the results are nearly complete.

Theorem 1.1. *Let $m \geq 1$ and $r \geq 2$.*
(i) *The manifold $P(m; n_1, \dots, n_r)$ is stably parallelizable if and only if $n_j = 1$ for all j and $2^{\varphi(m)}$ divides $\left(m + 1 + \binom{n}{2}\right)$.*
(ii) *Suppose that $P := P(m; 1, \dots, 1)$ is stably parallelizable. Then it is parallelizable if $\rho(m + 1) > \rho(m + 1 + n(n - 1))$. If m is even, then P is not parallelizable.*

The case when the flag manifold is a complex projective space corresponds to the classical Dold manifold $P(m, n - 1)$. In this special case the above result is due to J. Korbaš [9]. See also [21] in which J. Ucci characterized classical Dold manifolds which admit codimension-one embeddings in the Euclidean space.

Theorem 1.2. *Let $1 \leq k \leq n/2$ and let $m \geq 1$.*
(i) *If $\nu_2(k) < \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] = 0$ in $\mathfrak{N}$.*
(ii) *If $m \equiv 0 \mod 2$ and if $\nu_2(k) \geq \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] \neq 0$.*

The above theorem leaves out the case when $m \geq 1$ is odd and $\nu_2(k) \geq \nu_2(n)$. See Remark 3.8 for results on the vanishing of $[P(m; n_1, \dots, n_r)]$.

Our proofs make use of basic concepts in the theory of vector bundles and characteristic classes. We first introduce, in §2, the notion of a σ-conjugate complex vector bundle over X where σ is an involution on X and associate to each such complex vector bundle ω a real vector bundle over $\hat{\omega}$. We establish a splitting principle to obtain a formula for the Stiefel-Whitney classes of $\hat{\omega}$ in terms of certain 'cohomology extensions' of Stiefel-Whitney classes of ω, assuming that $H^1(X; \mathbb{Z}_2) = 0$. This leads to a formula for the Stiefel-Whitney classes of $P(m, X)$ when X is a smooth almost complex manifold and σ is a complex conjugation. Proof of Theorem 1.1 uses the main result of [18], the Bredon-Kosiński's theorem [3], and a certain functor μ^2 introduced by Lam [11] to study immersions of flag manifolds. Proof of Theorem 1.2 uses basic facts from the theory of Clifford algebras, a result of Conner and Floyd [5, Theorem 30.1] concerning cobordism of manifolds admitting stationary point free action of elementary abelian 2-group, and the main theorem of [17].

2. Vector bundles over $P(m, X, \sigma)$

Let $\sigma : X \to X$ be an involution of a path connected paracompact Hausdorff topological space and let ω be a complex vector bundle over X. Denote by $\omega^\vee$ the dual vector bundle $\mathrm{Hom}_{\mathbb{C}}(\omega, \epsilon_{\mathbb{C}})$. Here $\epsilon_{\mathbb{F}}$ denotes the the trivial $\mathbb{F}$-line bundle over X where $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Note that, since X is paracompact, ω admits a Hermitian metric and so $\omega^\vee$ is isomorphic to the conjugate bundle $\bar{\omega}$. The following definition generalises the notion of a conjugation of an almost complex manifold in the sense of Conner and Floyd [5, §24].

DEFINITION 2.1. Let $\sigma : X \to X$ be an involution and let ω be a complex vector bundle over X. A *σ-conjugation* on ω is an involutive bundle map $\hat{\sigma} : E(\omega) \to E(\omega)$ that covers σ which is conjugate complex linear on the fibres of ω. If such a $\hat{\sigma}$ exists, we say that $(\omega, \hat{\sigma})$ (or more briefly ω) is a *σ-conjugate bundle*.

Note that if ω is a σ-conjugate bundle, then $\bar{\omega} \cong \sigma^*(\omega)$.

EXAMPLE 2.2. (i) Let σ be any involution on X. When $\omega = n\epsilon_{\mathbb{C}}$, the trivial complex vector bundle of rank n, we have $E(\omega) = X \times \mathbb{C}^n$. The *standard* σ-conjugation on ω is defined as $\hat{\sigma}(x, \sum z_j e_j) = (\sigma(x), \sum \bar{z}_j e_j)$. Here $\{e_j\}_{1 \leq j \leq n}$ is the standard basis of $\mathbb{C}^n$. Thus $(n\epsilon_{\mathbb{C}}, \hat{\sigma})$ is σ-conjugate bundle.

(ii) Let $X = \mathbb{C}G_{n,k}$ and let $\sigma : X \to X$ be the involution $L \mapsto \bar{L}$. Then the standard σ-conjugation on $n\epsilon_{\mathbb{C}}$ defines, by restriction, a σ-conjugation of the canonical k-plane bundle $\gamma_{n,k}$. Explicitly, $v \mapsto \bar{v}$, $v \in L \in \mathbb{C}G_{n,k}$, is the required involutive bundle map $\hat{\sigma} : E(\gamma_{n,k}) \to E(\gamma_{n,k})$ that covers σ. Similarly the orthogonal complement $\beta_{n,k} := \gamma_{n,k}^\perp$ is also a σ-conjugate bundle.

(iii) If $X \subset \mathbb{C}P^N$ is a complex projective manifold defined over $\mathbb{R}$ and $\sigma : X \to X$ is the restriction of complex conjugation $[z] \mapsto [\bar{z}]$, then the tangent bundle τX of X is a σ-conjugate bundle. Indeed the differential of σ, namely $T\sigma : TX \to TX$ is the required bundle map $\hat{\sigma}$ of τX that covers σ. As mentioned above, this classical case was generalized by Conner and Floyd [5, §24] to the case when X is an almost complex manifold.

(iv) If ω, η are σ-conjugate vector bundles over X, then so are $\Lambda^r(\omega), \mathrm{Hom}_{\mathbb{C}}(\omega, \eta), \omega \otimes \eta$, and $\omega \oplus \eta$. For example, if $\hat{\sigma}$ and $\tilde{\sigma}$ are σ-conjugations on ω and η respectively, both covering σ, then $\mathrm{Hom}_{\mathbb{C}}(\omega, \eta) \ni f \mapsto \tilde{\sigma} \circ f \circ \hat{\sigma} \in \mathrm{Hom}_{\mathbb{C}}(\omega, \eta)$ is verified to be a conjugate complex linear bundle involution of $\mathrm{Hom}_{\mathbb{C}}(\omega, \eta)$ that covers σ.

(v) Any subbundle η of a σ-conjugate complex vector bundle ω over X is also σ-conjugate provided $\hat{\sigma} : E(\omega) \to E(\omega)$ satisfies $\hat{\sigma}(E(\eta)) = E(\eta)$.

2.1. Vector bundle associated to $(\eta, \hat{\sigma})$. Let η be a *real* vector bundle over X with projection $p_\eta : E(\eta) \to X$ and let $\hat{\sigma} : E(\eta) \to E(\eta)$ be an involutive bundle isomorphism that covers σ. We obtain a real vector bundle, denoted $\hat{\eta}$, over $P(m, X, \sigma)$ as follows: $(v, e) \mapsto (-v, \hat{\sigma}(e))$ defines a fixed point free involution of $\mathbb{S}^m \times E(\eta)$ with orbit space $P(m, E(\eta), \hat{\sigma})$. The map $p_{\hat{\eta}} : P(m, E(\eta), \hat{\sigma}) \to P(m, X, \sigma)$ defined as $[v, e] \mapsto [v, p_\eta(e)]$ is the projection of the required bundle $\hat{\eta}$.

This construction is applicable when $\eta = \rho(\omega)$, the underlying real vector bundle of a σ-

conjugate complex vector bundle $(\omega, \hat{\sigma})$. If β is a (real) subbundle of η such that $\hat{\sigma}(E(\beta)) = E(\beta)$, then the restriction of $\hat{\sigma}$ to $E(\beta)$ defines a bundle $\hat{\beta}$ which is evidently a subbundle of $\hat{\eta}$.

We shall denote by ξ the real line bundle over $P(m, X, \sigma)$, often referred to as the Hopf bundle, associated to the double cover $\mathbb{S}^m \times X \to P(m, X, \sigma)$. Its total space has the description $\mathbb{S}^m \times X \times_{\mathbb{Z}_2} \mathbb{R}$ consisting of elements $[v, x, t] = \{(v, x, t), (-v, \sigma(x), -t)\}, v \in \mathbb{S}^m, x \in X, t \in \mathbb{R}$. Denote by $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ the map $[v, x] \mapsto [v]$. Then π is the projection of a fibre bundle with fibre X. The map $E(\xi) \to E(\zeta)$ defined as $[v, x, t] \mapsto [v, t]$ is a bundle map that covers the projection $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$ and so $\xi \cong \pi^*(\zeta)$.

If $\sigma(x_0) = x_0 \in X$, then we have a cross-section $s : \mathbb{R}P^m \to P(m, X)$ defined as $[v] \mapsto [v, x_0]$. Note that $s^*(\xi) = \zeta$.

2.2. Dependence of $\hat{\omega}$ on $\hat{\sigma}$. *It should be noted that the definition of $\hat{\eta}$ depends not only on the real vector bundle η but also on the bundle map $\hat{\sigma}$ that covers σ.* For example, on the trivial line bundle $\epsilon_{\mathbb{R}}$, if $\hat{\sigma}(x, t) = (\sigma(x), t)$, then $\hat{\epsilon}_{\mathbb{R}} \cong \epsilon_{\mathbb{R}}$, whereas if $\hat{\sigma}(x, t) = (\sigma(x), -t)$, then $\hat{\epsilon}_{\mathbb{R}}$ is isomorphic to ξ.

When $\omega = \tau X$ is the tangent bundle over an almost complex manifold (X, J) and $\hat{\sigma} = T\sigma$ where σ is a conjugation on X, (i.e., satisfies $J_{\sigma(x)} \circ T_x\sigma = -T_x\sigma \circ J_x \ \forall x \in X$), the vector bundle $\hat{\tau}X$ is understood to be defined with respect to the pair $(\tau X, T\sigma)$.

