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ON THE GEVREY STRONG HYPERBOLICITY

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Abstract

In this paper we are concerned with a homogeneous differential operator p of order m of which characteristic set of order m is assumed to be a smooth manifold. We define the Gevrey strong hyperbolicity index as the largest number s such that the Cauchy problem for $p + Q$ is well-posed in the Gevrey class of order s for any differential operator Q of order less than m. We study the case of the largest index and we discuss in which way the Gevrey strong hyperbolicity index relates with the geometry of bicharacteristics of p near the characteristic manifold.

1. Introduction

Let

$$P = D_0^m + \sum_{|\alpha|\le m,\alpha_0<m} a_\alpha(x)D^\alpha = p(x, D) + P_{m-1}(x, D) + \cdots$$

be a differential operator of order m defined near the origin of $\mathbb{R}^{n+1}$ where $x = (x_0, \dots, x_n) = (x_0, x')$ and

$$D_j = -i\partial/\partial x_j, \quad D = (D_0, D'), \quad D' = (D_1, \dots, D_n).$$

Here $p(x, \xi)$ is the principal symbol of P;

$$p(x, \xi) = \xi_0^m + \sum_{|\alpha|=m,\alpha_0<m} a_\alpha(x)\xi^\alpha.$$

We assume that the coefficients $a_\alpha(x)$ are in the Gevrey class of order $s > 1$, sufficiently close to 1, which are constant outside $|x'| \le R$. We say that $f(x) \in \gamma^{(s)}(\mathbb{R}^{n+1})$, the Gevrey class of order s, if for any compact set $K \subset \mathbb{R}^{n+1}$ there exist $C > 0, A > 0$ such that we have

$$|D^\alpha f(x)| \le CA^{|\alpha|}|\alpha|!^s, \quad x \in K, \quad \forall\alpha \in \mathbb{N}^{n+1}.$$

Definition 1.1. We say that the Cauchy problem for P is $\gamma^{(s)}$ well-posed at the origin if for any $\Phi = (u_0, u_1, \dots, u_{m-1}) \in (\gamma^{(s)}(\mathbb{R}^n))^m$ there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 \quad \text{in} \quad U_\Phi, \\ D_0^j u(0, x') = u_j(x'), \ j = 0, 1, \dots, m-1, \quad x' \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a unique solution $u(x) \in C^\infty(U_\Phi)$.

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It is a fundamental fact that if $p(x,\xi)$ is strictly hyperbolic near the origin, that is $p(x,\xi_0,\xi') = 0$ has m real distinct roots for any x, near the origin and any $\xi' \neq 0$ then the Cauchy problem for $p + Q$ with any differential operator Q of order less than m is C^∞ well-posed near the origin. In particular, $\gamma^{(s)}$ well-posed for any $s > 1$. On the other hand the Lax-Mizohata theorem in the Gevrey classes asserts:

Proposition 1.1 ([16, Theorem 2.2]). *If the Cauchy problem for P is $\gamma^{(s)}$ $(s > 1)$ well-posed at the origin then $p(0,\xi_0,\xi') = 0$ has only real roots ξ_0 for any $\xi' \in \mathbb{R}^n$.*

Taking this result into account we assume, throughout the paper, that $p(x,\xi_0,\xi') = 0$ has only real roots for any x near the origin and any $\xi' \in \mathbb{R}^n$.

DEFINITION 1.2. We define $G(p)$ (the Gevrey strong hyperbolicity index) by

$$G(p) = \sup\left\{1 \le s \;\middle|\; \begin{array}{l}\text{Cauchy problem for } p+Q \text{ is } \gamma^{(s)} \text{ well-posed at the}\\ \text{origin for any differential operator } Q \text{ of order} < m\end{array}\right\}.$$

We first recall a basic result of Bronshtein [4].

Theorem 1.1 ([4, Theorem 1]). *Let p be a homogeneous differential operator of order m with real characteristic roots. Then for any differential operator Q of order less than m, the Cauchy problem for $p + Q$ is $\gamma^{(m/(m-1))}$ well-posed.*

This implies that for differential operators p of order m with real characteristic roots we have

$$G(p) \ge m/(m-1).$$

We also recall a result which bounds $G(p)$ from above. The following result is a special case of Ivrii [10, Theorem 1]. Recall that $(x,\xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ is called a characteristic of order r of p if

$$\partial_x^\alpha \partial_\xi^\beta p(x,\xi) = 0, \quad \forall |\alpha+\beta| < r.$$

Theorem 1.2 ([10, Theorem 1]). *Let p be a homogeneous differential operator of order m with real analytic coefficients and let $(0,\bar\xi)$, $\bar\xi = (0,\dots,0,1) \in \mathbb{R}^{n+1}$ be a characteristic of order m. If the Cauchy problem for $P = p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed at the origin we have*

$$\partial_\xi^\alpha \partial_x^\beta P_{m-1}(0,\bar\xi) = 0$$

for any $|\alpha+\beta| \le m - 2\kappa/(\kappa-1)$.

Assume that p has a characteristic $(0,\bar\xi)$ of order m and that the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed for any P_{m-1}. Then from Theorem 1.2 it follows that $m - 2\kappa/(\kappa-1) < 0$, that is $\kappa < m/(m-2)$ which yields

$$G(p) \le m/(m-2).$$

Let ρ be a characteristic of order m. Then the localization $p_\rho(X)$ of p at ρ is defined by $p(\rho + \mu X) = \mu^m(p_\rho(X) + o(1))$ with $X = (x,\xi)$ as $\mu \to 0$ which is nothing but the first non-vanishing term of the Taylor expansion of p around ρ. Note that p_ρ is a hyperbolic polynomial in X in the direction $(0,\theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ where $\theta = (1,\dots,0) \in \mathbb{R}^{n+1}$ (for example [7, Lemma 8.7.2]). The hyperbolic cone Γ_ρ of p_ρ is the connected component of $(0,\theta)$ in the set (for example [7, Lemma 8.7.3])

$$\Gamma_\rho = \{X \in \mathbb{R}^{2(n+1)} \mid p_\rho(X) \neq 0\}$$

and the propagation cone C_ρ of the localization p_ρ is the dual cone with respect to the symplectic two form $\sigma = d\xi \wedge dx = \sum_{j=0}^{n} d\xi_j \wedge dx_j$;

$$C_\rho = \{X \in \mathbb{R}^{2(n+1)} \mid (d\xi \wedge dx)(X,Y) \leq 0, \forall Y \in \Gamma_\rho\}.$$

Let

$$H_p = \sum_{j=0}^{n} (\partial p/\partial \xi_j)\partial/\partial x_j - (\partial p/\partial x_j)\partial/\partial \xi_j$$

be the Hamilton vector field of p then integral curves of H_p, along which $p = 0$, are called bicharacteristics of p. We note that C_ρ is the *minimal* cone including every bicharacteristic which has ρ as a limit point in the following sense:

Lemma 1.1 ([12, Lemma 1.1.1])**.** *Let $\rho \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ be a multiple characteristic of p. Assume that there are simple characteristics ρ_j and non-zero real numbers γ_j with $\gamma_j p_{\rho_j}(0,\theta) > 0$ such that*

$$\rho_j \to \rho \quad \textit{and} \quad \gamma_j H_p(\rho_j) \to X, \quad j \to \infty.$$

Then $X \in C_\rho$.

We now introduce assumptions of which motivation will be discussed in the next section. Denote by Σ the set of characteristics of order m of $p(x,\xi)$;

$$\Sigma = \{(x,\xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \mid \partial_x^\alpha \partial_\xi^\beta p(x,\xi) = 0, \forall |\alpha + \beta| < m\}$$

which is assumed to be a $\gamma^{(s)}$ manifold. Note that p_ρ is a function on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ because $p_\rho(X+Y) = p_\rho(Y)$ for any $X \in T_\rho\Sigma$ and any $Y \in \mathbb{R}^{2(n+1)}$ where $T_\rho\Sigma$ denotes the tangent space of Σ at $\rho \in \Sigma$. We assume that

$$p_\rho \ \text{ is a strictly hyperbolic polynomial on } \ \mathbb{R}^{2(n+1)}/T_\rho\Sigma, \quad \rho \in \Sigma. \tag{1.1}$$

We also assume that the propagation cone C_ρ is transversal to the characteristic manifold Σ;

$$C_\rho \cap T_\rho\Sigma = \{0\}, \quad \rho \in \Sigma. \tag{1.2}$$

Denoting $(T_\rho\Sigma)^\sigma = \{X \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid (d\xi \wedge dx)(X,Y) = 0, \forall Y \in T_\rho\Sigma\}$ we note that (1.2) is equivalent to $\Gamma_\rho \cap (T_\rho\Sigma)^\sigma \neq \emptyset$.

Our aim in this paper is to prove

Theorem 1.3. *Assume* (1.1) *and* (1.2)*. Then the Cauchy problem for $p + Q$ is $\gamma^{(s)}$ well-posed at the origin for any differential operator Q of order less than m and for any $1 < s < m/(m-2)$. In particular we have $G(p) = m/(m-2)$.*

Example 1.1. Let

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} c_\alpha \zeta^\alpha, \quad \zeta = (\zeta_0, \zeta_1, \dots, \zeta_k)$$

be a strictly hyperbolic polynomial in the direction ζ_0 where $k \le n$. Let $b_j(x,\xi')$, $j = 1,\dots,k$ be smooth functions in a conic neighborhood of $(0,\hat{\xi}')$ which are homogeneous of degree 1 in ξ' with linearly independent differentials at $(0,\hat{\xi}')$. We define

$$p(x,\xi) = q(b(x,\xi)), \quad b = (b_0, b_1, \dots, b_k)$$

where we set $b_0(x,\xi) = \xi_0$ for notational convenience. Then it is easy to see that $p(x,\xi)$ verifies the condition (1.1) near $\rho = (0,0,\hat{\xi}')$ with $\Sigma = \{(x,\xi) \mid b_j(x,\xi) = 0, j = 0,\dots,k\}$ and $p_\rho(x,\xi) = q(db_\rho(x,\xi))$, that is

$$p_\rho(x,\xi) = q(\hat{b}(x,\xi)), \quad \hat{b} = (\hat{b}_0, \hat{b}_1, \dots, \hat{b}_k)$$

where $\hat{b}_j(x,\xi)$ is the linear part of $b_j(x,\xi)$ at ρ. Therefor $\Gamma_\rho = \{X \mid \hat{b}(X) \in \Gamma\}$ where Γ is the hyperbolic cone of q. If

$$(\{b_i, b_j\})_{0\le i,j\le k} \quad \text{is non-singular at } \rho \tag{1.3}$$

then $p(x,\xi)$ verifies the condition (1.2) near ρ where $\{b_i, b_j\}$ denotes the Poisson bracket

$$\sum_{\mu=0}^{n} (\partial b_i/\partial \xi_\mu)(\partial b_j/\partial x_\mu) - (\partial b_i/\partial x_\mu)(\partial b_j/\partial \xi_\mu).$$

Indeed since $(T_\rho\Sigma)^\sigma$ is spanned by $H_{b_0}(\rho), H_{b_1}(\rho), \dots, H_{b_k}(\rho)$ it suffices to show that there are c_j such that $0 \ne X = \sum_{j=0}^{k} c_j H_{b_j}(\rho) \in \Gamma_\rho$. From

$$\hat{b}_j(X) = \sum_{i=0}^{k} c_i\{b_i, b_j\}(\rho), \quad j = 0,\dots,k$$

one can choose c_j so that $\hat{b}(X) = (1,0,\dots,0)$ by assumption (1.3) and hence the result.

EXAMPLE 1.2. Consider

$$q(\zeta_0,\zeta_1,\zeta_2) = \prod_{j=1}^{\ell} (\zeta_0 - c_j(\zeta_1^2 + \zeta_2^2)) \tag{1.4}$$

where c_j are real positive constants different from each other and $2\ell = m$. Take $b_1 = (x_0 - x_1)\xi_n$, $b_2 = \xi_1$ and consider

$$p(x,\xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j((x_0 - x_1)^2\xi_n^2 + \xi_1^2))$$

in a conic neighborhood of $\rho = (0,0,\dots,0,1)$. The 3×3 anti-symmetric matrix $(\{b_i, b_j\})$ is obviously singular. If $\max\{c_j\} = c < 1$ then $C_\rho \cap T_\rho\Sigma = \{0\}$. To see this take any $X = (t,t,x_2,\dots,x_n,0,0,\xi_2,\dots,\xi_n) \in T_\rho\Sigma$. Assume $X \in C_\rho$ so that $(d\xi\wedge dx)(X,Y) \le 0$ for any $Y = (y,\eta) \in \Gamma_\rho$, that is for any $(y,\eta) \in \mathbb{R}^{2(n+1)}$ with $\eta_0^2 > c((y_0 - y_1)^2 + \eta_1^2)$ and $\eta_0 > 0$. This implies that $x_2 = \cdots = x_n = 0$, $\xi_2 = \cdots = \xi_n = 0$ and $-t(\eta_0 + \eta_1) \le 0$ for any $\eta_0 > \sqrt{c}\,|\eta_1|$. Since $c < 1$ this gives $t = 0$ so that $X = 0$.

On the other hand if $\max\{c_j\} = c \ge 1$ then $C_\rho \cap T_\rho\Sigma \ne \{0\}$. Indeed let $X = (1,1,0,\dots,0,0,\dots,0) \in T_\rho\Sigma$. Noting that $\eta_0 > \sqrt{c}\,|\eta_1|$ if $Y = (y,\eta) \in \Gamma_\rho$ we see $(d\xi\wedge dx)(X,Y) = -\eta_0 - \eta_1 < 0$ for any $Y \in \Gamma_\rho$ which proves $X \in C_\rho$.

Example 1.3. Take q in (1.4) and choose $b_1 = x_0\xi_n$, $b_2 = \xi_1$ and consider

$$p(x,\xi) = \prod_{j=1}^{\ell}(\xi_0^2 - c_j(x_0^2\xi_n^2 + \xi_1^2))$$

near $\rho = (0,0,\ldots,0,1)$. As remarked in Example 1.2 the matrix $(\{b_i, b_j\})$ is singular. Suppose $X = (0, x_1, \ldots, x_n, 0, 0, \xi_2, \ldots, \xi_n) \in T_\rho\Sigma \cap C_\rho$. As in Example 1.2 we conclude $x_2 = \cdots = x_n = 0$, $\xi_1 = \cdots = \xi_n = 0$ and $-x_1\eta_1 \le 0$ for any $\eta_0^2 > c(y_0^2 + \eta_1^2)$. This gives $x_1 = 0$ so that $X = 0$. Thus we conclude $C_\rho \cap T_\rho\Sigma = \{0\}$.

Example 1.4. We specialize Example 1.1 with

$$q(\zeta_0,\zeta_1) = \prod_{j=1}^{m}(\zeta_0 - \alpha_j\zeta_1), \quad q(\zeta_0,\zeta_1) = \prod_{j=1}^{\ell}(\zeta_0^2 - c_j\zeta_1^2)$$

where α_j are real constants different from each other such that $\sum_{j=1}^m \alpha_j = 0$ and c_j are positive constant different from each other and $m = 2\ell$. For these q choosing $b_1 = x_0\xi_1$ and $b_1 = x_0|\xi'|$ respectively we get

$$p(x,\xi) = \prod_{j=1}^{m}(\xi_0 - \alpha_j x_0\xi_1), \quad p(x,\xi) = \prod_{j=1}^{\ell}(\xi_0^2 - c_j x_0^2|\xi'|^2).$$

It is clear that $\{b_0, b_1\} = \xi_1 \neq 0$ and $\{b_0, b_1\} = |\xi'| \neq 0$ respectively and hence $C_\rho \cap T_\rho\Sigma = \{0\}$. We find these examples in [5] where they studied Levi type conditions for differential operators of order m with coefficients depending only on the time variable.

2. Motivation, the doubly characteristic case

In this section we provide the motivation to introduce $G(p)$ and assumptions (1.1), (1.2). Let $m = 2$ and we consider differential operators of second order

$$P(x,D) = p(x,D) + P_1(x,D) + P_0(x)$$

of principal symbol $p(x,\xi)$. Let ρ be a double characteristic of p and hence singular (stationary) point of H_p. We linearize the Hamilton equation $\dot{X} = H_p(X)$ at ρ, the linearized equation turns to be $\dot{Y} = F_p(\rho)Y$ where $F_p(\rho)$ is given by

$$F_p(\rho) = \begin{pmatrix} \dfrac{\partial^2 p}{\partial x\partial\xi}(\rho) & \dfrac{\partial^2 p}{\partial\xi\partial\xi}(\rho) \\ -\dfrac{\partial^2 p}{\partial x\partial x}(\rho) & -\dfrac{\partial^2 p}{\partial\xi\partial x}(\rho) \end{pmatrix}$$

and called the Hamilton map (fundamental matrix) of p at ρ.

The following special structure of $F_p(\rho)$ results from the fact that $p(x,\xi_0,\xi') = 0$ has only real roots ξ_0 for any (x,ξ').

Lemma 2.1 ([9, Lemma 9.2, 9.4]). *All eigenvalues of the Hamilton map $F_p(\rho)$ are on the imaginary axis, possibly one exception of a pair of non-zero real eigenvalues.*

We assume that the doubly characteristic set $\Sigma = \{(x,\xi) \mid \partial_\xi^\alpha\partial_x^\beta p(x,\xi) = 0, \forall|\alpha+\beta| < 2\}$

verifies the following conditions:

$$
(2.1)\qquad \begin{cases} \Sigma \text{ is a } \gamma^{(s)} \text{ manifold,} \\ p \text{ vanishes on } \Sigma \text{ of order exactly } 2, \\ \operatorname{rank}(d\xi \wedge dx) = \text{const. on } \Sigma. \end{cases}
$$

Note that $p_\rho(X)$ is *always a strictly hyperbolic polynomial* on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ as far as p vanishes on Σ of order exactly 2. We also assume that the codimension Σ is 3 and no transition of spectral type of F_p occur on Σ, that is we assume

$$
(2.2)\qquad \text{either} \quad \operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \{0\} \ \text{ or } \ \operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \neq \{0\}
$$

throughout Σ. The following table sums up a general picture of the Gevrey strong hyperbolicity for differential operators with double characteristics ([2, 3, 17, 11]) where $W = \operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2$.

Spectrum of F_p	W	Geometry of bicharacteristics near Σ	$G(p)$
Exists non-zero real eigenvalue	$W = \{0\}$	At every point on Σ exactly two bicharacteristics intersect Σ transversally	$G(p) = \infty$
No non-zero real eigenvalue	$W \neq \{0\}$	No bicharacteristic intersects Σ	$G(p) = 4$
		Exists a bicharacteristic tangent to Σ	$G(p) = 3$
	$W = \{0\}$	No bicharacteristic intersects Σ	$G(p) = 2$

This table shows that, assuming (2.1), (2.2) and the codimension Σ is 3, the Gevrey strong hyperbolicity index $G(p)$ takes only the values 2, 3, 4 and ∞ and that these values completely determine the structure of the Hamilton map and the geometry of bicharacteristics near Σ and vice versa.

Lemma 2.2 ([6, Corollary 1.4.7], [12, Lemma 1.1.3]). *Let ρ be a double characteristic. Then the following two conditions are equivalent.*
(i) $F_p(\rho)$ *has non-zero real eigenvalues,*
(ii) $C_\rho \cap T_\rho\Sigma = \{0\}$.

Note that the condition (ii) is well defined for characteristics of *any order* while $F_p(\rho) \equiv 0$ if ρ is a characteristic of order larger than 2.

REMARK 2.1. Based on the table, it is quite natural to ask whether the converse of Theorem 1.3 is true. That is if $G(p) = m/(m-2)$ then (1.1) and (1.2) hold?

REMARK 2.2. Consider the case $C_\rho \subset T_\rho\Sigma$ that would be considered as a opposite case to $C_\rho \cap T_\rho\Sigma = \{0\}$. Here we note

Lemma 2.3 [18, Lemma 2.11] . *We have $C_\rho \subset T_\rho\Sigma$ if and only if $T_\rho\Sigma$ is involutive, that is $(T_\rho\Sigma)^\sigma \subset T_\rho\Sigma$.*

It is also natural to ask whether $G(p) = m/(m-1)$ if $C_\rho \subset T_\rho\Sigma$, $\rho \in \Sigma$. When Σ is involutive one can choose homogeneous symplectic coordinates x, ξ in a conic neighborhood of $\rho \in \Sigma$ such that Σ is defined by ([8, Theorem 21.2.4], for example)

$$\xi_0 = \xi_1 = \cdots = \xi_k = 0.$$

Thus by conjugation of a Fourier integral operator $p(x, \xi)$ can be written

$$p(x, \xi) = \xi_0^m + \sum_{\alpha_0 \le m-2, |\alpha| = m} a_\alpha(x, \xi)\tilde{\xi}^\alpha$$

where $\tilde{\xi} = (\xi_0, \xi_1, \dots, \xi_k)$. Thus $\partial_\xi^\alpha \partial_x^\beta p(0, \bar{\xi}) = 0$ for $|\alpha| < m$ and any β. If the resulting $p(x, D)$ is a *differential operator* so that $a_\alpha(x, \xi) = a_\alpha(x)$ then from [10, Theorem 1] we conclude that if the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed then $\partial_\xi^\alpha \partial_x^\beta P_{m-1}(0, \bar{\xi}) = 0$ for any $|\alpha| \le m - \kappa/(\kappa - 1)$ and any β. This proves $G(p) \le m/(m-1)$ and hence $G(p) = m/(m-1)$.

Example 2.1. When $m \ge 3$ the geometry of p with the limit point ρ becomes to be complicated comparing with the case $m = 2$, even (1.1) and (1.2) are satisfied. We give an example. Let us consider

$$p(x, \xi) = \xi_0^3 - 3a\{(x_0^2 + x_1^2)\xi_n^2 + \xi_1^2\}\xi_0 - 2bx_0x_1\xi_1\xi_n^2$$

near $\rho = (0, \dots, 0, 1)$ which is obtained from Example 1.2 with

$$q(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \zeta_0^3 - 3a(\zeta_1^2 + \zeta_2^2 + \zeta_3^2)\zeta_0 - 2b\zeta_1\zeta_2\zeta_3$$

and $b_1 = x_0\xi_n$, $b_2 = x_1\xi_n$, $b_3 = \xi_1$ where $a > 0$, b are real constants. Choosing $b = \delta a^{3/2}$ with $|\delta| < 1$ and repeating similar arguments as in Example 1.3 it is easily seen that $p(x, \xi)$ satisfies (1.1) and (1.2).

Consider the Hamilton equations

$$\dot{x}_j = \partial p/\partial \xi_j, \quad \dot{\xi}_j = -\partial p/\partial x_j, \quad j = 0, \dots, n. \tag{2.3}$$

Since $\dot{\xi}_n = 0$ we take $\xi_n = 1$ and $x_1 = \xi_0 = 0$ in (2.3) so that the resulting equations reduce to:

$$\dot{x}_0 = -3a(x_0^2 + \xi_1^2), \quad \dot{\xi}_1 = 2bx_0\xi_1. \tag{2.4}$$

We fix $-1 < \delta < 0$ and take $a > 0$ so that $2b/(3a) < -1$. Then any integral curve of (2.4) passing a point in the cone $|\xi_1| < |1 + (2b/3a)|^{1/2}|x_0|$, $x_0 < 0$ arrives at the origin inside the cone (see, for example [20]). In particular there are infinitely many bicharacteristics with the limit point ρ.