Let $k, l \geq 0$ be integers and let $n = k + l \geq 1$ and let $s_1, \ldots, s_n$ be everywhere linearly independent sections of the trivial bundle $n\epsilon_{\mathbb{R}}$. Denote by $\varepsilon_{k,l} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ the involutive bundle map $n\epsilon_{\mathbb{R}}$ covering σ defined as $\varepsilon_{k,l}(x, \sum_j t_j s_j(x)) = (\sigma(x), -\sum_{1\leq j\leq k} t_j s_j(x) + \sum_{k<j\leq n} t_j s_j(x))$. Then the bundle over $P(m, X, \sigma)$ associated to $(n\epsilon_{\mathbb{R}}, \varepsilon_{k,l})$ is isomorphic to $k\xi \oplus l\epsilon_{\mathbb{R}}$. When $n = 2d, k = l = d, n\epsilon_{\mathbb{R}} = \rho(d\epsilon_{\mathbb{C}})$ then the standard conjugation on $d\epsilon_{\mathbb{C}}$ equals $\varepsilon_{d,d}$ (for an obvious choice of $s_j, 1 \leq j \leq n$).

Let $(\omega, \hat{\sigma})$ be a σ-conjugate complex vector bundle and let η be a real vector bundle which is isomorphic to the real vector bundle $\rho(\omega)$ underlying ω. Suppose that $f : \rho(\omega) \to \eta$ is a bundle isomorphism that covers the identity map of X. Set $\tilde{\sigma} := f \circ \hat{\sigma} \circ f^{-1}$. Then $\widetilde{\sigma}$ is an involution of η that covers σ and hence defines a vector bundle $\hat{\eta}$ over $P(m, X, \sigma)$.

Lemma 2.3. *We keep the above notations.* (i) *The real vector bundles $\hat{\omega}$ and $\hat{\eta}$ over $P(m, X, \sigma)$ associated to the pairs $(\omega, \hat{\sigma})$ and $(\eta, \tilde{\sigma})$ are isomorphic. In particular $\hat{\omega} \cong \hat{\bar{\omega}}$.*
(ii) *Suppose that $\rho(\omega) = \eta_0 \oplus \eta_1$ where $\eta_j, j = 0, 1$ are real vector bundles. Suppose that $\hat{\sigma}(E(\eta_j)) = E(\eta_j)$, then $\hat{\omega}$ is isomorphic to $\hat{\eta}_0 \oplus \hat{\eta}_1$ where $\hat{\eta}_j$ is defined with respect to the pair $(\eta_j, \hat{\sigma}|_{E(\eta_j)})$, $j = 0, 1$.*
(iii) *Let $n = k + l \geq 1$. Suppose that $\rho(\omega) \oplus n\epsilon_{\mathbb{R}} \cong N\epsilon_{\mathbb{R}}$, where $N := 2d + n$, and that $\varepsilon_{d+k,d+l}$ on $N\epsilon_{\mathbb{R}}$ restricts to $\hat{\sigma}$ on $\rho(\omega)$ and to $\varepsilon_{k,l}$ on $n\epsilon_{\mathbb{R}}$. Then $\hat{\omega} \oplus k\xi \oplus l\epsilon_{\mathbb{R}} \cong (d + k)\xi \oplus (d + l)\epsilon_l$.*

Proof. We will only prove (i); the proofs of remaining parts are likewise straightforward. Consider the map $\phi : \mathbb{S}^m \times E(\omega) \to \mathbb{S}^m \times E(\eta)$ defined as $\phi(v, e) = (v, f(e)) \ \forall v \in \mathbb{S}^m, e \in E(\omega)$. The $\phi((-v, \sigma(e))) = (-v, f(\hat{\sigma}(e))) = (-v, \tilde{\sigma}(f(e)))$. Thus ϕ is $\mathbb{Z}_2$-equivariant and so induces a vector bundle homomorphism $\bar{\phi} : P(m, E(\omega), \hat{\sigma}) \to P(m, E(\eta), \tilde{\sigma})$ that covers the identity map of $P(m, X, \sigma)$. Restricted to each fibre, the map $\bar{\phi}$ is an $\mathbb{R}$-linear isomorphism since this is true of f. Therefore $\hat{\omega}$ and $\hat{\eta}$ are isomorphic vector bundles. Finally, let $\eta = \bar{\omega}, \tilde{\sigma} = \hat{\sigma}$ and $f = id$. Then $\hat{\omega} \cong \hat{\bar{\omega}}$. □

Example 2.4. (i) Consider the Riemann sphere $\mathbb{S}^2 = \mathbb{C}P^1$. Let $\gamma \subset 2\epsilon_{\mathbb{C}}$ be the tautological (complex) line bundle over $\mathbb{C}P^1$ and let β be its orthogonal complement. As complex line bundles one has the isomorphism $\beta \cong \bar{\gamma}$. It follows that from the above lemma that $2\hat{\gamma} \cong 2\hat{\beta} \cong 2\xi \oplus 2\epsilon_{\mathbb{R}}$.
(ii) Suppose that $X = \mathbb{C}G_{n,k}$ and let $\sigma : X \to X$ be the conjugation $L \to \bar{L}$. As seen in Example 2.2(ii), $v \mapsto \bar{v}$ define conjugations of $\gamma_{n,k}, \beta_{n,k}$ that cover σ. Note that $\gamma_{n,k} \oplus \beta_{n,k} = n\epsilon_{\mathbb{C}}$. By the above lemma we obtain that $\hat{\gamma}_{n,k} \oplus \hat{\beta}_{n,k} \cong d\hat{\epsilon}_{\mathbb{C}} \cong d\epsilon_{\mathbb{R}} \oplus d\xi$. Also, the conjugations on $\gamma_{n,k}, \beta_{n,k}$ induce an involution, denoted $\hat{\sigma}$, on $\omega := \mathrm{Hom}(\gamma_{n,k}, \beta_{n,k})$; see Example 2.2(iv). One has the isomorphism $\tau\mathbb{C}G_{n,k} \cong \omega$ of complex vector bundles ([11]). Under this isomorphism, the bundle involution $\hat{\sigma}$ corresponds to $T\sigma : T\mathbb{C}G_{n,k} \to T\mathbb{C}G_{n,k}$. Therefore $\hat{\omega} \cong \hat{\tau}\mathbb{C}G_{n,k}$.

2.3. Splitting principle. Denote by $\mathrm{Flag}(\mathbb{C}^r)$ the complete flag manifold $\mathbb{C}G(1, \dots, 1)$. Let ω be a complex vector bundle over X of rank $r \geq 1$ endowed with a Hermitian metric and let $q : \mathrm{Flag}(\omega) \to X$ be the $\mathrm{Flag}(\mathbb{C}^r)$-bundle associated to ω. Thus the fibre over an $x \in X$ is the space $\{(L_1, \dots, L_r) \mid L_1 + \cdots + L_r = p_\omega^{-1}(x), L_j \perp L_k, 1 \leq j < k \leq r, \dim_{\mathbb{C}} L_j = 1\} \cong \mathrm{Flag}(\mathbb{C}^r)$ of complete flags in $p_\omega^{-1}(x) \subset E(\omega)$. The vector bundle $q^*(\omega)$ splits as a Whitney sum $q^*(\omega) = \oplus_{1 \leq j \leq r} \omega_j$ of complex line bundles ω_j over $\mathrm{Flag}(\omega)$ with projection $p_j : E(\omega_j) \to \mathrm{Flag}(\omega)$. The fibre over a point $\mathbf{L} = (L_1, \dots, L_r) \in \mathrm{Flag}(\omega)$ of the bundle ω_j is the vector space $L_j \subset p_\omega^{-1}(q(\mathbf{L}))$.

Suppose that $\sigma : X \to X$ is an involution and that $\hat{\sigma} : E(\omega) \to E(\omega)$ is a σ-conjugation on ω. We shall write $\bar{e}$ for $\hat{\sigma}(e), e \in E(\omega)$. One has the involution $\theta : \mathrm{Flag}(\omega) \to \mathrm{Flag}(\omega)$ defined as $\mathbf{L} = (L_1, \dots, L_r) \mapsto (\bar{L}_1, \dots, \bar{L}_r) =: \bar{\mathbf{L}}$. Here $\bar{V}$ denotes the subspace $\hat{\sigma}(V) \subset p_\omega^{-1}(\sigma(x))$ when $V \subset p_\omega^{-1}(x)$. Then $\hat{\theta} : E(q^*(\omega)) \to E(q^*(\omega))$ defined as $\hat{\theta}(\mathbf{L}, e) = (\bar{\mathbf{L}}, \bar{e})$ is a θ-conjugation on $q^*(\omega)$. Moreover, it restricts to a θ-conjugation $\hat{\theta}_j$ on the subbundle ω_j for each $j \leq r$.

Recall from §2.1 that $\hat{\omega}$ is the real vector bundle with projection $p_{\hat{\omega}} : P(m, E(\omega), \hat{\omega}) \to P(m, X, \sigma)$. Likewise, we have the real 2-plane bundle $\hat{\omega}_j$ over $P(m, \mathrm{Flag}(\omega), \theta)$ with projection $p_{\hat{\omega}_j} : P(m, E(\omega_j), \hat{\theta}_j) \to P(m, \mathrm{Flag}(\omega), \theta)$. Since $q \circ \theta = \sigma \circ q$, we have the induced map $\hat{q} : P(m, \mathrm{Flag}(\omega), \theta) \to P(m, X, \sigma)$ defined as $[v, \mathbf{L}] \mapsto [v, q(\mathbf{L})]$. The map $\hat{q}$ is in fact the projection of a fibre bundle with fibre the flag manifold $\mathrm{Flag}(\mathbb{C}^r)$. Since $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$, applying Lemma 2.3 (ii) we see that $\hat{q}^*(\hat{\omega}) \cong \oplus_{1 \leq j \leq r} \hat{\omega}_j$.

Recall that the first Chern classes mod 2 of the canonical complex line bundles ξ_j over $\mathrm{Flag}(\mathbb{C}^r)$, $1 \leq j \leq r$, generate the $\mathbb{Z}_2$-cohomology algebra $H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2)$. In fact $H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_r]/I$ where I is the ideal generated by the elementary symmetric polynomials in $c_1, \dots, c_r$. Here the generators $c_j + I$ may be identified with the (integral) Chern class $c_1(\xi_j)$. In particular $H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z})^{S_r} = H^0(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}) \cong \mathbb{Z}$ and so a similar isomorphism holds for mod 2 cohomology.