3. Preliminaries

Choosing a new system of local coordinates leaving $x_0 =$ const. to be invariant one can assume that

$$p(x, \xi) = \xi_0^m + a_2(x, \xi')\xi_0^{m-2} + \cdots + a_m(x, \xi')$$

and hence $\Sigma \subset \{\xi_0 = 0\}$. Thus near ρ we may assume that Σ is defined by $b_0(x, \xi) = \cdots =$

$b_k(x,\xi) = 0$ where $b_0 = \xi_0$, $b_j = b_j(x,\xi')$, $1 \le j \le k$ and db_j are linearly independent at ρ' where ρ' stands for $(\bar{x},\bar{\xi}')$ when $\rho = (\bar{x},\bar{\xi})$. Recall that the localization $p_\rho(x,\xi)$ is a homogeneous hyperbolic polynomial of degree m in (x,ξ) in the direction $(0,\theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Lemma 3.1 ([12, Lemma 1.1.3]). *The next two conditions are equivalent.*
(i) $C_\rho \cap T_\rho\Sigma = \{0\}$,
(ii) $\Gamma_\rho \cap (T_\rho\Sigma)^\sigma \cap \langle(0,\theta)\rangle^\sigma \neq \emptyset$ *where* $\langle(0,\theta)\rangle = \{t(0,\theta) \mid t \in \mathbb{R}\}$.

Assume $C_\rho \cap T_\rho\Sigma = \{0\}$ then thanks to Lemma 3.1 there exists $0 \neq X \in \Gamma_\rho \cap (T_\rho\Sigma)^\sigma \cap \langle(0,\theta)\rangle^\sigma$. Since $(T_\rho\Sigma)^\sigma$ is spanned by $H_{b_j}(\rho)$, $j = 0,\ldots,k$ one can write

$$X = \sum_{j=0}^{k} \alpha_j H_{b_j}(\rho) \tag{3.1}$$

where $\alpha_0 = 0$ because $X \in \langle(0,\theta)\rangle^\sigma$. This proves $\partial_{x_0} b_j(\rho') \neq 0$ with some $1 \le j \le k$. Indeed if not we would have $X = (x,0,\xi')$ while denoting $p_\rho(x,\xi) = \prod_{j=1}^m(\xi_0 - \Lambda_j(x,\xi'))$ we see $\Gamma_\rho = \{(x,\xi) \mid \xi_0 > \max_j \Lambda_j(x,\xi')\}$ (for example [7, Lemma 8.7.3]) and we would have $\Lambda_j(x,\xi') < 0$ which contradicts $\sum_{j=1}^m \Lambda_j(x,\xi') = 0$. Renumbering, if necessary, one can assume $\partial_{x_0} b_1(\rho') \neq 0$ so that

$$b_1(x,\xi') = (x_0 - f_1(x',\xi'))e_1(x,\xi'), \quad e_1(x,\xi') \neq 0.$$

Writing $b_j(x,\xi') = b_j(f_1(x',\xi'),x_1,\ldots,x_n,\xi') + c_j(x,\xi')b_1(x,\xi')$ we may assume $b_j(x,\xi')$, $2 \le j \le k$ are independent of x_0. Since $p(x,\xi)$ vanishes on Σ of order m one can write with $b = (b_0,b_1,\ldots,b_k) = (b_0,b')$

$$p(x,\xi) = b_0^m + \sum_{|\alpha|=m,\alpha_0\le m-2} \tilde{a}_\alpha(x,\xi')b(x,\xi)^\alpha. \tag{3.2}$$

Let $\hat{b}_j$ be defined by $b_j(\rho+\mu X) = \mu\hat{b}_j(X) + O(\mu^2)$ and with $\hat{b} = (\xi_0,\hat{b}_1,\ldots,\hat{b}_k)$ we have $p_\rho(X) = q(\hat{b}(X))$ where

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m,\alpha_0\le m-2} \tilde{a}_\alpha(\rho')\zeta^\alpha, \quad \zeta = (\zeta_0,\zeta_1,\ldots,\zeta_k) = (\zeta_0,\zeta')$$

is a strictly hyperbolic polynomial in the direction $(1,0,\ldots,0) \in \mathbb{R}^{k+1}$ by (1.1). Denote $\tilde{q}(\zeta;x,\xi') = q(\zeta)+\sum a_\alpha(x,\xi')\zeta^\alpha$ with $a_\alpha(x,\xi') = \tilde{a}_\alpha(x,\xi')-\tilde{a}(\rho')$ and hence we have $p(x,\xi) = \tilde{q}(b(x,\xi);x,\xi')$.

Lemma 3.2. *There are m real valued functions* $\lambda_1(x,\xi') \le \lambda_2(x,\xi') \le \cdots \le \lambda_m(x,\xi')$ *defined in a conic neighborhood of* ρ' *such that*

$$p(x,\xi) = \prod_{j=1}^m(\xi_0 - \lambda_j(x,\xi')), \quad |\lambda_j(x,\xi')| \le C|b'(x,\xi')|,$$
$$|\lambda_i(x,\xi') - \lambda_j(x,\xi')| \ge c\,|b'(x,\xi')|, \quad (i \neq j)$$

with some $c > 0$, $C > 0$.

Proof. The first assertion is clear because $p(x,\xi)$ is a hyperbolic polynomial in the direction ξ_0. Note that $\tilde{q}(\zeta;\rho') = 0$ has m real distinct roots for $\zeta' \neq 0$ then by Rouché's theorem $\tilde{q}(\zeta_0, \zeta'; x, \xi') = 0$ has m real distinct roots $\zeta_0 = \lambda_j(\zeta'; x, \xi')$ if $|\xi' - \rho'|$ is sufficiently small which are of homogeneous of degree 1 in ζ' and 0 in ξ'. It is easy to check that $|\lambda_j(\zeta'; x, \xi')| \le C|\zeta'|$ and $|\lambda_i(\zeta'; x, \xi') - \lambda_j(\zeta'; x, \xi')| \ge c\,|\zeta'|$ $(i \neq j)$ with some $c > 0, C > 0$. Since $\{|\zeta'| = 1\}$ is compact we end the proof. □

4. Basic weights (energy estimates)

We first introduce symbol classes of pseudodifferential operators which will be used in this paper. Denote $\langle\xi\rangle_\gamma^2 = \gamma^2 + |\xi|^2$ where $\gamma \ge 1$ is a positive parameter.

Definition 4.1. Let $W = W(x, \xi; \gamma) > 0$ be a positive function and let $s > 1$, $0 \le \delta \le \rho \le 1$. We define $S^{(s)}_{\rho,\delta}(W)$ to be the set of all $a(x, \xi; \gamma) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that one can find $A, C > 0$ so that

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi; \gamma)| \le CA^{|\alpha+\beta|}|\alpha + \beta|!^s W\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^{n+1} \tag{4.1}$$

holds with some $A, C > 0$ independent of $\gamma \ge 1$ and $S_{\rho,\delta}(W)$ to be the set of all $a(x, \xi; \gamma)$ satisfying (4.1) with $C_{\alpha\beta}$ in place of $CA^{|\alpha+\beta|}|\alpha + \beta|!^s$ which may depend on α, β but not on $\gamma \ge 1$. We denote $S^{(s)}_{1,0}(W)$, $S_{1,0}(W)$ simply by $S^{(s)}(W)$, $S(W)$ respectively. We define $S^{\langle 1,s\rangle}_{\rho,\delta}(W)$ to be the set of all $a(x, \xi; \gamma)$ such that we have

$$|\partial_x^\beta \partial_\xi^\alpha a| \le CA^{|\alpha+\beta|}W(|\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} \tag{4.2}$$

for any $\alpha, \beta \in \mathbb{N}^{n+1}$ with positive constants $C, A > 0$ independent of $\gamma \ge 1$. If $a(x, \xi; \gamma)$ satisfies (4.1) (resp.(4.2)) in a conic open set $U \subset \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ we say $a(x, \xi; \gamma) \in S^{(s)}_{\rho,\delta}(W)$ (resp. $S^{\langle 1,s\rangle}_{\rho,\delta}(W)$) in U. We often write $a(x, \xi)$ for $a(x, \xi; \gamma)$ dropping γ.

It is clear $S^{(s)}(W) \subset S^{\langle 1,s\rangle}_{\rho,\delta}(W)$ if $1 - \rho \ge \delta/2$. It is also clear that one may replace $(|\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ by $|\alpha + \beta|!(1 + |\alpha + \beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ in (4.2), still defining the same symbol class.

Since $p(x, \xi)$ is a polynomial in ξ of degree m it is clear that $p(x, \xi) \in S^{(s)}(\langle\xi\rangle_\gamma^m)$. Since $b_j(x, \xi')$ are defined only in a conic neighborhood of $\rho' = (\bar{x}, \bar{\xi}')$ we extend such symbols to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let $\chi(t) \in \gamma^{(s)}(\mathbb{R})$ be 1 for $|t| < c/2$ and 0 for $|t| > c$ with small $0 < c < 1/2$ and set

$$\begin{cases} y(x) = \chi(|x - \bar{x}|)(x - \bar{x}) + \bar{x}, \\ \eta'(\xi) = \chi(|\xi'\langle\xi\rangle_\gamma^{-1} - \bar{\xi}'|)(\xi' - \langle\xi\rangle_\gamma\bar{\xi}') + \langle\xi\rangle_\gamma\bar{\xi}'. \end{cases}$$

Then it is easy to see η', $b_j(y, \eta') \in S^{(s)}(\langle\xi\rangle_\gamma)$ and $\langle\xi\rangle_\gamma/C \le |\eta'| \le C\langle\xi\rangle_\gamma$ with some $C > 0$. In what follows we denote $b_j(y, \eta')$ by $b_j(x, \xi)$.

We now define $w(x, \xi)$, $\omega(x, \xi)$ by

$$\begin{cases} w(x, \xi) = \left(\sum_{j=1}^k b_j(x, \xi)^2\langle\xi\rangle_\gamma^{-2} + \langle\xi\rangle_\gamma^{-2\delta}\right)^{1/2}, \\ \omega(x, \xi) = (\phi(x, \xi)^2 + \langle\xi\rangle_\gamma^{-2\delta})^{1/2}, \quad \phi(x, \xi) = \sum_{j=1}^k \alpha_j b_j(x, \xi)\langle\xi\rangle_\gamma^{-1}. \end{cases}$$

Here we recall (3.1), that is

$$H_\phi(\rho) \in \Gamma_\rho. \tag{4.3}$$

In what follows we always assume that

$$
\left\{\begin{array}{l} 0<\delta<\rho<1, \quad \rho+\delta=1, \\ 0<s-1<(1-\rho)/2\rho. \end{array}\right. \tag{4.4}
$$

Lemma 4.1. *There exist* $C, A>0$ *such that*

$$
|\partial_x^\beta \partial_\xi^\alpha w| \le CA^{|\alpha+\beta|}(|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} w\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}
$$

that is we have $w\in S^{\langle 1,s\rangle}_{\rho,\delta}(w)$ *and* $\omega^{\pm 1}\in S^{\langle 1,s\rangle}_{\rho,\delta}(\omega^{\pm 1})$.

We first remark an easy lemma.

Lemma 4.2. *Let* $M>0$ *be such that* $2(1+4\sum_{j=0}^\infty (j+1)^{-2})M\le 1/2$ *and* $\Gamma_1(k)=Mk!/k^3$, $k\in\mathbb{N}$ *where* $\Gamma(0)=M$. *Then we have*

$$
\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\Gamma_1(|\alpha'|)\Gamma_1(|\alpha''|)\le \Gamma_1(|\alpha|)/2.
$$

Proof of Lemma 4.1. It suffices to prove the assertion for ϵw with small $\epsilon>0$ so that one can assume $|w|\le 1$. Thus with $w^2=F$ there is $A_1>0$ such that

$$
|\partial_x^\alpha\partial_\xi^\beta F|\le A_1^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)|\alpha+\beta|^{(s-1)|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}
$$

holds for any α,β. Noting $|\partial_x^\alpha\partial_\xi^\beta w|\le C_{\alpha\beta}w^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}$ for any α,β we choose $A\ge 2A_1$ so that $C_{\alpha\beta}\le A^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)$ for $|\alpha+\beta|\le 4$ then we have

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta w| &\le A^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)w\langle\xi\rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}\\
&\quad\times(w^{-1}\langle\xi\rangle_\gamma^{-\delta}+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}.
\end{aligned} \tag{4.5}
$$

Suppose that (4.5) holds for $|\alpha+\beta|\le k$, $4\le k$ and let $|\alpha+\beta|=k+1\ge 4$. Noting

$$
2w\partial_x^\alpha\partial_\xi^\beta w=-\sum_{1\le|\alpha'+\beta'|\le k}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\partial_x^{\alpha'}\partial_\xi^{\beta'}w\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'}w+\partial_x^\alpha\partial_\xi^\beta F
$$

and $w^{-1}\ge 1$, applying Lemma 4.2 we see that $w|\partial_x^\alpha\partial_\xi^\beta w|$ is bounded by

$$
\begin{aligned}
&\frac{1}{2}A^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)w^2\langle\xi\rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(w^{-1}\langle\xi\rangle_\gamma^{-\delta}+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\\
&\quad+A_1^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)w^2|\alpha+\beta|^{(s-1)|\alpha+\beta|}(w^{-2}\langle\xi\rangle_\gamma^{-\delta|\alpha+\beta|})\langle\xi\rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}.
\end{aligned}
$$

Since we have $w^{-2}\langle\xi\rangle_\gamma^{-\delta|\alpha+\beta|}\le\langle\xi\rangle_\gamma^{-\delta(|\alpha+\beta|-2)}\le\langle\xi\rangle_\gamma^{-\delta|\alpha+\beta|/2}$ if $|\alpha+\beta|\ge 4$ then taking $A^{|\alpha+\beta|}/2+A_1^{|\alpha+\beta|}\le A^{|\alpha+\beta|}$ into account we conclude that (4.5) holds for $|\alpha+\beta|=k+1$. Therefore noting $w^{-1}\langle\xi\rangle_\gamma^{-\delta}\le 1$ we get

$$
|\partial_x^\alpha\partial_\xi^\beta w|\le A^{|\alpha+\beta|}\Gamma_1(|\alpha+\beta|)w\langle\xi\rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}.
$$

The assertion for ω is proved similarly. As for ω^{-1}, using

$$
\begin{aligned}
\omega|\partial_x^\alpha\partial_\xi^\beta\omega^{-1}|&\le\sum\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}A^{|\alpha+\beta|}\Gamma_1(|\alpha'+\beta'|)\Gamma_1(|\alpha+\beta|-|\alpha'+\beta'|)\\
&\quad\times\langle\xi\rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(w^{-1}\langle\xi\rangle_\gamma^{-\delta}+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}
\end{aligned}
$$

the proof follows from induction on $|\alpha+\beta|$. □

We now introduce a basic weight symbol which plays a key role in obtaining energy estimates:

$$\psi = \langle\xi\rangle_\gamma^\kappa \log(\phi+\omega), \quad \kappa = \rho - \delta. \tag{4.6}$$

Lemma 4.3. *We have* $(\phi+\omega)^{\pm 1} \in S^{\langle 1,s\rangle}_{\rho,\delta}((\phi+\omega)^{\pm 1})$. *We have also* $\psi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^\kappa \log\langle\xi\rangle_\gamma)$. *Moreover* $\partial_x^\beta\partial_\xi^\alpha\psi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ *for* $|\alpha+\beta| = 1$.

Proof. With $W = \phi+\omega$ we put for $|\alpha+\beta| = 1$

$$\partial_x^\beta\partial_\xi^\alpha W = \frac{\partial_x^\beta\partial_\xi^\alpha\phi}{\omega}W + \frac{\partial_x^\beta\partial_\xi^\alpha\langle\xi\rangle_\gamma^{-2\delta}}{2\omega} = \Phi_\beta^\alpha W + \Psi_\beta^\alpha. \tag{4.7}$$

We examine $\partial_x^\beta\partial_\xi^\alpha\phi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\omega\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|})$ for $|\alpha+\beta| = 1$. Indeed noting $\omega^{-1}\langle\xi\rangle_\gamma^{-\delta} \le 1$ we have

$$\begin{aligned}|\partial_x^{\nu+\beta}\partial_\xi^{\mu+\alpha}\phi| &\le CA^{|\mu+\nu|}\omega|\mu+\nu|^{s|\mu+\nu|}\langle\xi\rangle_\gamma^{-\delta|\mu+\nu|}\langle\xi\rangle_\gamma^{-\rho|\mu+\alpha|+\delta|\nu+\beta|}\\ &\le CA^{|\mu+\nu|}\omega\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}(|\mu+\nu|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}.\end{aligned}$$

Since $\omega^{-1} \in S^{\langle 1,s\rangle}_{\rho,\delta}(\omega^{-1})$ one can find $A_1 > 0$ such that

$$|\partial_x^\nu\partial_\xi^\mu\Phi_\beta^\alpha| \le A_1^{|\mu+\nu|+1}\langle\xi\rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|}|\mu+\nu|!(1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}, \quad \forall\mu,\nu$$

holds for $|\alpha+\beta| = 1$. Since $\langle\xi\rangle_\gamma^{-2\delta} \le W$ similar arguments prove

$$|\partial_x^\nu\partial_\xi^\mu\Psi_\beta^\alpha| \le A_1^{|\mu+\nu|+1}W\langle\xi\rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|}|\mu+\nu|!(1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}, \quad \forall\mu,\nu. \tag{4.8}$$

Now suppose

$$|\partial_x^\beta\partial_\xi^\alpha W| \le CA_2^{|\alpha+\beta|}W\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}|\alpha+\beta|!(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \tag{4.9}$$

holds for $|\alpha+\beta| \le \ell$ and letting $|\alpha+\beta+e_1+e_2| = \ell+1$ we see

$$\begin{aligned}|\partial_x^{\beta+e_2}\partial_\xi^{\alpha+e_1}W| &\le C\sum\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}A_1^{|\alpha-\alpha'+\beta-\beta'|+1}A_2^{|\alpha'+\beta'|}\\ &\qquad\times|\alpha-\alpha'+\beta-\beta'|!|\alpha'+\beta'|!W\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\\ &\qquad+A_1^{|\alpha+\beta|+1}|\alpha+\beta|!W(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}\\ &\le (CA_2^{|\alpha+\beta|+1}A_1(A_2-A_1)^{-1}+A_1^{|\alpha+\beta|+1})\\ &\qquad\times|\alpha+\beta|!W(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}.\end{aligned}$$

Thus it suffices to choose A_2 so that $A_1(A_2-A_1)^{-1}+C^{-1}(A_1A_2^{-1}) \le 1$ to conclude $\phi+\omega \in S^{\langle 1,s\rangle}_{\rho,\delta}(\phi+\omega)$. As for $(\phi+\omega)^{-1}$ it suffices to repeat the proof of Lemma 4.1.

We turn to the next assertion. From $\langle\xi\rangle_\gamma^{-2\delta}/C \le \phi+\omega \le C$ it is clear $|\psi| \le \langle\xi\rangle_\gamma^\kappa\log\langle\xi\rangle_\gamma$. Since $\partial_x^\beta\partial_\xi^\alpha\log(\phi+\omega) = \partial_x^\beta\partial_\xi^\alpha(\phi+\omega)/(\phi+\omega)$ for $|\alpha+\beta| = 1$ and $(\phi+\omega)^{\pm 1} \in S^{\langle 1,s\rangle}_{\rho,\delta}((\phi+\omega)^{\pm 1})$ we see $\psi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^\kappa\log\langle\xi\rangle_\gamma)$. Since $\partial_x^\beta\partial_\xi^\alpha\phi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{-|\alpha|})$ for $|\alpha+\beta| = 1$ and $\omega^{-1} \in S^{\langle 1,s\rangle}_{\rho,\delta}(\omega^{-1})$ it follows from (4.7) and (4.9) that

$$\begin{aligned}|\partial_x^\nu\partial_\xi^\mu(\partial_x^\beta\partial_\xi^\alpha W)| &\le CA^{|\mu+\nu|}\omega^{-1}W\langle\xi\rangle_\gamma^{-|\alpha|}|\mu+\nu|!\\ &\qquad\times(1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{\mu+\nu|}\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}\end{aligned}$$

which proves the second assertion. $\square$

5. Composition formula (energy estimates)

In studying $\mathrm{Op}(e^{\psi})P\mathrm{Op}(e^{-\psi}) = \mathrm{Op}(e^{\psi}\#P\#e^{-\psi})$, if $\psi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{\kappa'})$ with $\kappa' < \rho-\delta$ one can apply the calculus obtained in [19] to get an asymptotic formula of $e^{\psi}\#P\#e^{-\psi}$, where the proof is based on the almost analytic extension of symbols and the Stokes' formula using a space $\rho-\delta-\kappa' > 0$. In the present case $\psi \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{\kappa})$ there is no space between κ and $\rho-\delta$ and then, introducing a small parameter $\epsilon > 0$, we carefully estimate $e^{\epsilon\psi}\#p\#e^{-\epsilon\psi}$ directly to obtain the composition formula in Theorem 5.1 below.

We denote $a(x,\xi;\gamma,\epsilon) \in \epsilon^{\kappa_1}S^{\langle 1,s\rangle}_{\rho,\delta}(W)$ if $\epsilon^{-\kappa_1}a \in S^{\langle 1,s\rangle}_{\rho,\delta}(W)$ uniformly in $0<\epsilon\ll 1$. Our aim in this section is to give a sketch of the proof of

Theorem 5.1. *Let $p(x,\xi) \in S^{(s)}(\langle\xi\rangle_\gamma^m)$. Then there exists $\epsilon_0 > 0$ such that one can find $K = 1+r$, $r \in \sqrt{\epsilon}S_{\rho,\delta}(1)$ and $\gamma_0(\epsilon) > 0$ for $0<\epsilon\le\epsilon_0$ so that we have for $\gamma \ge \gamma_0(\epsilon)$*

$$\begin{aligned} e^{\epsilon\psi}\#p\#e^{-\epsilon\psi}\#K &= \sum_{|\alpha+\beta|\le m}\frac{\epsilon^{|\alpha+\beta|}}{\alpha!\beta!}p^{(\beta)}_{(\alpha)}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta \\ &\quad+\sum_{1\le|\alpha+\beta|\le m}\epsilon^{|\alpha+\beta|+1/2}\,c^\alpha_\beta p^{(\beta)}_{(\alpha)}+R \end{aligned} \tag{5.1}$$

where $p^{(\beta)}_{(\alpha)} = \partial_\xi^\beta\partial_x^\alpha p$ and $c^\alpha_\beta \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$, $R \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$. Moreover we have $c^\alpha_\beta \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta|=1$. In particular $e^{\epsilon\psi}\#p\#e^{-\epsilon\psi} \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^m)$.

Denote

$$p_\psi(x,\xi) = \sum_{|\alpha+\beta|\le m}\frac{\epsilon^{|\alpha+\beta|}}{\alpha!\beta!}p^{(\beta)}_{(\alpha)}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta \tag{5.2}$$

which will be the principal part of $e^{\epsilon\psi}\#p\#e^{-\epsilon\psi}\#K$ and differs from the second term on the right-hand side of (5.1) by multiplicative factor $O(\epsilon^{1/2})$.