Since $\hat{\omega}_j$ restricts to the (real) 2-plane bundle $\rho(\xi_j)$, we have $c_1(\xi_j) = i^*(w_2(\omega_j))$ where $i : \mathrm{Flag}(\mathbb{C}^r) \cong \hat{q}^{-1}([v, x]) \to P(m, \mathrm{Flag}(\omega), \theta)$ is fibre inclusion, we see that the $\mathrm{Flag}(\mathbb{C}^r)$-bundle $(P(m, \mathrm{Flag}(\omega), \theta), P(m, X, \sigma), \hat{q})$ admits a $\mathbb{Z}_2$-cohomology extension of the fibre. By Leray-Hirsch theorem [19, §7, Ch.V], we have $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2) \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2)$. Thus $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)$ is a free module over the algebra $H^*(P(m, X, \sigma); \mathbb{Z}_2)$ of rank $\dim_{\mathbb{Z}_2} H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2) = r!$. In particular, it follows that $\hat{q}$ induces a monomorphism in mod 2 cohomology.

The symmetric group S_r operates on $\mathrm{Flag}(\omega)$ by permuting the components of each flag $\mathbf{L} = (L_1, \ldots, L_r)$ and the projection $q : \mathrm{Flag}(\omega) \to X$ is constant on the S_r-orbits. Moreover, $\theta \circ \lambda = \lambda \circ \theta$ for each $\lambda \in S_r$. This implies that the S_r action on $\mathrm{Flag}(\omega)$ extends to an action on $P(m, \mathrm{Flag}(\omega), \theta)$ where $\lambda([v, \mathbf{L}]) = [v, \lambda(\mathbf{L})]$. The projection $\hat{q} : P(m, \mathrm{Flag}(\omega), \theta) \to P(m, X, \sigma)$ is constant on S_r-orbits. It follows that the image of the ring homomorphism $\hat{q}^* : H^*(P(m, X, \sigma); \mathbb{Z}_2) \to H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)$ is contained in the subring $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)^{S_r}$ of elements fixed by the induced action of S_r on $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)$. As the S_r-action induces the identity map of $P(m, X, \sigma)$ we see that it acts as $H^*(P(m, X, \sigma); \mathbb{Z}_2)$-module automorphisms on $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)$. Since $H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2)^{S_r} = H^0(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \cong \mathbb{Z}_2$, we have $H^*(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2)^{S_r} \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^*(\mathrm{Flag}(\mathbb{C}^r); \mathbb{Z}_2)^{S_r} = H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^0(\mathrm{Flag}(\mathbb{C}^r; \mathbb{Z}_2) \cong H^*(P(m, X, \sigma); \mathbb{Z}_2)$.

We summarise the above discussion in the proposition below.

Proposition 2.5 (Splitting principle)**.** *Let ω be a σ-conjugate complex vector bundle of rank r and let $q : Flag(\omega) \to X$ be the associated $Flag(\mathbb{C}^r)$-bundle over X. Then, with the above notations,*
(i) *the ω_j are θ-conjugate line bundles for $1 \leq j \leq r$, and, $\hat{q}^*(\hat{\omega}) = \oplus_{1 \leq j \leq r} \hat{\omega}_j$.*
(ii) *$\hat{q} : P(m, Flag(\omega), \theta) \to P(m, X, \sigma)$ induces a monomorphism in cohomology, moreover, $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$ is isomorphic, as an $H^*(P(m, X, \sigma); \mathbb{Z}_2)$-module, to a free module with basis a $\mathbb{Z}_2$-basis of $H^*(Flag(\mathbb{C}^r); \mathbb{Z}_2)$.*
(iii) *The image of $\hat{q}^*$ equals the subalgebra invariant under the action of the symmetric group S_r on $H^*(P(m, Flag(\omega), \theta); \mathbb{Z}_2)$.* □

We end this section with the following lemma which will be used in the sequel.

Lemma 2.6. *We keep the above notations. Let ω be a σ-conjugate complex vector bundle over X. Suppose that $Fix(\sigma) \neq \emptyset$ and that $H^1(X; \mathbb{Z}_2) = 0$. Then $Fix(\theta) \neq \emptyset$ and $H^1(P(m, Flag(\omega), \theta); \mathbb{Z}_2) \cong H^1(P(m, X, \sigma); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Proof. Let $\sigma(x) = x \in X$ and set $V := p_\omega^{-1}(x)$. Then $\hat{\sigma}$ restricts to a conjugate complex isomorphism $\hat{\sigma}_x$ of V onto itself. Thus $V \cong \bar{V}$. Then, setting $\mathrm{Fix}(\hat{\sigma}_x) =: U \subset V$, we see that V is the $\mathbb{C}$-linear extension of U, that is, $V = U \otimes_{\mathbb{R}} \mathbb{C}$. The Hermitian product on V restricts to a (real) inner product on U. Let $(K_1, \ldots, K_r)$ be a complete real flag in U and define $L_j := K_j \otimes_{\mathbb{R}} \mathbb{C} \subset V$. Then it is readily seen that $\mathbf{L} = (L_1, \ldots, L_r)$ belongs to $\mathrm{Flag}(\omega)$ and is fixed by θ.

Since $H^1(X; \mathbb{Z}_2) = 0$, we have $H^1(P(m, X, \sigma); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2$, using the Serre spectral sequence of the X-bundle with projection $\pi : P(m, X, \sigma) \to \mathbb{R}P^m$. The same argument applied to the $\mathrm{Flag}(\mathbb{C}^r)$-bundle with projection $q : \mathrm{Flag}(\omega) \to X$ yields that $H^1(\mathrm{Flag}(\omega); \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2) = 0$. Now using the $\mathrm{Flag}(\omega)$-bundle with projection $\hat{q} : P(m, \mathrm{Flag}(\omega), \theta) \to P(m, X, \sigma)$, we obtain that $H^1(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2) \cong H^1(P(m, X, \sigma); \mathbb{Z}_2) \cong \mathbb{Z}_2$. □

We shall identify $H^1(P(m, \mathrm{Flag}(\omega), \theta); \mathbb{Z}_2), H^1(P(m, X, \sigma); \mathbb{Z}_2), H^1(\mathbb{R}P^m; \mathbb{Z}_2)$ and denote the generator of any one of them by x. [1]

[1] This should however cause no confusion with the notation for a typical point of X.

2.4. A formula for Stiefel-Whitney classes of $\hat{\omega}$. Denote the Stiefel-Whitney polynomial $\sum_{0\leq i\leq q} w_i(\eta)t^i$ of a rank q real vector bundle η by $w(\eta;t)$ and similarly the Chern polynomial $\sum_{0\leq i\leq q} c_j(\alpha)t^j$ of a complex vector bundle α of rank q by $c(\alpha;t)$. Recall that when α is regarded as a real vector bundle, we have $w(\alpha;t) = c(\alpha;t^2) \mod 2$. (See [13].)

We shall make no notational distinction between $c_j(\alpha) \in H^{2j}(X;\mathbb{Z})$ and its reduction mod 2 in $H^{2j}(X;\mathbb{Z}_2)$. In fact, we will mostly be working with $\mathbb{Z}_2$-coefficients.

Since $\hat{\omega}$ restricted to any fibre of $\pi : P(m,X,\sigma) \to \mathbb{R}P^m$ is isomorphic to ω (regarded as a real vector bundle), we obtain that, the total Stiefel-Whitney polynomial $j^*(w(\hat{\omega};t)) = w(\omega;t) = c(\omega,t^2)$ where $j : X \to P(m,X,\sigma)$ is the fibre inclusion.

The following proposition yields the Stiefel-Whitney classes of $\hat{\omega}$ when ω is a complex line bundle. Using this and the splitting principle, we will obtain a formula for the Stiefel-Whitney classes when ω is of arbitrary rank. The proposition was obtained in the special case of Dold manifolds in [21, Prop. 1.4]. Recall that ξ is the line bundle associated to the double cover $\mathbb{S}^m \times X \to P(m,X,\sigma)$ and is isomorphic to $\pi^*(\zeta)$.

Lemma 2.7. *Let $\sigma : X \to X$ be an involution with non-empty fixed point set and let ω be a complex vector bundle of rank r over X. With the above notations, we have $\hat{\omega} \cong \xi \otimes \hat{\omega}$.*

Proof. The total space of the bundle $\xi \otimes \hat{\omega}$ has the description $E(\xi \otimes \hat{\omega}) = \{[v,x;t\otimes e] \mid [v,x] \in P(m,X;\sigma), t \in \mathbb{R}, e \in p_\omega^{-1}(x)\}$ where $[v,x;t\otimes e] = \{(v,x;t\otimes e),(-v,\sigma(x);-t\otimes \hat{\sigma}(e))\}$; here $\hat{\sigma} : E(\omega) \to E(\omega)$ is an *involutive* bundle map that covers σ and is conjugate linear isomorphism on each fibre. Thus we have the equality $\hat{\sigma}(\sqrt{-1}te) = -\sqrt{-1}t\hat{\sigma}(e)$. Observe that $[v,x;\sqrt{-1}te] = [-v,\sigma(x);\hat{\sigma}(\sqrt{-1}te)] = [-v,\sigma(x),-\sqrt{-1}t\hat{\sigma}(e)]$ and so the map $h : E(\xi\otimes\hat{\omega}) \to E(\hat{\omega})$, $[v,x;t\otimes e] \mapsto [v,x;\sqrt{-1}te] = [-v,\sigma(x);-\sqrt{-1}t\hat{\sigma}(e)]$ is a well-defined isomorphism of real vector bundles. □

Simplifying assumptions. We shall make the following simplifying assumptions.

(a) $\sigma : X \to X$ has a fixed point. As observed already, the X-bundle $\pi : P(m,X,\sigma) \to \mathbb{R}P^m$ admits a cross-section $s : \mathbb{R}P^m \to P(m,X,\sigma)$. It follows that $\pi^* : H^*(\mathbb{R}P^m;\mathbb{Z}_2) \to H^*(P(m,X,\sigma);\mathbb{Z}_2)$ is a monomorphism. We shall identify $H^*(\mathbb{R}P^m;\mathbb{Z}_2)$ with its image under π^*.

(b) $H^1(X;\mathbb{Z}_2) = 0$. This implies that $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2)$ induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ of the coefficient rings is surjective.