5.1. Estimates of symbol $e^{\epsilon\psi}$. Let $H = H(x,\xi;\gamma) > 0$ be a positive function. Assume that f satisfies

$$\begin{aligned}\left|\partial_x^\nu\partial_\xi^\mu f\right| &\le C_0A_0^{|\mu+\nu|}(|\mu+\nu|-1)! \\ &\quad\times(1+(|\mu+\nu|-1)^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|-1}H\langle\xi\rangle_\gamma^{\delta|\nu|-\rho|\mu|}\end{aligned}$$

for $|\mu+\nu|\ge 1$. Set $\Omega^\alpha_\beta = e^{-f}\partial_x^\beta\partial_\xi^\alpha e^f$ then we have

Lemma 5.1. *Notations being as above. There exist $A_i, C > 0$ such that the following estimate holds for $|\alpha+\beta|\ge 1$;*

$$\begin{aligned}\left|\partial_x^\nu\partial_\xi^\mu\Omega^\alpha_\beta\right| &\le CA_1^{|\nu+\mu|}A_2^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta+\nu|-\rho|\alpha+\mu|} \\ &\quad\times\sum_{j=1}^{|\alpha+\beta|}H^{|\alpha+\beta|-j+1}(|\mu+\nu|+j)!(1+(|\mu+\nu|+j)^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|+j}.\end{aligned}$$

Corollary 5.1. *We have with some $A, C > 0$*

$$|\partial_x^\beta\partial_\xi^\alpha e^f| \le CA^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta|-\rho|\alpha|}(H+|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}e^f.$$

Moreover for $|\alpha+\beta|\geq 1$, $|\partial_x^\beta\partial_\xi^\alpha e^f|$ *is bounded by*

$$CHA^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta|-\rho|\alpha|}(H+|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}e^f.$$

Corollary 5.2. *Notations being as above. We have for* $|\alpha+\beta|\geq 1$

$$\Omega_\beta^\alpha\in S_{\rho,\delta}^{\langle 1,s\rangle}(H(H+|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}).$$

Corollary 5.3. *Let* $\omega_\beta^\alpha = e^{-\epsilon\psi}\partial_x^\beta\partial_\xi^\alpha e^{\epsilon\psi}$. *Then there exists* $\gamma_0(\epsilon)>0$ *such that* $\omega_\beta^\alpha\in\epsilon^{|\alpha+\beta|}S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ *for* $\gamma\geq\gamma_0(\epsilon)$.

5.2. Estimates of $(be^{\epsilon\psi})\#e^{-\epsilon\psi}$**.** Let $\chi(r)\in\gamma^{(s)}(\mathbb{R})$ be 1 in $|r|\leq 1/4$ and 0 outside $|r|\leq 1/2$. Let $b\in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^t\langle\xi\rangle_\gamma^m)$ and consider

$$\begin{aligned}(be^{\epsilon\psi})\#e^{-\epsilon\psi} &= \int e^{-2i(\eta z-y\zeta)}b(X+Y)e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}dYdZ\\ &= b(X)+\int e^{-2i(\eta z-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)dYdZ\end{aligned}$$

where $Y=(y,\eta)$, $Z=(z,\zeta)$. Denoting $\hat{\chi}=\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$, $\tilde{\chi}=\chi(|y|/4)\chi(|z|/4)$ we write

$$\begin{aligned}&\int e^{-2i(\eta z-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)\{\tilde{\chi}\hat{\chi}+(1-\tilde{\chi})\hat{\chi}\}dYdZ\\ &\quad+\int e^{-2i(\eta z-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)(1-\hat{\chi})dYdZ.\end{aligned}$$

After the change of variables $Z\to Z+Y$ the first integral turns to

$$\int e^{-2i(\eta z-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Y+Z)}-1)\hat{\chi}_0dYdZ \tag{5.3}$$

where we have set $\hat{\chi}_0=\tilde{\chi}(y,z)\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\eta+\zeta\rangle\langle\xi\rangle_\gamma^{-1})$.

Lemma 5.2. *Let* $\Psi(X,Y,Z)=\psi(X+Y)-\psi(X+Y+Z)$ *then on the support of* $\hat{\chi}_0$ *one has*

$$|\Psi(X,Y,Z)|\leq C\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)$$

where $g_{(x,\xi)}(y,\eta)=\langle\xi\rangle_\gamma^{2\delta}|y|^2+\langle\xi\rangle_\gamma^{-2\rho}|\eta|^2$ *and for* $|\alpha+\beta|\geq 1$

$$\begin{aligned}|\partial_{(x,y)}^\beta\partial_{(\xi,\eta)}^\alpha e^{\epsilon\Psi}| &\leq \epsilon CA^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)\\ &\quad\times(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)+|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}e^{\epsilon\Psi}.\end{aligned}$$

Proof. The assertions follow from Lemma 4.3 and Corollary 5.2. □

Introducing the following differential operators and symbols

$$\begin{cases} L=1+4^{-1}\langle\xi\rangle_\gamma^{2\rho}|D_\eta|^2+4^{-1}\langle\xi\rangle_\gamma^{-2\delta}|D_y|^2,\\ M=1+4^{-1}\langle\xi\rangle_\gamma^{2\delta}|D_\zeta|^2+4^{-1}\langle\xi\rangle_\gamma^{-2\rho}|D_z|^2,\\ \Phi=1+\langle\xi\rangle_\gamma^{2\rho}|z|^2+\langle\xi\rangle_\gamma^{-2\delta}|\zeta|^2=1+\langle\xi\rangle_\gamma^{2\kappa}g_X(Z),\\ \Theta=1+\langle\xi\rangle_\gamma^{2\delta}|y|^2+\langle\xi\rangle_\gamma^{-2\rho}|\eta|^2=1+g_X(Y)\end{cases}$$

so that $\Phi^{-N}L^Ne^{-2i(\eta z-y\zeta)}=e^{-2i(\eta z-y\zeta)}$, $\Theta^{-\ell}M^\ell e^{-2i(\eta z-y\zeta)}=e^{-2i(\eta z-y\zeta)}$ we make integration by parts in (5.3). Let $F = b(X+Y)(e^{\epsilon\Psi}-1)$, $\chi^* = \chi(\epsilon\Phi)$, $\chi_* = 1-\chi^*$ and note $|(\langle\xi\rangle_\gamma^{-\rho}\partial_z)^\alpha(\langle\xi\rangle_\gamma^{\delta}\partial_\zeta)^\beta\chi^*| \le C_{\alpha\beta}$ with $C_{\alpha\beta}$ independent of ϵ. Consider

$$
\begin{aligned}
&\int e^{-2i(\eta z - y\zeta)}\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ \\
&= \int e^{-2i(\eta z - y\zeta)}\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)dYdZ.
\end{aligned}
\tag{5.4}
$$

Applying Corollary 5.1 we can estimate the integrand of the right-hand side of (5.4);

$$
\begin{aligned}
&|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)| \le C_\ell A^{2N+|\alpha+\beta|+\ell}\Phi^{-N}\Theta^{-\ell}\Big\{\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) \\
&\times(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + |\alpha+\beta| + (2N+|\alpha+\beta|)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|-1}e^{\epsilon\Psi} \\
&+ |e^{\epsilon\Psi}-1|(2N+|\alpha+\beta| + (2N+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|}\Big\}\omega^t(X+Y)\langle\xi\rangle_\gamma^m.
\end{aligned}
\tag{5.5}
$$

Here we remark the following easy lemma.

Lemma 5.3. *Let $A \ge 0$, $B \ge 0$. Then there exists $C > 0$ independent of $n, m \in \mathbb{N}$, A, B such that*

$$(A+n+m+(n+m)^sB)^{n+m} \le C^{n+m}(A+n+n^sB)^n(A+m+m^sB)^m.$$

Since $|e^{\epsilon\Psi}-1| \le C|\epsilon\Psi| \le C\epsilon\Phi^{1/2} \le C\sqrt{\epsilon}$ on the support of χ^*, the right-hand side of (5.5) can be estimated by

$$
\begin{aligned}
&C_\ell A_1^{2N+|\alpha+\beta|}\Phi^{-N}\Theta^{-\ell}\Big\{\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N-1 \\
&+(2N-1)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha+\beta| \\
&+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} + \sqrt{\epsilon}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N} \\
&\times(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\Big\}\omega^t(X+Y)\langle\xi\rangle_\gamma^m
\end{aligned}
$$

where we remark $\omega^{\pm1}(X+Y) \le C\omega^{\pm1}(X)(1+g_X^{1/2}(Y))$ on the support of $\hat{\chi}_0$ and hence we have $\omega^t(X+Y) \le C\omega^t(X)\Theta^{t'}$ with some t'. Noting

$$
\begin{aligned}
&A_1^{2N}\Phi^{-N}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N} \\
&= \Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N}
\end{aligned}
$$

and

$$
\begin{aligned}
&A_1^{2N}\Phi^{-N}\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N-1 + (2N-1)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1} \\
&\le \epsilon A_1\Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N-1}
\end{aligned}
$$

we choose $N = N(z,\zeta,\xi)$ so that $2A_1N = \bar{c}\Phi^{1/2}$ with a small $\bar{c} > 0$. Then noting that $\Phi \le 1 + 2\langle\xi\rangle_\gamma^{2\rho}|z|^2 + 2\langle\xi\rangle_\gamma^{-2\delta}|\zeta|^2 \le C\langle\xi\rangle_\gamma^{2\rho}$ on the support of $\hat{\chi}_0$ and therefore $\Phi^{(s-1)/2}\langle\xi\rangle_\gamma^{-\delta/2} \le C\langle\xi\rangle_\gamma^{-\epsilon_1} \le C\gamma^{-\epsilon_1}$ with some $\epsilon_1 > 0$ we have

$$
\begin{aligned}
&\Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N-1} \\
&\le (A_1\epsilon + \bar{c} + \bar{c}^s\Phi^{(s-1)/2}\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1} \le Ce^{-c_1\Phi^{1/2}}
\end{aligned}
$$

choosing $\bar{c}$ small and $\gamma \geq \gamma_0(\epsilon)$ large. On the other hand since $\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) \leq \Phi^{1/2}$ it is clear

$$
\begin{aligned}
&(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}e^{-c\,\Phi^{1/2}} \\
&\qquad \leq CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}e^{-c'\Phi^{1/2}}.
\end{aligned}
$$

Set $\ell' = \ell - t'$. Then noting $e^{-c'\Phi^{1/2}} \leq C\Phi^{-\ell'}$ we have

$$
\begin{aligned}
&|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)| \\
&\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}\Theta^{-\ell'}\Phi^{-\ell'}.
\end{aligned}
\tag{5.6}
$$

Finally choosing $\ell > t' + (n+1)/2$ and recalling $\int \Theta^{-\ell'}\Phi^{-\ell'}dYdZ = C$ we conclude

$$
\begin{aligned}
&\Big|\int e^{-2i(\eta z - y\zeta)}\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ\Big| \\
&\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.
\end{aligned}
$$

We next consider

$$
\begin{aligned}
&\int e^{-2i(\eta z - y\zeta)}\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ \\
&= \int e^{-2i(\eta z - y\zeta)}L^N\Phi^{-N}M^\ell\Theta^{-\ell}\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ.
\end{aligned}
$$

Similar arguments obtaining (5.6) show that

$$
\begin{aligned}
|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)| &\leq CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\
&\quad \times\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}\Theta^{-\ell'}\Phi^{-\ell'}e^{-c\,\Phi^{1/2}}.
\end{aligned}
$$

Since $\Phi^{1/2} \geq \epsilon^{-1/2}$ on the support of χ_* we see $e^{-c\,\Phi^{1/2}} \leq e^{-c\,\epsilon^{-1/2}} \leq C\sqrt{\epsilon}$ and this proves

$$
\begin{aligned}
&\Big|\partial_x^\beta\partial_\xi^\alpha\int e^{-2i(\eta z - y\zeta)}F\hat{\chi}_0 dYdZ\Big| \\
&\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.
\end{aligned}
$$