EXAMPLE 2.8. (i) Let X be the complex flag manifold $\mathbb{C}G(n_1,\ldots,n_r)$ and let $\sigma : X \to X$ be defined by the complex conjugation on $\mathbb{C}^n$, $n = \sum n_j$. Then $\mathrm{Fix}(\sigma)$ is the *real* flag manifold $\mathbb{R}G(n_1,\ldots,n_r) = O(n)/(O(n_1)\times\cdots\times O(n_r))$ so assumption (a) holds. Since X is simply connected, (b) also holds.

(ii) Let ω be a σ-conjugate complex vector bundle of rank r. Suppose that $\mathrm{Fix}(\sigma) \neq \emptyset$ and that $H^1(X;\mathbb{Z}_2) = 0$. Let $\theta : \mathrm{Flag}(\omega) \to \mathrm{Flag}(\omega)$ be the associated involution of the $\mathrm{Flag}(\mathbb{C}^r)$-manifold bundle over X. (See §2.3.) Then $\mathrm{Fix}(\theta) \neq \emptyset$ and $H^1(\mathrm{Flag}(\omega);\mathbb{Z}_2) = 0$.

In the Serre spectral sequence of the bundle $(P(m,X),\mathbb{R}P^m,X,\pi)$, we have $E_2^{0,k} = H^0(\mathbb{R}P^m;\mathcal{H}^k(X;\mathbb{Z}_2))$ where $\mathcal{H}^k(X;\mathbb{Z}_2)$ denotes the local coefficient system on $\mathbb{R}P^m$. The action of the fundamental group of $\mathbb{R}P^m$ on $H^*(X;\mathbb{Z}_2)$ is generated by the involution $\sigma^* : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$. Hence $E_2^{0,2} = H^2(X;\mathbb{Z}_2)^{\mathbb{Z}_2} = \mathrm{Fix}(\sigma^*)$. In order to emphasise the dimension, we shall write $H^2(\sigma;\mathbb{Z}_2)$ instead of σ^*. Also (b) implies that $E_3^{0,2} = E_2^{0,2}$ and (a) implies that the transgression $E_3^{0,2} = \mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2)) \to E_3^{3,0} = H^3(\mathbb{R}P^3;\mathbb{Z}_2)$ is zero. It follows that $E_3^{0,2} = E_\infty^{0,2}$ and that the image $j^* : H^2(P(m,X);\mathbb{Z}_2) \to H^2(X;\mathbb{Z}_2)$ equals $\mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2))$, where $j : X \hookrightarrow P(m,X)$ is the fibre inclusion. We have the exact sequence:

$$(1) \qquad 0 \to H^2(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{\pi^*} H^2(P(m,X,\sigma);\mathbb{Z}_2) \xrightarrow{j^*} \mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2)) \to 0.$$

The homomorphism $s^* : H^2(P(m,X,\sigma);\mathbb{Z}_2) \to H^2(\mathbb{R}P^m;\mathbb{Z}_2)$ yields a splitting and allows us to identify $\mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2))$ as a *subspace* of $H^2(P(m,X,\sigma);\mathbb{Z}_2)$, namely the kernel of s^*. *We shall denote the image of an element $u \in Fix(H^2(\sigma;\mathbb{Z}_2))$ by $\tilde{u}$.*

Lemma 2.9. *Suppose that $\sigma(x_0) = x_0$ and $H^1(X;\mathbb{Z}_2) = 0$. Let $s : \mathbb{R}P^m \to P(m,X,\sigma)$ be defined as $v \mapsto [v,x_0]$ and let ω be a σ-conjugate complex vector bundle over X of rank r. Then* (i) *$s^*(\hat{\omega}) \cong r\epsilon_{\mathbb{R}} \oplus r\zeta$,* (ii) *$c_k(\omega) \in Fix(H^{2k}(\sigma;\mathbb{Z}_2))$, $k \le r$, and* (iii) *if $r = 1$, then $w(\hat{\omega}) = 1 + x + \tilde{c}_1(\omega)$.*

Proof. (i) Since $\sigma(x_0) = x_0$, $\hat{\sigma}$ restricts to a conjugate complex linear automorphism $\hat{\sigma}_0$ of $V := p_\omega^{-1}(x_0)$. Let $U \subset V$ is the eigenspace of $\hat{\sigma}_0$ corresponding to eigenvalue 1 of $\hat{\sigma}_0$. Then $\sqrt{-1}U$ is the -1 eigenspace. The vector bundle $s^*(\hat{\omega})$ is isomorphic to the Whitney sum of the bundles $\mathbb{S}^m \times_{\mathbb{Z}_2} U \to \mathbb{R}P^m$ and $\mathbb{S}^m \times_{\mathbb{Z}_2} \sqrt{-1}U \to \mathbb{R}P^m$. Evidently these bundles are isomorphic to $r\epsilon_{\mathbb{R}}$ and $r\xi$ respectively.

(ii) Since $\hat{\sigma} : E(\omega) \to E(\omega)$ is a *conjugate* complex linear bundle map covering σ, we have $\sigma^*(\omega) \cong \bar{\omega}$. So $\sigma^*(c_k(\omega)) = c_k(\sigma^*(\omega)) = (c_k(\bar{\omega})) = (-1)^k c_1(\omega) \in H^{2k}(X;\mathbb{Z})$. Therefore $c_k(\omega) \in \mathrm{Fix}(H^{2k}(\sigma;\mathbb{Z}_2))$, $k \le r$.

(iii) Using the isomorphism $s^* : H^1(P(m,X);\mathbb{Z}_2) \cong H^1(\mathbb{R}P^m;\mathbb{Z}_2)$, it follows from (i) that $w_1(\hat{\omega}) = w_1(\xi) = x$. Since $c_1(\omega) \in \mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2))$, the element $\tilde{c}_1(\omega)$ is meaningful. It remains to show that $w_2(\hat{\omega}) = \tilde{c}_1(\omega)$. Since $j^*(\hat{\omega}) = \omega$, we see that $j^*(w_2(\hat{\omega})) = w_2(\omega) = c_1(\omega) \in \mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2))$. On the other hand, $w_2(s^*(\hat{\omega})) = 0$. So, under our identification of $\mathrm{Fix}(H^2(\sigma;\mathbb{Z}_2))$ with the kernel of s^*, we have $w_2(\hat{\omega}) = \tilde{c}_1(\omega)$. □

REMARK 2.10. The above lemma shows that the element $\tilde{c}_1(\omega) \in H^2(P(m,X);\mathbb{Z}_2)$ is independent of the choice of the fixed point $x_0 \in X$ (used in the definition of s^*) since it equals $w_2(\hat{\omega})$.

Suppose that ω is a σ-conjugate complex vector bundle of rank r over X. Since $q^*(\omega)$ splits as a Whitney sum $q^*(\omega) = \oplus_{1\le j\le r}\omega_j$, where $q : \mathrm{Flag}(\omega) \to X$ is the $\mathrm{Flag}(\mathbb{C}^r)$-bundle, in view of Example 2.8, we have $c_1(\omega_j) \in \mathrm{Fix}(H^2(\theta;\mathbb{Z}_2))$. Therefore we obtain their 'lifts' $\tilde{c}_1(\omega_j) \in H^2(P(m,\mathrm{Flag}(\omega);\theta);\mathbb{Z}_2)$. The bundle $\hat{q}^*(\hat{\omega})$ splits as $\hat{q}^*(\hat{\omega}) = \oplus_{1\le j\le r}\hat{\omega}_j$ (see Proposition 2.5(i)), where $\hat{q} : P(m,\mathrm{Flag}(\omega),\theta) \to P(m,X,\sigma)$ is the projection of the $\mathrm{Flag}(\mathbb{C}^r)$-bundle. Therefore $e_j(\tilde{c}_1(\omega_1),\ldots,\tilde{c}_1(\omega_r)) = e_j(w_2(\hat{\omega}_1),\ldots,w_2(\hat{\omega}_r))$ is in $H^{2j}(P(m,X,\sigma);\mathbb{Z}_2)$. Here e_j stands for the j-th elementary symmetric polynomial.

NOTATION. Set $\tilde{c}_j(\omega) := e_j(w_2(\hat{\omega}_1),\ldots,w_2(\hat{\omega}_r)) \in H^{2j}(P(m,X,\sigma);\mathbb{Z}_2)$, $1 \le j \le r$.

When $j > r$, $\tilde{c}_j = 0$. Observe that $\tilde{c}_j(\omega)$ restricts to $c_j(\omega) \in H^{2j}(X;\mathbb{Z}_2)$ on any fibre of $\pi : P(m,X,\sigma);\mathbb{Z}_2) \to \mathbb{R}P^m$.

We have the following formula for the Stiefel-Whitney classes of $\hat{\omega}$.

Proposition 2.11. *We keep the above notations. Let ω be a σ-conjugate complex vector bundle over X. Suppose that $H^1(X;\mathbb{Z}_2) = 0$ and that $Fix(\sigma) \neq \emptyset$. Then,*

$$w(\hat{\omega};t) = \sum_{0\leq j\leq r} (1+xt)^{r-j}\tilde{c}_j(\omega)t^{2j}. \tag{2}$$

Proof. The case when ω is a line bundle was settled in Lemma 2.9. In the more general case, we apply the splitting principle, Proposition 2.5(i). The bundle isomorphism $\hat{q}^*(\hat{\omega}) = \hat{\omega}_1 \oplus \cdots \hat{\omega}_r$ given in Proposition 2.5(i) leads to the formula

$$w(\hat{\omega};t) = \prod_{1\leq j\leq r} (1+xt+\tilde{c}_1(\omega_j)t^2).$$

The proposition follows from Lemma 2.9 and the definition of $\tilde{c}_j(\omega)$ since $w_2(\hat{\omega}_j) = \tilde{c}_1(\omega_j)$. □

3. The tangent bundle of $P(m,X)$

Let X be a connected almost complex manifold and let $\sigma : X \to X$ be a complex conjugation. Thus $\hat{\sigma} = T\sigma$ is a σ-conjugation. The manifold $P(m,X,\sigma)$ will be more briefly denoted $P(m,X)$. The bundle $\hat{\tau}X$ restricts to the tangent bundle along any fibre of $\pi : P(m,X) \to \mathbb{R}P^m$ and so is a subbundle of $\tau P(m,X)$. Clearly $\hat{\tau}X$ is contained in the kernel of $T\pi : TP(m,X) \to T\mathbb{R}P^m$. In fact $\hat{\tau}X = \ker(T\pi)$ since their ranks are equal. Therefore we have a Whitney sum decomposition

$$\tau P(m,X) = \pi^*(\tau\mathbb{R}P^m) \oplus \hat{\tau}X. \tag{3}$$

We assume that Fix(σ) is non-empty and hence a smooth manifold of dimension $d = (1/2)\dim X$. Also we assume that $H^1(X;\mathbb{Z}_2) = 0$. Using the fact that $w(\mathbb{R}P^m) = (1+x)^{m+1}$, and applying Proposition 2.11, we have

Theorem 3.1. *Let X be a connected compact almost complex manifold with complex conjugation σ. Suppose that $Fix(\sigma) \neq \emptyset$ and that $H^1(X;\mathbb{Z}_2) = 0$. Then:*

$$w(P(m,X);t) = (1+xt)^{m+1} \cdot \sum_{0\leq j\leq d} (1+xt)^{d-j}\tilde{c}_j(X)t^{2j}. \tag{4}$$ □

As an application of the above theorem we obtain

Corollary 3.2. (i) *$P(m,X)$ is orientable if and only if $m+d$ is odd.*
(ii) *$P(m,X)$ admits a spin structure if and only if X admits a spin structure and $m+1 \equiv d$* mod 4.