We then consider

$$
\int e^{-2i(\eta z - y\zeta)}(|y|^2 + |z|^2)^{-N}(|D_\zeta|^2 + |D_\eta|^2)^N F\hat{\chi}_1 dYdZ
$$

where $F = b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)$ and $\hat{\chi}_1 = (1-\tilde{\chi})\hat{\chi}$. Let $\kappa < \kappa_1 < \rho$ then since $|\psi(X+Y)| + |\psi(X+Z)|$ is bounded by $C\langle\xi\rangle_\gamma^{\kappa_1}$ and $C^{-1} \leq \langle\xi+\eta\rangle_\gamma/\langle\xi\rangle_\gamma, \langle\xi+\zeta\rangle_\gamma/\langle\xi\rangle_\gamma \leq C$ with some $C > 0$ on the support of $\hat{\chi}_1$ thanks to Corollary 5.1 it is not difficult to show

$$
\begin{aligned}
\Big|(|D_\zeta|^2 + |D_\eta|^2)^N\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_1\Big| &\leq CA^{2N+|\alpha+\beta|}\omega^t(X+Y)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|} \\
&\quad\times(\langle\xi\rangle_\gamma^{\kappa_1} + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\
&\quad\times\langle\xi\rangle_\gamma^{-2\rho N}(\langle\xi\rangle_\gamma^{\kappa_1} + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}e^{c\langle\xi\rangle_\gamma^{\kappa_1}}.
\end{aligned}
$$

Choose $N = c_1\langle\xi\rangle_\gamma^\rho$ with small $c_1 > 0$ so that

$$
A^{2N}\langle\xi\rangle_\gamma^{-2\rho N}(\langle\xi\rangle_\gamma^{\kappa_1} + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}
$$

is bounded by $Ce^{-c\langle\xi\rangle_\gamma^\rho}$ and $\langle\xi\rangle_\gamma^{\kappa_1|\alpha+\beta|}e^{-c\langle\xi\rangle_\gamma^\rho}$ is bounded by $CA^{|\alpha+\beta|}e^{-c'\langle\xi\rangle_\gamma^\rho}$. Then noting $\omega^t(X+$

$Y) \le C\omega^t(X)\langle\xi\rangle_\gamma^{t'}$ and $e^{-c'\langle\xi\rangle_\gamma^\rho} \le \sqrt{\epsilon}\, C\langle\xi\rangle_\gamma^{-2(n+1)-t'}$ for $\gamma \ge \gamma_0(\epsilon)$ and that $\langle\xi\rangle_\gamma^{-2(n+1)}\int(|y|^2 + |z|^2)^{-N}\hat{\chi}_1 dYdZ \le C$ we conclude

Lemma 5.4. *Let $\hat{\chi} = \chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$. Then we have for $\gamma \ge \gamma_0(\epsilon)$*

$$\Big|\partial_x^\beta\partial_\xi^\alpha \int e^{-2i(z\eta-y\zeta)} b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)\hat{\chi}dYdZ\Big|$$
$$\le \sqrt{\epsilon}\, CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.$$

Let us write

$$1-\hat{\chi} = (1-\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}))(1-\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})) + (1-\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}))\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$$
$$+(1-\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1}))\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}) = \hat{\chi}_2 + \hat{\chi}_3 + \hat{\chi}_4.$$

Denoting $F = b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)$ again we consider

$$\int e^{-2i(\eta z-y\zeta)}\langle\eta\rangle^{-2N_2}\langle\zeta\rangle^{-2N_1}\langle D_z\rangle^{2N_2}\langle D_y\rangle^{2N_1}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_2\chi^* dYdZ$$

where χ^* is either $\chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ or $1-\chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$. If $\chi^* = \chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ we choose $N_1 = \ell$, $N_2 = N$ and noting $\omega^t(X+Y) \le C\langle\eta\rangle^{t'}$ with some $t' \ge 0$ and $|\psi(X+Y)|+|\psi(X+Z)| \le C\langle\eta\rangle^{\kappa_1}$ with $\kappa < \kappa_1 < \rho$ on the support of $\hat{\chi}_2\chi^*$ it is not difficult to see that

$$|\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle D_z\rangle^{2N}\langle D_y\rangle^{2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_2\chi^*|$$

is bounded by

$$\begin{aligned}&C_\ell A^{2N+|\alpha+\beta|}\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle\eta\rangle^{m+t'+2\delta\ell}\langle\eta\rangle^{6\ell\rho}\\ &\times(\langle\eta\rangle^{\kappa_1} + 2N\langle\eta\rangle^\delta + N^s\langle\eta\rangle^{\delta/2})^{2N}\\ &\times(\langle\eta\rangle^{\kappa_1} + \langle\eta\rangle^\delta|\alpha+\beta| + \langle\eta\rangle^{\delta/2}|\alpha+\beta|^s)^{|\alpha+\beta|}e^{C\langle\eta\rangle^{\kappa_1}}.\end{aligned} \tag{5.7}$$

Here writing

$$A^{2N}\langle\eta\rangle^{-2N}(\langle\eta\rangle^{\kappa_1} + 2N\langle\eta\rangle^\delta + N^s\langle\eta\rangle^{\delta/2})^{2N}$$
$$= \Big(\frac{A\langle\eta\rangle^{\kappa_1}}{\langle\eta\rangle} + \frac{2AN}{\langle\eta\rangle^\rho} + \frac{AN^s\langle\eta\rangle^{\delta/2}}{\langle\eta\rangle}\Big)^{2N}$$

we take $2N = c_1\langle\eta\rangle^\rho$ with small $c_1 > 0$ so that the right-hand side is bounded by $Ce^{-c\langle\eta\rangle^\rho}$. Noting $\langle\eta\rangle^{\delta|\alpha+\beta|}e^{-c\langle\eta\rangle^\rho} \le CA_1^{|\alpha+\beta|}|\alpha+\beta|^{\delta|\alpha+\beta|/\rho}e^{-c_1\langle\eta\rangle^\rho}$ and $\langle\eta\rangle^{\kappa_1|\alpha+\beta|}e^{-c\langle\eta\rangle^\rho} \le CA_1^{|\alpha+\beta|}|\alpha+\beta|^{|\alpha+\beta|}e^{-c_1\langle\eta\rangle^\rho}$ one sees that (5.7) is bounded by

$$C_\ell A_1^{|\alpha+\beta|}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}(|\alpha+\beta|^{1+\delta/\rho} + |\alpha+\beta|^{s+\delta/2\rho})^{|\alpha+\beta|}e^{-c_1\langle\eta\rangle^\rho}.$$

Similarly if $\chi^* = 1-\chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ choosing $N_1 = N$, $N_2 = \ell$ it is proved that (5.7) is estimated by

$$C_\ell A_1^{|\alpha+\beta|}\langle\eta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}(|\alpha+\beta|^{1+\delta/\rho} + |\alpha+\beta|^{s+\delta/2\rho})^{|\alpha+\beta|}e^{-c_1\langle\zeta\rangle^\rho}.$$

Thus taking $1+\delta/\rho = 1/\rho$ and $s+\delta/2\rho \le 1/\rho$ into account and recalling that $\langle\xi\rangle_\gamma \le \langle\eta\rangle$, $\langle\xi\rangle_\gamma \le \langle\zeta\rangle$ on the support of $\hat{\chi}_2$ we get

Lemma 5.5. *We have*

$$\Big|\partial_x^\beta\partial_\xi^\alpha\int e^{-2i(z\eta-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)\hat{\chi}_2 dYdZ\Big|$$
$$\le CA^{|\alpha+\beta|}|\alpha+\beta|^{|\alpha+\beta|/\rho}e^{-c_1\langle\xi\rangle_\gamma^\rho}.$$

Repeating similar arguments we can prove

$$\Big|\partial_x^\beta\partial_\xi^\alpha\int e^{-2i(\eta z-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)\hat{\chi}_i dYdZ\Big|$$
$$\le CA^{|\alpha+\beta|}|\alpha+\beta|^{|\alpha+\beta|/\rho}e^{-c_1\langle\xi\rangle_\gamma^\rho}$$

for $i=3,4$. We summarize what we have proved in

Proposition 5.1. *Let $b\in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^t\langle\xi\rangle_\gamma^m)$ then we have*

$$(be^{\epsilon\psi})\#e^{-\epsilon\psi}=b+\omega^t\,\hat{b}+R$$

where $\hat{b}\in\sqrt{\epsilon}\,S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^m)$ and $R\in S_{0,0}^{(1/\rho)}(e^{-c_1\langle\xi\rangle_\gamma^\rho})$, that is

$$|\partial_x^\beta\partial_\xi^\alpha R|\le CA^{|\alpha+\beta|}|\alpha+\beta|^{|\alpha+\beta|/\rho}e^{-c_1\langle\xi\rangle_\gamma^\rho}.$$

5.3. Proo of Theorem 5.1. We start with the next lemma which is proved repeating similar arguments in the preceding subsection.

Lemma 5.6. *Let $a\in S^{(s)}(\langle\xi\rangle_\gamma^d)$ and $b\in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^h)$. Then we have*

$$(be^{\epsilon\psi})\#a=\sum_{|\alpha+\beta|<N}\frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}a_{(\alpha)}^{(\beta)}(be^{\epsilon\psi})_{(\beta)}^{(\alpha)}+b_Ne^{\epsilon\psi}+R$$

where $b_N\in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{d+h-\delta N})$, $R\in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. For $a\#(be^{\epsilon\psi})$ similar assertion holds, where $(-1)^{|\beta|}$ is replaced by $(-1)^{|\alpha|}$.

We can also prove

Lemma 5.7. *Let $R\in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. Then we have*

$$R\#e^{\pm\epsilon\psi},\quad e^{\pm\epsilon\psi}\#R\in S_{0,0}^{(1/\rho)}(e^{-c'\langle\xi\rangle_\gamma^\rho}).$$

Corollary 5.4. *Let $R\in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. Then for any $t\in\mathbb{R}$ we have*

$$e^{\pm\epsilon\psi}\#R\#e^{\mp\epsilon\psi}\in S(\langle\xi\rangle_\gamma^t).$$

Lemma 5.8. *Let $p\in S^{(s)}(\langle\xi\rangle_\gamma^d)$. Then one can write*

$$(e^{\epsilon\psi})\#p=\sum_{|\alpha+\beta|<N}\frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!}p_{(\alpha)}^{(\beta)}\#(\omega_\beta^\alpha e^{\epsilon\psi})+r_Ne^{\epsilon\psi}+R$$

where $r_N\in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{d-\delta N})$, $R\in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$ and $\omega_\beta^\alpha=e^{-\epsilon\psi}\partial_x^\beta\partial_\xi^\alpha e^{\epsilon\psi}$.