Proof. Since $P(m,X) = (\mathbb{S}^m \times X)/\mathbb{Z}_2$, it is readily seen that $P(m,X)$ is orientable if and only if the antipodal map of $\mathbb{S}^m$ and the conjugation involution σ on X are simultaneously either orientation preserving or orientation reversing. The latter condition is equivalent to $m+1 \equiv d$ mod 2. Alternatively, from Theorem 3.1, we obtain that $w_1(P(m,X)) = (m+1+d)x$, which is zero precisely if $m+d$ is odd.

Using the same formula, we have $w_2(P(m,X)) = \left(\binom{m+1}{2} + \binom{d}{2}\right)x^2 + \tilde{c}_1(X)$. The existence of a spin structure being equivalent to vanishing of the first and the second Stiefel-Whitney classes, we see that $P(m,X)$ admits a spin structure if and only if X admits a spin structure and $\binom{m+1}{2} \equiv \binom{d}{2}$ mod 2 with $m+d$ odd. The latter condition is equivalent to $m+1 \equiv d$ mod 4. □

The notions of stable parallelizability and parallelizability were recalled in the Introduction. Recall from §2.2 the σ-conjugation $\varepsilon_{k,n-k} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$, defined with respect to a set of everywhere linearly independent sections $s_1, \ldots, s_n$.

Theorem 3.3. *Let σ be a conjugation on a connected almost complex manifold X and let $\dim_{\mathbb{R}} X = 2d$. Suppose that $Fix(\sigma) \neq \emptyset$. Then:*
(i) *If $P(m,X)$ is stably parallelizable, then X is stably parallelizable and $2^{\varphi(m)}|(m+1+d)$.*
(ii) *Suppose that $\rho(\tau X) \oplus n\epsilon_{\mathbb{R}} \cong (2d+n)\epsilon_{\mathbb{R}}$ as* real *vector bundle. Suppose that the bundle map $\varepsilon_{d+k,d+n-k}$ of $(2d+n)\epsilon_{\mathbb{R}}$ covering σ restricts to $\hat{\sigma} = T\sigma$ on TX and to $\varepsilon_{k,n-k}$ on $n\epsilon_{\mathbb{R}}$. If $2^{\varphi(m)}|(m+1+d)$, then $P(m,X)$ is stably parallelizable.*
(iii) *Suppose that m is even and that $P(m,X)$ is stably parallelizable. Then $P(m,X)$ is parallelizable if and only if $\chi(X) = 0$.*

Proof. (i) If $E \to B$ is *any* smooth fibre bundle with fibre X, the normal bundle to the fibre inclusion $X \hookrightarrow E$ is trivial. So if E is stably parallelizable, then so is X. It follows that stable parallelizability of $P(m,X)$ implies that of X.

Let $x_0 \in \mathrm{Fix}(\sigma)$ and let $s : \mathbb{R}P^m \to P(m,X)$ be the corresponding cross-section defined as $[v] \mapsto [v, x_0]$. In view of Lemma 2.9 and the bundle isomorphism (3), we see that $s^*(\tau P(m,X)) = s^*(\pi^*\tau\mathbb{R}P^m \oplus \hat{\tau}X) = \tau\mathbb{R}P^m \oplus d\epsilon_{\mathbb{R}} \oplus d\zeta \cong (m+1+d)\zeta \oplus (d-1)\epsilon_{\mathbb{R}}$. Thus the stable parallelizability of $P(m,X)$ implies that $(m+1+d)([\zeta]-1) = 0$ in $KO(\mathbb{R}P^m)$. By the result of Adams [1] (recalled in §1) it follows that $2^{\varphi(m)}|(m+1+d)$.

(ii) Our hypothesis implies, using Lemma 2.3, that $\hat{\tau}X \oplus (k\xi \oplus (n-k)\epsilon_{\mathbb{R}}) \cong (d+n-k)\epsilon_{\mathbb{R}} \oplus (d+k)\xi$. Therefore, using the isomorphism (3), $\tau P(m,X) \oplus k\xi \oplus (n-k+1)\epsilon_{\mathbb{R}} \cong k\xi \oplus (n-k+1)\epsilon_{\mathbb{R}} \oplus \pi^*(\tau\mathbb{R}P^m) \oplus \hat{\tau}X \cong (m+1)\xi \oplus \hat{\tau}X \oplus k\xi \oplus (n-k)\epsilon_{\mathbb{R}} \cong (m+1)\xi \oplus (d+k)\xi \oplus (d+n-k)\epsilon_{\mathbb{R}}$. Since $\dim P(m,X) = 2d+m < 2d+n+1+m$, we may cancel the factor $k\xi \oplus (n-k)\epsilon_{\mathbb{R}}$ on both sides [7, Theorem 1.1, Ch. 9], leading to an isomorphism $\tau P(m,X) \oplus \epsilon_{\mathbb{R}} \cong (d+m+1)\xi \oplus d\epsilon_{\mathbb{R}}$. Since $\xi = \pi^*(\zeta)$, again using Adams' result it follows that $P(m,X)$ is stably parallelizable if $2^{\varphi(m)}$ divides $(m+d+1)$.

(iii) Since m is even, $P(m,X)$ is even dimensional. By Bredon-Kosiński's theorem [3], it follows that $P(m,X)$ is parallelizable if and only if its span is at least 1. By Hopf's theorem, span $P(m,X) \geq 1$ if and only if $\chi(P(m,X))$ vanishes. Since $\chi(P(m,X)) = \chi(\mathbb{R}P^m) \cdot \chi(X) = \chi(X)$ as m is even, the assertion follows. □

The *stable span* of a smooth manifold M is the largest number $s \geq 0$ such that $\tau M \oplus \epsilon_{\mathbb{R}} \cong (s+1)\epsilon_{\mathbb{R}} \oplus \eta$ for some real vector bundle η. We extend the notion of span and stable span to a (real) vector bundle γ over a base space B in an obvious mannner; thus span(α) is the largest number $r \geq 0$ so that $\gamma \cong \alpha \oplus r\epsilon_{\mathbb{R}}$ for some vector bundle α. If rank of γ equals n and if B is a CW complex of dimension $d \leq n$, then span$(\gamma) \geq n-d$. See [7, Theorem 1.1, Ch. 9]. It follows that if $n > d$, then span(γ) = stable span(γ).

Remark 3.4. (i) Suppose that $P(m,X)$ is stably parallelizable. If m is odd, then $\chi(P(m,X)) = 0$ as $\chi(\mathbb{R}P^m) = 0$. Consequently we obtain no information about $\chi(X)$ from the equality $\chi(P(m,X)) = \chi(\mathbb{R}P^m)\chi(X)$. Let us suppose that $\chi(X) \neq 0$. Since span$(\mathbb{R}P^m)$ = span$(\mathbb{S}^m)$, we obtain the lower bound $\mathrm{span}(P(m,X)) \geq \mathrm{span}(\mathbb{S}^m) = \rho(m+1)-1$, where $\rho(m+1)$ is the Hurwitz-Radon function defined as $\rho(2^{4a+b}(2c+1)) = 8a+2^b$, $0 \leq b < 4$, $a, c \geq 0$. From Bredon-Kosiński's theorem [3], we obtain that $P(m,X)$ is parallelizable if $\rho(m+1) > \rho(m+2d+1)$. For example if $m = (2c+1)2^r-1$ and $d = 2^s(2k+1)$ with $s < r-1$ then $m+1+2d = ((2c+1)2^{r-1-s}+2k+1)2^{s+1}$ and so $\rho(m+1) = \rho(2^r) > \rho(2^{s+1}) = \rho(m+2d+1)$; consequently $P(m,X)$ is parallelizable.

(ii) The following bounds for the span and stable span of $P(m,X)$ are easily obtained.

- stable $\mathrm{span}(P(m,X)) \leq \min\{d + \mathrm{span}(m+d+1)\zeta, m + \text{stable span}(X)\}$,
- $\mathrm{span}(P(m,X)) \geq \mathrm{span}(\mathbb{R}P^m)$.

If m is even and $\chi(X) = 0$, then $\chi(P(m,X)) = 0$. Since $\dim P(m,X)$ is even, it follows by [10, Theorem 20.1], that $\mathrm{span}(P(m,X)) = \text{stable span}(P(m,X))$.

We illustrate Theorem 3.3 in the case when X is the complex flag manifold $\mathbb{C}G(n_1,\ldots,n_r)$, where the $n_j \geq 1$ are positive integers and $n = \sum_{1\leq j\leq r} n_j$, with its usual differentiable structure. It admits an $U(n)$-invariant complex structure and the smooth involution $\sigma : X \to X$ defined by the complex conjugation on $\mathbb{C}^n$ is a conjugation, as remarked in Example 2.8(i). We assume, without loss of generality, that $n_1 \geq \cdots \geq n_r$. We denote by $P(m; n_1,\ldots,n_r)$ the space $P(m, \mathbb{C}G(n_1,\ldots,n_r))$. Note that $\mathbb{C}G(1,\ldots,1)$ is the complete flag manifold $\mathrm{Flag}(\mathbb{C}^n)$.

The classical Dold manifold corresponds to $r = 2$ and $n_1 \geq n_2 = 1$. Theorem 1.1 in this special case is due to J. Korbaš [9]. (Cf. [21], [12].)

Proof of Theorem 1.1. When $n_j > 1$ for some j, the flag manifold $X = \mathbb{C}G(n_1,\ldots,n_r)$ is well-known to be *not* stably parallelizable; see, for example, [18]. (Cf. [8].) So, by Theorem 3.3, the non-trivial part of theorem concerns the case when the flag manifold is stably parallelizable, namely, $n_j = 1$ for all j. It remains to determine the values of m for which $P = P(m; 1,\ldots,1)$ is stably parallelizable. This is done in Proposition 3.5 below.

The manifold $X = \mathbb{C}G(1,\ldots,1)$ has non-vanishing Euler characteristic; in fact, $\chi(X) = n!$, the order of the Weyl group of $U(n)$. When m is even, it follows that $\chi(P) = n!$ and so $\mathrm{span}(P) = 0$.

Suppose that $\rho(m+1) > \rho\left(m+1+\binom{n}{2}\right)$. Then $\mathrm{span}(P) \geq \mathrm{span}(\mathbb{R}P^m) \geq \rho(m+1)-1$ whereas the span of the sphere of dimension $\dim P = m+2d = m+n(n-1)$ equals $\rho(m+1+n(n-1))-1$. So, by Bredon-Kosiński theorem [3], P is parallelizable if it is stably parallelizable and $\rho(m+1) > \rho(m+1+n(n-1))$. □

It is known that $\mathrm{Flag}(\mathbb{C}^n)$ is stably parallelizable, but not parallelizable, as a real manifold (Cf. [11, p.313].) (The non-parallelizability of $\mathrm{Flag}(\mathbb{C}^n)$ follows immediately from the fact that $\chi(\mathrm{Flag}(\mathbb{C}^n)) \neq 0$.)

As a preparation for the proof of Proposition 3.5 we recall a certain functor μ^2 introduced by Lam [11, §§4-5]. This allows us to apply Lemma 2.3(iii).

The functor $\mu^2 = \mu^2_{\mathbb{C}}$ associates a real vector bundle to a complex vector bundle.[2] We assume the base space to be paracompact so that every complex vector bundle over it admits a Hermitian metric. If V is any complex vector space $\mu^2(V)$ is defined as $\mu^2(V) = \bar{V}\otimes_{\mathbb{C}} V/\mathrm{Fix}(\theta)$ where $\theta : \bar{V}\otimes V \to \bar{V}\otimes V$ is the conjugate complex linear automorphism defined as $\theta(u\otimes v) = -v\otimes u$. As with any *continuous* functor ([13, §3(f)]), μ^2 is determined by its

[2] Lam defined μ^2 in a more general setting that includes (left) vector bundles over quaternions as well.

restriction to the category of finite dimensional complex vector spaces and their isomorphisms. The functor μ^2 has the following properties where $\omega, \omega_1, \omega_2$ are all complex vector bundles over a base space X. The first three were established by Lam.
(i) $\mathrm{rank}(\mu^2(\omega)) = n^2$ where n is the rank of ω as a complex vector bundle.
(ii) $\mu^2(\omega) \cong \epsilon_{\mathbb{R}}$ if ω is a complex line bundle. Indeed, choosing a positive Hermitian metric on ω, the map $E(\mu^2(\omega)) \ni [u \otimes zu] \mapsto (p_\omega(u), Re(z)\|u\|^2) \in X \times \mathbb{R}, z \in \mathbb{C}$ is a well-defined real vector bundle homomorphism. It is clearly non-zero and since the ranks agree, it is an isomorphism.
(iii) $\mu^2(\omega_1 \oplus \omega_2) = \mu^2(\omega_1) \oplus (\bar{\omega}_1 \otimes_{\mathbb{C}} \omega_2) \oplus \mu^2(\omega_2)$.
(iv) If $\hat{\sigma} : E(\omega) \to E(\omega)$ is a conjugation of ω covering an involution $\sigma : X \to X$, then $\mu^2(\hat{\sigma}) : E(\mu^2(\omega)) \to E(\mu^2(\omega))$ is a bundle map covering σ. In particular $\mu^2(\bar{\omega}) \cong \mu^2(\omega)$.
(v) If $\hat{\sigma}$ is a conjugation of a complex *line* bundle ω with a Hermitian metric $\langle .,. \rangle$ covering an involution σ such that $\langle u, v\rangle_x = \overline{\langle \hat{\sigma}(u), \hat{\sigma}(v)\rangle}_{\sigma(x)}, u, v \in p_\omega^{-1}(x), x \in X$, then $\mu^2(\hat{\sigma}) : \mu^2(\omega) \to \mu^2(\omega)$ is the identity on each fibre under the isomorphism $\mu^2(\omega) \cong \epsilon_{\mathbb{R}}$ of (ii) since $\|\hat{\sigma}(u)\| = \|u\|$.

Proposition 3.5. *The manifold* $P(m; 1, \ldots, 1) = P(m, Flag(\mathbb{C}^n))$ *is stably parallelizable if and only if* $2^{\varphi(m)}$ *divides* $(m + 1 + \binom{n}{2})$.

Proof. Recall ([11, Corollary 1.2]) that $\tau\mathbb{C}G(n_1, \ldots, n_r) \cong \oplus_{1\le i<j\le r}\bar{\gamma}_i \otimes \gamma_j$ where γ_j is the j-th canonical bundle of rank n_j whose fibre over $(L_1, \ldots, L_r) \in \mathbb{C}G(n_1, \ldots, n_r)$ is the complex vector space L_j. We have

$$\gamma_1 \oplus \cdots \oplus \gamma_r \cong n\epsilon_{\mathbb{C}}.$$

Applying μ^2 and using the above description of $\tau\mathbb{C}G(n_1, \ldots, n_r)$ we obtain the following isomorphism of *real* vector bundles by repeated use of property (iii) of μ^2 listed above:

$$\bigoplus \mu^2(\gamma_j) \oplus \tau(\mathbb{C}G(n_1, \ldots, n_r)) \cong n\epsilon_{\mathbb{R}} \oplus (\bigoplus_{1\le i<j\le n} \epsilon_{\mathbb{C}}(\bar{e}_i \otimes e_j)) \cong n^2\epsilon_{\mathbb{R}}. \tag{5}$$

(Cf. [11, Theorem 5.1].) Specialising to the case of $X = \mathrm{Flag}(\mathbb{C}^n)$ we have $\mu^2(\gamma_j) \cong \epsilon_{\mathbb{R}}$. The involution $\sigma : X \to X$ defined as $\mathbf{L} \mapsto \bar{\mathbf{L}}$ induces a complex conjugation of $\hat{\sigma} = T\sigma$ on τX which preserves the summands $\omega_{ij} := \bar{\gamma}_i \otimes \gamma_j, i < j$, yielding a conjugation $\hat{\sigma}_{ij}$ on it. The bundle involution $\varepsilon_{d,d}$ (covering σ) on the summand on the right $\oplus_{1\le i<j\le n}\rho(\epsilon_{\mathbb{C}})$, defined with respect to the basis $\bar{e}_i \otimes e_j, \bar{e}_i \otimes \sqrt{-1}e_j, 1 \le i < j \le n$, and $\varepsilon_{0,n}$ on the summand $\oplus_{1\le i\le n}\epsilon_{\mathbb{R}}(\bar{e}_i \otimes e_i)$ defined with respect to $\bar{e}_i \otimes e_i, 1 \le i \le n$, together define an involution, denoted ε, that covers σ. Under the isomorphism, ε restricts to $T\sigma$ on τX and to $\varepsilon_{0,n}$ on $\oplus_{1\le i\le n}\mu^2(\gamma_i)$ defined with respect to a basis $\bar{u}_i \otimes u_i, 1 \le i \le n$, where $u_i \in L_i$ with $\|u_i\| = 1$. It follows, by using (v) above and Lemma 2.3, that

$$n\epsilon_{\mathbb{R}} \oplus \hat{\tau}\mathrm{Flag}(\mathbb{C}^n) \cong n\epsilon_{\mathbb{R}} \oplus \binom{n}{2}(\epsilon_{\mathbb{R}} \oplus \xi).$$

Therefore $(n + 1)\epsilon_{\mathbb{R}} \oplus \tau P \cong (m + 1)\xi \oplus \hat{\tau}\mathrm{Flag}(\mathbb{C}^n) \oplus n\epsilon_{\mathbb{R}} \cong \left(m + 1 + \binom{n}{2}\right)\xi \oplus \binom{n+1}{2}\epsilon_{\mathbb{R}}$. Hence τP is stably trivial if and only if $\left(m + 1 + \binom{n}{2}\right)\xi$ is stably trivial if and only if $\left(m + 1 + \binom{n}{2}\right)\zeta$ on $\mathbb{R}P^m$ is stably trivial if and only if $2^{\varphi(m)}$ divides $\left(m + 1 + \binom{n}{2}\right)$. This completes the proof. $\square$

REMARK 3.6. It is clear that for a given $n \geq 2$, there are only finitely many values $m \geq 1$ for which $P = P(m, \mathrm{Flag}(\mathbb{C}^n))$ is parallelizable. In fact, since $2^{\varphi(m)} \geq 2m$ for $m \geq 8$, we must have $m \leq \max\left\{8, \binom{n}{2}\right\}$. However the required values of m are highly restricted. For example when $n = 2^s$, $s \geq 4$, P is parallelizable only when $m \in \{1, 3, 7\}$ and when $n = 2^s - 2$, $s \geq 5$, $m \in \{2, 6\}$. When $n = 6$, P is not parallelizable for any m.

3.1. More examples of parallelizable generalized Dold manifolds. We give examples of parallelizable manifolds $P(m, X)$ for some other classes of X. Specifically, we take X to be certain (i) Hopf manifold, (ii) complex torus, and (iii) compact Clifford-Klein form of a (non-compact) complex Lie group. In all these case, it turns out that $\mathrm{Fix}(\sigma) \neq \emptyset$ and $\hat{\tau}X \cong d\xi \oplus d\epsilon_{\mathbb{R}}$. In particular $\mathrm{span}(P(m, X)) \geq d$. If $2^{\varphi(m)}$ divides $(m + 1 + d)$, then $P(m, X)$ is stably parallelizable. Furthermore, if $d > \rho(m + 2d)$, then $P(m, X)$ is parallelizable.