Proof. We first examine

$$p_{(\alpha)}^{(\beta)}\omega_\beta^\alpha e^{\epsilon\psi}-\sum_{|\gamma+\delta|<N}\frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|}\gamma!\delta!}p_{(\alpha+\delta)}^{(\beta+\gamma)}\#(\omega_{\beta+\gamma}^{\alpha+\delta}e^{\epsilon\psi})=r_{N,|\alpha+\beta|}e^{\epsilon\psi}+R \tag{5.8}$$

with $r_{N,|\alpha+\beta|} \in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{d-\delta N})$. Indeed since $\partial_x^\nu\partial_\xi^\mu(\omega^\alpha_\beta e^{\epsilon\psi}) = \omega^{\alpha+\mu}_{\beta+\nu}e^{\epsilon\psi}$ thanks to Lemma 5.6 one can write

$$\sum_{|\gamma+\delta|<N}\frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|}\gamma!\delta!}p^{(\beta+\gamma)}_{(\alpha+\delta)}\#(\omega^{\alpha+\delta}_{\beta+\gamma}e^{\epsilon\psi})$$
$$=\sum_{|\gamma'+\delta'|<2N}\frac{(-1)^{\gamma'|}}{(2i)^{|\gamma'+\delta'|}\gamma'!\delta'!}\left(\sum\binom{\gamma'}{\mu}\binom{\delta'}{\nu}(-1)^{|\mu+\nu|}\right)p^{(\beta+\gamma')}_{(\alpha+\delta')}\omega^{\alpha+\delta'}_{\beta+\gamma'}e^{\epsilon\psi}$$
$$+\,r_{N,|\alpha+\beta|}e^{\epsilon\psi}+R$$

where $\sum\binom{\gamma'}{\mu}\binom{\delta'}{\nu}(-1)^{|\mu+\nu|}=0$ if $|\gamma'+\delta'|>0$ so that the right-hand side is

$$p^{(\beta)}_{(\alpha)}\omega^\alpha_\beta e^{\epsilon\psi}+r_{N,|\alpha+\beta|}e^{\epsilon\psi}+R,\quad r_{N,|\alpha+\beta|}\in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{d-\delta N})$$

which proves (5.8). Now insert the expression of $p^{(\beta)}_{(\alpha)}\omega^\alpha_\beta e^{\epsilon\psi}$ in (5.8) into

$$(e^{\epsilon\psi})\#p=\sum_{|\alpha+\beta|<N}\frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}p^{(\beta)}_{(\alpha)}\omega^\alpha_\beta e^{\epsilon\psi}+r_Ne^{\epsilon\psi}+R$$

which follows from Lemma 5.6 to get

$$\sum_{|\alpha'+\beta'|<2N}\frac{(-1)^{|\beta'|}}{(2i)^{|\alpha'+\beta'|}\alpha'!\beta'!}\left(\sum\binom{\alpha'}{\delta}\binom{\beta'}{\gamma}\right)p^{(\beta')}_{(\alpha')}\#(\omega^{\alpha'}_{\beta'}e^{\epsilon\psi})+\tilde r_Ne^{\epsilon\psi}+R$$

where $\tilde r_N\in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{d-\delta N})$. Here we note $\sum\binom{\alpha'}{\delta}\binom{\beta'}{\gamma}=2^{|\alpha'+\beta'|}$. It is clear that $p^{(\beta')}_{(\alpha')}\#(\omega^{\alpha'}_{\beta'}e^{\epsilon\psi})=r'e^{\epsilon\psi}+R$ with $r'\in S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{d-\delta N})$ for $|\alpha'+\beta'|\geq N$ and hence we get the assertion. □

Proof of Theorem 5.1. From Lemma 5.8 we see

$$(e^{\epsilon\psi})\#p\#e^{-\epsilon\psi}=\sum_{|\alpha+\beta|\leq m}\frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!}p^{(\beta)}_{(\alpha)}\#((\omega^\alpha_\beta e^{\epsilon\psi})\#e^{-\epsilon\psi})+(r_me^{\epsilon\psi}+R)\#e^{-\epsilon\psi}$$

where $(r_me^{\epsilon\psi}+R)\#e^{-\epsilon\psi}\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$ which follows from Propositions 5.1 and Lemma 5.7. Therefore Proposition 5.1 together with Corollary 5.3 gives

$$(e^{\epsilon\psi})\#p\#e^{-\epsilon\psi}=\sum_{|\alpha+\beta|\leq m}\frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!}p^{(\beta)}_{(\alpha)}\#(\omega^\alpha_\beta+\bar\omega^\alpha_\beta)+R$$

where $\bar\omega^\alpha_\beta\in\epsilon^{|\alpha+\beta|+1/2}S^{\langle 1,s\rangle}_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$, $R\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$ and

$$\bar\omega^\alpha_\beta\in\epsilon^{3/2}S^{\langle 1,s\rangle}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|}),\quad |\alpha+\beta|=1.$$

Since $e^{\epsilon\psi}\#e^{-\epsilon\psi}=1-r_1$, $r_1\in\sqrt{\epsilon}\,S_{\rho,\delta}(1)$ by Proposition 5.1 there exists $K=1+r$, $r\in\sqrt{\epsilon}\,S_{\rho,\delta}(1)$ such that $e^{\epsilon\psi}\#e^{-\epsilon\psi}\#K=1$ if $0<\epsilon\leq\epsilon_0$ is small ([1, Theorem 3.2] and [15, Theorem 2.6.27] for example). Thus we have

$$(e^{\epsilon\psi})\#p\#e^{-\epsilon\psi}\#K=p+\sum_{1\leq|\alpha+\beta|\leq m}\frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!}p^{(\beta)}_{(\alpha)}\#(\omega^\alpha_\beta+\bar\omega^\alpha_\beta)\#K+R.$$

On the other hand it is clear $(\omega^\alpha_\beta+\bar\omega^\alpha_\beta)\#(1+r)=\omega^\alpha_\beta+\tilde\omega^\alpha_\beta$ with $\tilde\omega^\alpha_\beta\in\epsilon^{|\alpha+\beta|+1/2}S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$.

Since $\omega_\beta^\alpha \in \epsilon S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$ it is also clear that $\tilde{\omega}_\beta^\alpha \in \epsilon^{3/2} S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Note

$$p_{(\alpha)}^{(\beta)}\#(\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha) - \sum_{|\mu+\nu|\le m-|\alpha+\beta|} \frac{(-1)^{|\nu|}}{i^{|\mu+\nu|}\mu!\nu!} p_{(\alpha+\nu)}^{(\beta+\mu)}(\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha)_{(\mu)}^{(\nu)}$$
$$\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta|\alpha+\beta|-\rho(m+1-|\alpha+\beta|)}) \subset S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$$

and $(\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha)_{(\mu)}^{(\nu)} \in \gamma^{-\kappa|\mu+\nu|} S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|})$ which is contained in $\epsilon^{|\alpha+\beta|+|\mu+\nu|+1/2} S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|})$ if $|\mu+\nu| \ge 1$, $\gamma \ge \gamma_0(\epsilon)$ so that

$$(e^{\epsilon\psi})\#p\#e^{-\epsilon\psi}\#K = p + \sum_{1\le|\alpha+\beta|\le m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)}(\omega_\beta^\alpha + \hat{\omega}_\beta^\alpha) + R$$

where $\hat{\omega}_\beta^\alpha \in \epsilon^{|\alpha+\beta|+1/2} S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ and $\hat{\omega}_\beta^\alpha \in \epsilon^{3/2} S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Now check ω_β^α. For $|\alpha+\beta| = 1$ we have $\omega_\beta^\alpha = \epsilon(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta$. Let $|\alpha+\beta| \ge 2$ then ω_β^α is a linear combination of terms $(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)}\cdots(\epsilon\psi)_{(\beta_s)}^{(\alpha_s)}$ with $\alpha_1+\cdots+\alpha_s = \alpha$, $\beta_1+\cdots+\beta_s = \beta$, $|\alpha_j+\beta_j| \ge 1$. If $|\alpha_j+\beta_j| = 1$ for all j it is clear $\omega_\beta^\alpha = \epsilon^{|\alpha+\beta|}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta$. If $|\alpha_j+\beta_j| \ge 2$ for some j so that $s \le |\alpha+\beta|-2$ then one has

$$(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)}\cdots(\epsilon\psi)_{(\beta_s)}^{(\alpha_s)} \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{-\kappa+\rho|\beta|-\delta|\alpha|}) \subset \gamma^{-\kappa} S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|}).$$

Since we can assume $\gamma^{-\kappa} \le \epsilon^{|\alpha+\beta|+1/2}$ for $\gamma \ge \gamma_0(\epsilon)$ we get the assertion. □

6. Energy estimates

To obtain energy estimates we follow [13] where the main point is to derive microlocal energy estimates. We sketch how to get microlocal energy estimates. Let us denote

$$P_\psi = \mathrm{Op}(e^{\epsilon\psi})P\mathrm{Op}(e^{-\epsilon\psi})\mathrm{Op}(K)$$

of which principal symbol is given by $p_\psi = e^{\epsilon\psi}\#p\#e^{-\epsilon\psi}\#K$. In this section we say $a(x,\xi;\gamma,\epsilon) \in \tilde{S}_{\rho,\delta}(W)$ if $a \in S_{\rho,\delta}(W)$ for each fixed $0 < \epsilon \ll 1$. Let $a \in \tilde{S}_{\rho,\delta}(W)$ and let $N \in \mathbb{N}$ be given. Then with a fixed small $0 < 2\tau < \rho - \delta$ we have

$$|\partial_x^\beta\partial_\xi^\alpha a| \le C_{\alpha\beta}(\epsilon)W\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} \le C_{\alpha\beta}\gamma^{-2\tau|\alpha+\beta|}W\langle\xi\rangle_\gamma^{-(\rho-\tau)|\alpha|+(\delta+\tau)|\beta|}$$

where one can assume that $C_{\alpha\beta}(\epsilon)\gamma^{-2\tau|\alpha+\beta|}$ are arbitrarily small for $1 \le |\alpha+\beta| \le N$ taking γ large.

6.1. Symbol of P_ψ.

Define $h_j(x,\xi)$ by

$$h_j(x,\xi) = \sum_{1\le\ell_1<\ell_2<\cdots<\ell_j\le m} |q_{\ell_1}|^2\cdots|q_{\ell_j}|^2, \quad q_j = \xi_0 - \lambda_j(x,\xi').$$

Lemma 6.1. *There exists $c > 0$ such that*

$$h_{m-k}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta) \ge c(\epsilon\omega)^{2(j-k)}\langle\xi\rangle_\gamma^{2(j-k)} h_{m-j}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$$

for $j = k,\ldots,m$ where $h_0 = 1$ and $1 \le k \le m$.

Proof. We show the case $k = 1$. By definition $h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$ is bounded from below by

$$2^{-1}(|q_i(x,\xi)|^2 + |q_j(x,\xi)|^2 + \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa}) \prod_{k\neq i,j} |q_k(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)|^2.$$

From Lemma 3.2 we have $|q_i(x,\xi)|^2 + |q_j(x,\xi)|^2 \geq c\,|b'(x,\xi)|^2$. Since

$$\begin{aligned}&|b'(x,\xi)|^2 + \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa}\\ &= \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^2(\epsilon^{-2}|b'(x,\xi)|^2\omega^2\langle\xi\rangle_\gamma^{-2} + \langle\xi\rangle_\gamma^{-4\delta}) \geq c\,\epsilon^2\omega^2\langle\xi\rangle_\gamma^2\end{aligned}$$

with some $c > 0$ because $C|b'(x,\xi)|^2\langle\xi\rangle_\gamma^{-2} \geq \phi^2$ and $\omega^2 \geq \phi^2$ then it is clear that $h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$ is bounded from below by

$$c\epsilon^2\omega^2\langle\xi\rangle_\gamma^2 \prod_{k\neq i,j} |q_k(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)|^2.$$

Summing up over all pair i, j ($i \neq j$) we get the assertion for the case $j = 2$. Continuing this argument one can prove the case $j \geq 3$. □

Let us put

$$h(x,\xi) = h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}.$$

Lemma 6.2. *There exists $C > 0$ such that we have*

$$\begin{cases} |p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\alpha|}h, & 1 \leq |\alpha+\beta| \leq m,\\ |p\,p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{2-|\alpha+\beta|}\langle\xi\rangle_\gamma^{2-|\alpha|}h^2, & 2 \leq |\alpha+\beta| \leq m.\end{cases}$$

Proof. From [4, Proposition 3] one has

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \leq Ch_{m-|\alpha+\beta|}(x,\xi)^{1/2}|\xi|^{|\beta|}$$

for $|\alpha+\beta| \leq m$ which is bounded by $Ch_{m-|\alpha+\beta|}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}|\xi|^{|\beta|}$ clearly. On the other hand it follows from Lemma 6.1

$$Ch(x,\xi) \geq (\epsilon\omega)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{|\alpha+\beta|-1}h_{m-|\alpha+\beta|}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}$$

for $1 \leq |\alpha+\beta| \leq m$ which proves the assertion. The proof of the second assertion is similar. □

Lemma 6.3. *Assume that $c_\beta^\alpha \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ and $c_\beta^\alpha \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Then for $1 \leq |\alpha+\beta| \leq m$ we have $p_{(\beta)}^{(\alpha)}c_\alpha^\beta \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ and $\epsilon^{|\alpha+\beta|}|p_{(\beta)}^{(\alpha)}c_\alpha^\beta| \leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ with $C > 0$ independent of ϵ.*