(i) Let $\lambda > 1$. The infinite cyclic subgroup $\langle\lambda\rangle$ of the multiplicative group $\mathbb{R}^{\times}_{>0}$ acts on $\mathbb{C}^d_0 := \mathbb{C}^d \setminus \{0\}$ via scalar multiplication. Consider the Hopf manifold $X = X_\lambda := \mathbb{C}^d_0/\langle\lambda\rangle$. Then $X \cong \mathbb{S}^1 \times \mathbb{S}^{2d-1}$ is parallelizable. Although X_λ is defined for any complex number λ with $|\lambda| \neq 1$, our hypothesis that λ is real implies that complex conjugation on $\mathbb{C}^d$ induces an involution σ on X. Moreover $\mathrm{Fix}(\sigma) = (\mathbb{R}^d \setminus \{0\})/\langle\lambda\rangle$ is non-empty. In fact $\mathrm{Fix}(\sigma) \cong \mathbb{S}^1 \times \mathbb{S}^{d-1}$. We claim that τX is isomorphic to $d\epsilon_{\mathbb{C}}$ as a complex vector bundle. Indeed, scalar multiplication $\lambda : \mathbb{C}^d_0 \to \mathbb{C}^d_0$ induces multiplication by λ on the tangent space $T_z\mathbb{C}^d_0$ for any $z \in \mathbb{C}^d_0$. Therefore $TX = (\mathbb{C}^d_0 \times \mathbb{C}^d)/\langle\lambda\rangle$ where $\langle\lambda\rangle$ acts diagonally. The required isomorphism $\phi : TX \to X \times \mathbb{C}^n$ is then obtained as $[z, v] \mapsto ([z], v/\|z\|)$. We observe that this is well-defined since λ is positive. Moreover, $\phi(T\sigma([z, v])) = \phi([\bar{z}, \bar{v}]) = ([\bar{z}], \bar{v}/\|z\|)$. Thus $T\sigma$ corresponds to complex conjugation on $d\epsilon_{\mathbb{C}}$ and so $\hat{\tau}X \cong d\xi \oplus d\epsilon$ by Theorem 3.3(ii).

(ii) Let $X = X_\Lambda \cong (\mathbb{S}^1)^{2d}$ be the complex torus $\mathbb{C}^d/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2d}$ is stable under conjugation; equivalently $\Lambda = \Lambda_0 + \sqrt{-1}\Lambda_0$ where Λ_0 is a lattice in $\mathbb{R}^d$. Then complex conjugation on $\mathbb{C}^d$ induces a conjugation σ on X. It is readily seen that $\mathrm{Fix}(\sigma) = (\mathbb{R}^d + \frac{\sqrt{-1}}{2}\Lambda_0)/\Lambda_0$. Also $\tau X \cong d\epsilon_{\mathbb{C}}$ as a complex vector bundle. As in (i) above, $\hat{\tau}X \cong d\xi \oplus d\epsilon_{\mathbb{R}}$.

(iii) More generally, suppose that $G \subset GL(N, \mathbb{C})$ is a connected complex linear Lie group such that G is stable by conjugation $A \mapsto \bar{A}$ in $GL(n, \mathbb{C})$. Suppose that Λ a discrete subgroup of G such that $X = G/\Lambda$ is compact; that is, Λ is a uniform lattice in G. Assume that $\bar{\Lambda} = \Lambda$. (For example, G is the group of unipotent upper triangular matrices in $GL(N, \mathbb{C})$ with Γ the subgroup of G consisting matrices with entries in $\mathbb{Z}[\sqrt{-1}]$.) Then $X = G/\Lambda$ is *holomorphically parallelizable,* i.e., τX is trivial as a complex analytic vector bundle. See [2]. In particular, $\tau X \cong d\epsilon_{\mathbb{C}}$. Let $p : G \to X$ be the covering projection. Denoting by $\mathfrak{g}$ the Lie algebra of G, viewed as the space of vector fields on G invariant under right translation, we have a bundle isomorphism $f : X \times \mathfrak{g} \to TX$ defined as $(g\Gamma, V) \mapsto Tp_g(V_g)\ \forall V \in \mathfrak{g}$. This is well-defined since V is invariant under right-translation. Under this isomorphism, $T\sigma$ is the standard σ-conjugation on $d\epsilon_{\mathbb{C}}$. So $\hat{\tau}X \cong d\xi \oplus d\epsilon_{\mathbb{R}}$. As the identity coset is fixed by σ, $\mathrm{Fix}(\sigma) \neq \emptyset$.

3.2. Unoriented cobordism. Recall from the work of Thom and Pontrjagin ([13, Ch. 4]) that the (unoriented) cobordism class of a smooth closed manifold is determined by its Stiefel-Whitney numbers. Let σ be a complex conjugation on a connected almost complex manifold X and let $\dim_{\mathbb{R}} X = 2d$. Assume that $\mathrm{Fix}(\sigma) \neq \emptyset$ and that $H^1(X; \mathbb{Z}_2) = 0$. Proposition 2.11 allows us to compute certain Stiefel-Whitney numbers of $P(m, X)$ in terms of those of X, even without the knowledge of the cohomology algebra $H^*(P(m, X); \mathbb{Z}_2)$. Let $s : \mathbb{R}P^m \to P(m, X)$ be the cross-section corresponding to an $x_0 \in \mathrm{Fix}(\sigma)$. We identify $\mathbb{R}P^m$ with its image under s and X with the fibre over $[e_{m+1}] \in \mathbb{R}P^m$. Then $X \cap \mathbb{R}P^m = \{[e_{m+1}, x_0]\}$ and the intersection is transverse. Denoting the mod 2 Poincaré dual of a submanifold $M \hookrightarrow P(m, X)$ by $[M]$, we have $[\mathbb{R}P^m]\cdot[X] = [\mathbb{R}P^m \cap X] = [\{[e_{m+1}, x_0]\}]$, which is the generator of $H^{m+2d}(P(m, X); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

We claim that the class $[X] \in H^m(P(m, X); \mathbb{Z}_2)$ equals x^m. To see this, let S_j be the sphere $S_j = \{v \in \mathbb{S}^m \mid v \perp e_j\}, 1 \leq j \leq m$. and let X_j be the submanifold $\{[v, x] \mid v \in S_j, x \in X\} \cong P(m-1, X)$. Let $u_0 = (e_1 + \ldots + e_m)/\sqrt{m}$. Then $C := \{[\cos(t)u_0 + \sin(t)e_{m+1}, x_0] \in P(m, X) \mid 0 \leq t \leq \pi\} \cong \mathbb{R}P^1$ meets X_j transversally at $[e_{m+1}, x_0]$. So $[C]\cdot[X_j] \neq 0$. It follows that $[X_j] = x, 1 \leq j \leq m$, since $H^1(P(m, X); \mathbb{Z}_2) = \mathbb{Z}_2 x$. Also (i) $\cap_{1 \leq i < j} X_i$ intersects X_j transversely for any $j \leq m$, and, (ii) $\cap_{1 \leq j \leq m} X_j = X$. It follows that $[X] = [X_1] \cdots [X_m] = x^m$ as claimed.

Denote by $\mu_X, \mu_{P(m,X)}$ the mod 2 fundamental classes of $X, P(m, X)$ respectively. Note that $w_{2j}(P(m, X))$ is of the form $w_{2j}(P(m, X)) = \tilde{c}_j(X) + a_1 x^2 \tilde{c}_{j-1}(X) + \ldots + a_k x^{2k} \tilde{c}_{j-k}(X)$ for suitable $a_i \in \{0, 1\}, 1 \leq i \leq k$, where $k = \min\{\lfloor m/2 \rfloor, j\}$. Similarly $w_{2j+1}(P(m, X)) = b_0 x \tilde{c}_j(X) + b_1 x^3 \tilde{c}_{j-1}(X) + \ldots + b_k x^{2k+1} \tilde{c}_{j-k}$, $b_i \in \{0, 1\}, 0 \leq i \leq k$, with $k = \min\{\lfloor (m-1)/2 \rfloor, j\}$. A straightforward calculation using Theorem 3.1 reveals that $b_0 = m + 1 + d - j$. Let $J = j_1, \ldots, j_r$ be a sequence of positive integers with $|J| := j_1 + \cdots + j_r = m + 2d$. Then $w_J(P(m, X)) := w_{j_1}(P(m, X)) \ldots w_{j_r}(P(m, X))$ is a polynomial in x over the subring $\mathbb{Z}_2[\tilde{c}_1(X), \ldots, \tilde{c}_d(X)] \subset H^*(P(m, X); \mathbb{Z}_2)$. Since $x^{m+1} = 0$, we see that $w_J(P(m, X)) = 0$ if the number of odd numbers among $j_k, 1 \leq k \leq r$, exceeds m.

Suppose that $I = i_1, \ldots, i_k; J = 1^m, 2I = 1^m, 2i_1, \ldots, 2i_k$, (i.e., $j_t = 1, 1 \leq t \leq m$) and $P(m, X)$ is non-orientable, so that $w_1(P(m, X)) = x$, we have $w_J(P(m, X)) = x^m.\tilde{c}_I(X))$. Using $j^*(\tilde{c}_I(X)) = c_I(X) = w_{2I}(X)$, we obtain that $w_J[P(m, X)] := \langle w_J(P(m, X)), \mu_{P(m,X)} \rangle = \langle x^m.w_{2I}(P(m, X)), \mu_{P(m,X)} \rangle = \langle w_{2I}(X), \mu_X \rangle = w_{2I}[X] \in \mathbb{Z}_2$.

Theorem 3.7. *Suppose that $H^1(X; \mathbb{Z}_2) = 0$ and that $Fix(\sigma) \neq \emptyset$.*
(i) *Assume that $m \equiv d \mod 2$. If $[X] \neq 0$ in $\mathfrak{N}$, then $[P(m, X)] \neq 0$.*
(ii) *If $[P(1, X)] \neq 0$, then $[X] \neq 0$.*

Proof. (i) Since $m \equiv d \mod 2$, we have $w_1(P(m, X)) = x$. Since the odd Stiefel-Whitney classes $w_{2i+1}(X)$ vanish (as X is an almost complex manifold), $[X] \neq 0$ implies that we must have that $w_{2I}[X] \neq 0$ for some I with $|I| = d$. Then, by our above discussion $w_J[P(m, X)] \neq 0$ where $J = 1^m, 2I$. This proves the first assertion.

(ii) Let $m = 1$. $\dim P(1, X) = 1 + 2d$ is odd. Using $x^2 = 0$, we have, from the above discussion, that $w_{2j}(P(1, X)) = \tilde{c}_j(X)$ and $w_{2j+1}(P(1, X)) = (d - j)x\tilde{c}_j(X)$. Suppose that $w_J[P(1, X)] \neq 0$. Then we see that exactly one term, say j_k, in J must be odd. Write $j_k = 2s + 1$ where $s \geq 0$. If $d - s$ is even, then $w_J[P(1, X)] = 0$. So $d - s$ is odd and we have $w_J(P(1, X)) = x\tilde{c}_I(X)$ where $2I$ is obtained from J by replacing j_k by $j_k - 1$. Therefore $w_{2I}[X] = w_J[P(1, X)] \neq 0$. This completes the proof. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall use the structure of complex Clifford algebras to obtain an action of $G := \mathbb{Z}_2^r$ on $P(m, X)$ with $X := \mathbb{C}G_{n,k}$ such that $P(m, X)$ has no G-fixed points.

This implies, by [5, Theorem 30.1], that $[P(m, X)] = 0$. The required action of G on $P(m, X)$ arises from such an action on X via a linear representation of G on $\mathbb{C}^n$. In order to ensure the G-action on X leads to an action on $P(m, X)$, we need ensure that the representation of G is real, that is, it arises by extension of scalars from an action on G on $\mathbb{R}^n$.

Let $\nu_2(n) = r$. It is a basic fact that there exist orthogonal transformations $\phi_1, \ldots, \phi_r$ of $\mathbb{R}^{2^r}$ such that $\phi_i^2 = -id$ and $\phi_i \circ \phi_j = -\phi_j \circ \phi_i, 1 \leq i < j \leq r$. The $\mathbb{R}$-subalgebra of $M_{2^r}(\mathbb{R})$ generated by these transformations is the Clifford algebra C_r associated to the quadratic module $(\mathbb{R}^{2^r}, -\|\cdot\|^2)$. See [7, Ch. 12]. We shall denote by C_r^c the complex Clifford algebra $C_r \otimes_{\mathbb{R}} \mathbb{C}$. Evidently $\mathbb{R}^{2^r}$ is a C_r-module and $\mathbb{C}^{2^r}$ is a C_r^c-module. Then $\mathbb{C}^n = (\mathbb{C}^{2^r})^s$ is a C_r^c-module where $s := n/2^r$.

We denote by the same symbol $\phi_j : \mathbb{C}^{2^r} \to \mathbb{C}^{2^r}$ the $\mathbb{C}$-linear extension of ϕ_j. We further abuse notation by using the same symbol to denote the (diagonal) action of ϕ_j on $\mathbb{C}^n$. Since the ϕ_j are complexifications of real linear transformations, we have $\phi_j(\bar{z}) = \overline{\phi_j(z)}$, $\forall z \in \mathbb{C}^n$. Therefore $\phi_j(\bar{L}) = \overline{\phi_j(L)}$ for all complex vector subspaces $L \subset \mathbb{C}^n$. It follows that $[v, L] \mapsto [v, \phi_j(L)]$ is a well-defined smooth self-map $f_j : P(m, X) \to P(m, X)$, where $X := \mathbb{C}G_{n,k}$. We observe that the $f_j, 1 \leq j \leq r$, are pairwise commuting involutions. Therefore we obtain an action of $G = \mathbb{Z}_2^r$ on $P(m, X)$.

We claim that there are no G-fixed points for this action. Indeed $f_j([v, L]) = [v, \phi_j(L)] = [v, L]$ if and only if $L = \phi_j(L)$. So the G-fixed points $[v, L]$ are in bijective correspondence with C_r^c submodules $L \subset \mathbb{C}^n$. But C_r^c is isomorphic to $M_{2^r}(\mathbb{C})$ or to $M_{2^r}(\mathbb{C}) \oplus M_{2^r}(\mathbb{C})$. See [7, §5, Ch. 12]. It follows that any non-zero module over C_r^c has complex dimension divisible by 2^r. Our assumption that $\nu_2(k) < \nu_2(n) = r$ implies that there is no C_r^c-submodule of $\mathbb{C}^n$ having dimension (over $\mathbb{C}$) equal to k. This establishes the claim and the assertion of the lemma follows.

(ii) Suppose that $\nu_2(k) \geq \nu_2(n)$. Then $[\mathbb{C}G_{n,k}] \neq 0$ by the main theorem of [17]. (See also [16].) Note that $\dim_{\mathbb{C}} \mathbb{C}G_{n,k}$ is even in this case. If m is also even, then it follows that $[P(m, \mathbb{C}G_{n,k})] \neq 0$ by Theorem 3.7(i). □

REMARK 3.8. It appears to be unknown precisely which (real or complex) flag manifolds are unoriented boundaries. Let $n_1, \ldots, n_r \geq 1$ be integers and let $n = \sum_{1 \leq j \leq r} n_j$. Proceeding as in the case of the $P(m; n, k)$ it is readily seen that $[\mathbb{C}G(n_1, \ldots, n_r)]$ and $[P(m; n_1, \ldots, n_r)]$ in $\mathfrak{N}$ are zero if $\nu_2(n) > \nu_2(n_j)$ for some j. Also, if $n_i = n_j$ for some $i \neq j$, then $X := \mathbb{C}G(n_1, \ldots, n_r)$ admits a fixed point free involution $t_{i,j}$, which swaps the i-th and the j-component of each flag $\mathbf{L}$ in X. Clearly $t_{i,j}(\bar{\mathbf{L}}) = \overline{t_{i,j}(\mathbf{L})}, \mathbf{L} \in X$, and so we obtain an involution $[v, \mathbf{L}] \mapsto [v, t_{i,j}(\mathbf{L})]$ on $P(m; n_1, \ldots, n_r)$, which is again fixed point free. It follows that $[P(n_1, \ldots, n_r)] = 0$ in this case. If $m \equiv d \mod 2$ where $d = \dim_{\mathbb{C}} X = \sum_{1 \leq i < j \leq r} n_i n_j$ and if $[X] \neq 0$, then $[P(m; n_1, \ldots, n_r)] \neq 0$ by Theorem 3.7. For example, it is known that $\chi(X) = n!/(n_1! \ldots . n_r!)$. So if m and d are even and if $n!/(n_1! \ldots . n_r!)$ is odd, then $\chi(P(m; n_1, \ldots, n_r))$ is also odd and so $[P(m; n_1, \ldots, n_r)] \neq 0$.

ACKNOWLEDGEMENTS. Sankaran thanks Peter Zvengrowski for bringing to his attention the papers of Július Korbaš [9] and Peter Novotný [15].

References

[1] J.F. Adams: *Vector fields on spheres*, Ann. Math. **75** (1962), 603–632.

[2] D.N. Akhiezer: *Homogeneous complex manifolds*; in Several complex variables-IV, Translation edited by S.G. Gindikin and G.M. Khenkin, Encycl. Math. Sci. **10**, Springer, New York, 1990, 195–244.

[3] G.E. Bredon and A. Kosiński: *Vector fields on π-manifolds,* Ann. Math. (2) **84** (1966) 85–90.

[4] P. Chakraborty and A.S. Thakur: *Nonexistence of almost complex structures on the product* $S^{2m} \times M$, Topology Appl. **199** (2016), 102–110.

[5] P.E. Conner and E.E. Floyd: Differentiable periodic maps. Ergebnisse der Mathematik und Ihrer Grenzgebiete. **33**, Springer-Verlag, Berlin, 1963.

[6] A. Dold: *Erzeugende der Thomschen Algebra* $\mathfrak{N}$, Math. Zeit. **65** (1956), 25–35.

[7] D. Husemoller: Fibre bundles, Third Edition, Grad. Texts in Math. **20**, Springer-Verlag, N.Y. 1994.

[8] J. Korbaš: *Vector fields on real flag manifolds*, Ann. Global Anal. Geom. **3** (1985), 173–184.

[9] J. Korbaš: *On the parallelizability and span of Dold manifolds*, Proc. Amer. Math. Soc. **141** (2013), 2933–2939.

[10] U. Koschorke: Vector fields and other vector bundle morphisms–a singularity approach, Lecture Notes in Mathematics **847**, Springer, Berlin, 1981.

[11] K.-Y. Lam: *A formula for the tangent bundle of flag manifolds and related manifolds*, Trans. Amer. Math. Soc. **213** (1975), 305–314.

[12] B.H. Li: *Codimension* 1 *and* 2 *imbeddings of Dold manifolds*, Kexue Tongbao (English Ed.) **33** (1988), 182–185.

[13] J.W. Milnor and J.D. Stasheff: Characteristic classes, Annals of Mathematics Studies **76**, Princeton University Press, Princeton, N.J. 1974.

[14] A.C. Naolekar and A.S. Thakur: *Note on the characteristic rank of vector bundles*, Math. Slovaca **64** (2014), 1525–1540.

[15] P. Novotný: *Span of Dold manifolds*, Bull. Belg. Math. Soc. Simon Stevin, **15** (2008), 687–698.

[16] P. Sankaran: *Which Grassmannians bound?*, Arch. Math. (Basel) **50** (1988), 474–476.

[17] P. Sankaran: *Determination of Grassmann manifolds which are boundaries*, Canad. Math. Bull. **34** (1991), 119–122.

[18] P. Sankaran and P. Zvengrowski: *On stable parallelizability of flag manifolds*, Pacific J. Math. **122** (1986), 455–458.

[19] E.H. Spanier: Algebraic topology, Corrected reprint, Springer-Verlag, New York, 1981.

[20] A.S. Thakur: *On trivialities of Stiefel-Whitney classes of vector bundles over iterated suspensions of Dold manifolds*, Homology Homotopy Appl. **15** (2013), 223–233.

[21] J.J. Ucci: *Immersions and embeddings of Dold manifolds*, Topology **4** (1965) 283–293.

Avijit Nath
Institute of Mathematical Sciences, (HBNI)
CIT Campus, Taramani, Chennai 600113
India
e-mail: avijitnath@imsc.res.in

Parameswaran Sankaran
Institute of Mathematical Sciences, (HBNI)
CIT Campus, Taramani, Chennai 600113
India
e-mail: sankaran@imsc.res.in