Proof. Let $2 \leq |\alpha+\beta| \leq m$ then since $1 = \kappa + 2\delta$ by (4.4) and (4.6) we see by Lemma 6.2

$$\begin{aligned}\epsilon^{|\alpha+\beta|}|p_{(\beta)}^{(\alpha)}c_\alpha^\beta| &\leq C\epsilon\omega^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\alpha|+\rho|\alpha|-\delta|\beta|}h\\ &\leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa(\omega^{-1}\langle\xi\rangle_\gamma^{-\delta})^{|\alpha+\beta|-2}h \leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h.\end{aligned}$$

When $|\alpha+\beta| = 1$ noting $c_\alpha^\beta \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\beta|})$ we get the same assertion. We next estimate $\sum p_{(\beta+\nu')}^{(\alpha+\mu')}(c_\alpha^\beta)_{(\nu'')}^{(\mu'')}$. If $|\alpha+\mu'+\beta+\nu'| \geq m$ we have

$$\begin{aligned}\left|p^{(\alpha+\mu')}_{(\beta+\nu')}\omega^{\beta(\mu'')}_{\alpha(\nu'')}\right| &\le \langle\xi\rangle_\gamma^{m-|\alpha+\mu'|}\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|}\langle\xi\rangle_\gamma^{-\rho|\mu''|+\delta|\nu''|}\\ &\le C_\epsilon\omega^{-(m-1)}h\langle\xi\rangle_\gamma^{1-|\alpha+\mu'|}\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|+\rho|\mu'|-\delta|\nu'|}\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}\\ &\le C_\epsilon\omega^{-1}\langle\xi\rangle_\gamma^{\kappa}(\omega^{-(m-2)}\langle\xi\rangle_\gamma^{-\delta(|\alpha+\mu'+\beta+\nu'|-2)})\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}h\end{aligned}$$

where the right-hand side is bounded by $C_\epsilon\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-\rho|\mu|+\delta|\nu|}h$. We turn to the case $|\alpha+\mu'+\beta+\nu'|\le m$. From Lemma 6.2 it follows

$$\begin{aligned}|p^{(\alpha+\mu')}_{(\beta+\nu')}| &\le C_\epsilon\omega^{1-|\alpha+\mu'+\beta+\nu'|}\langle\xi\rangle_\gamma^{1-|\alpha+\mu'|}h\\ &\le C_\epsilon(\omega^{-1}\langle\xi\rangle_\gamma^{-\delta})^{|\mu'+\nu'|}\omega^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\alpha|}h\langle\xi\rangle_\gamma^{-\rho|\mu'|+\delta|\nu'|}\end{aligned}$$

therefore for $|\alpha+\beta|\ge 2$ we see easily

$$\begin{aligned}\left|p^{(\alpha+\mu')}_{(\beta+\nu')}c^{\beta(\mu'')}_{\alpha(\nu'')}\right| &\le C_\epsilon(\omega^{-1}\langle\xi\rangle_\gamma^{-\delta})^{|\alpha+\beta|-2}\omega^{-1}\langle\xi\rangle_\gamma^{\kappa}h\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}\\ &\le C_\epsilon\omega^{-1}\langle\xi\rangle_\gamma^{\kappa}h\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}\end{aligned}$$

which also holds for $|\alpha+\beta|=1$ because $c^\beta_\alpha\in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\beta|})$. □

6.2. Definition of $Q(z)$ which separates $P_\psi(z)$. We follow the arguments in [13]. Let us define $\tilde p(x+iy,\xi+i\eta)$ by

$$\tilde p(x+iy,\xi+i\eta)=\sum_{|\alpha+\beta|\le m}\frac{1}{\alpha!\beta!}\partial_x^\alpha\partial_\xi^\beta p(x,\xi)(iy)^\alpha(i\eta)^\beta.$$

Then p_ψ given in (5.2) is expressed as $\tilde p(z-i\epsilon H_\psi)$, which one can also write as

$$\tilde p(z-i\epsilon H_\psi)=\sum_{j=0}^m\left(i\frac{\partial}{\partial t}\right)^j p(z-\epsilon tH_\psi)/j!\Big|_{t=0}.$$

Using this expression we define $Q(z)$ which separates $\tilde p(z-i\epsilon H_\psi)$ by

$$Q(z)=\epsilon^{-1}|\tilde H_\psi|^{-1}\left(\frac{\partial}{\partial t}\right)\sum_{j=0}^m\left(i\frac{\partial}{\partial t}\right)^j p(z-\epsilon tH_\psi)/j!\Big|_{t=0}$$

where $\tilde H_\psi=(\langle\xi\rangle_\gamma\nabla_\xi\psi,-\nabla_x\psi)$. By the homogeneity it is clear that

$$\tilde p(z-i\epsilon H_\psi)=\langle\xi\rangle_\gamma^m\tilde p(\tilde z-i\lambda(z)\tilde H_\psi/|\tilde H_\psi|),\ \lambda(x,\xi)=\epsilon|\tilde H_\psi|\langle\xi\rangle_\gamma^{-1}$$

where $\tilde z=(x,\xi\langle\xi\rangle_\gamma^{-1})$. It is not difficult to check $\tilde p(z-i\epsilon H_\psi)\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^m)$ and $Q\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-1})$. We study $\tilde p(z-i\epsilon H_\psi)$ and $Q(z)$ in a conic neighborhood of ρ. We first recall

Proposition 6.1 ([14, Lemma 5.8]). *Let ρ be a characteristic of p of order m and let $K\subset\Gamma_\rho$ be a compact set. Then one can find a conic neighborhood V of ρ and positive $C>0$ such that for any $(x,\xi)\in V$, $\zeta\in K$ and small $s\in\mathbb{R}$ one can write*

$$p(z-s\zeta)=e(z,\zeta,s)\prod_{j=1}^m(s-\mu_j(z,\zeta))$$

where $\mu_j(z,\zeta)$ are real valued and $e(z,\zeta,s)\neq 0$ for $(z,\zeta,s)\in V\times K\times\{|s|<s_0\}$. Moreover there exists $C>0$ such that we have

$$|\mu_j(z,(0,\theta))|/C \le |\mu_j(z,\zeta)| \le C|\mu_j(z,(0,\theta))|, \quad j=1,2,\dots,m \tag{6.1}$$

for any $(x,\xi)\in V$, $\zeta\in K$.

Writing $\prod_{j=1}^m (t-\mu_j) = \sum_{\ell=0}^m p_\ell t^\ell$ we see that $\tilde p(z-is\zeta)$ is written

$$\sum_{j=0}^m \frac{1}{j!}\Big(is\frac{\partial}{\partial t}\Big)^j \Big(e\prod_{j=1}^m (t-\mu_j)\Big)\Big|_{t=0} = \sum_{\ell=0}^m \sum_{k=0}^{m-\ell} \frac{1}{k!}\Big(is\frac{\partial}{\partial t}\Big)^k e\Big|_{t=0} p_\ell (is)^\ell$$

which is equal to

$$\begin{aligned}\sum_{\ell=0}^m \Big(\sum_{k=0}^m \frac{1}{k!}\Big(is\frac{\partial}{\partial t}\Big)^k e\Big|_{t=0} - \sum_{m-\ell+1\le k\le m} \frac{1}{k!}\Big(is\frac{\partial}{\partial t}\Big)^k e\Big|_{t=0}\Big) p_\ell (is)^\ell \\ = \sum_{k=0}^m \frac{1}{k!}\Big(is\frac{\partial}{\partial t}\Big)^k e\Big|_{t=0} \prod_{j=1}^m (is-\mu_j) + O(s^{m+1})\end{aligned}$$

which proves

$$\tilde p(z-is\zeta) = e_0(z,\zeta,s)\prod_{k=1}^m (is-\mu_j(z,\zeta)) + O(s^{m+1}). \tag{6.2}$$

Note that $e_0(z,\zeta,s) = \sum_{k=0}^m (is\partial/\partial t)^k e(z,\zeta,t)/k!\big|_{t=0}$ and hence we have $e_0(z,\zeta,0) = e(z,\zeta,0) \ne 0$.

Lemma 6.4. *There exist a conic neighborhood U of ρ and a compact convex set $K\subset\Gamma_\rho$ such that $\tilde H_\psi/|\tilde H_\psi| \in K$ for $(x,\xi)\in U$, $\gamma\ge\gamma_0$.*

Proof. From $\psi = \langle\xi\rangle_\gamma^\kappa \log(\phi+\omega)$ it is easy to see

$$\begin{cases} \nabla_x\psi = \omega^{-1}\langle\xi\rangle_\gamma^\kappa \nabla_x\phi, \\ \nabla_\xi\psi = \omega^{-1}\langle\xi\rangle_\gamma^\kappa \nabla_\xi\phi + O(\langle\xi\rangle_\gamma^{\kappa-1})\log(\phi+\omega) + O(\langle\xi\rangle_\gamma^{\kappa-1}). \end{cases}$$

Therefore one has

$$\tilde H_\psi = \omega^{-1}\langle\xi\rangle_\gamma^\kappa (\tilde H_\phi + (O(1)\omega\log(\phi+\omega), 0)).$$

Since $|\phi+\omega|\le 2\omega$ we can assume $\omega\log(\phi+\omega)$ is enough small taking U small. In particular we have $\omega^{-1}\langle\xi\rangle_\gamma^\kappa/C \le |\tilde H_\psi| \le C\omega^{-1}\langle\xi\rangle_\gamma^\kappa$. Then noting $\tilde H_\phi(\rho)/|\tilde H_\phi(\rho)| \in \Gamma_\rho$ which follows from (4.3) we get the assertion. □

We rewrite $Q(z)$ according to (6.2).

Lemma 6.5. *Let $\tilde\omega = \tilde H_\psi/|\tilde H_\psi|$. Then we have*

$$\begin{aligned} Q(z) = \langle\xi\rangle_\gamma^{m-1}\Big\{ -i\partial e_0(\tilde z,\tilde\omega,\lambda)/\partial\lambda \prod_{j=1}^m (i\lambda - \mu_j(\tilde z,\tilde\omega)) \\ + e_0(\tilde z,\tilde\omega,\lambda)\sum_{j=1}^m \prod_{k=1,k\ne j}^m (i\lambda-\mu_k(\tilde z,\tilde\omega)) + O(\lambda^m)\Big\}. \end{aligned}$$

Proof. Noting $\lambda(z)|\tilde H_\psi|^{-1}\langle\xi\rangle_\gamma = \epsilon$ one can write

$$Q(z) = \epsilon^{-1}|\tilde{H}_\psi|^{-1}\langle\xi\rangle_\gamma^m \sum_{j=0}^{m}\left(\frac{\partial}{\partial t}\right)\left(i\frac{\partial}{\partial t}\right)^j p(\tilde{z} - t\lambda(z)\tilde{\omega})/j!\Big|_{t=0}$$
$$= \langle\xi\rangle_\gamma^{m-1}\sum_{j=0}^{m}\lambda(z)^j\left(\frac{\partial}{\partial t}\right)\left(i\frac{\partial}{\partial t}\right)^j p(\tilde{z} - t\tilde{\omega})/j!\Big|_{t=0}$$

which is equal to

$$\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial s}\frac{1}{i}\sum_{j=0}^{m}\frac{1}{(j+1)!}\left(is\frac{\partial}{\partial t}\right)^{j+1} p(\tilde{z} - t\tilde{\omega})\Big|_{t=0, s=\lambda(z)}$$
$$= \frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial s}\{\tilde{p}(\tilde{z} - is\tilde{\omega}) - p(\tilde{z}) + O(s^{m+1})\}_{s=\lambda(z)} \tag{6.3}$$
$$= \frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\left\{\frac{\partial}{\partial s}\tilde{p}(\tilde{z} - is\tilde{\omega})\Big|_{s=\lambda(z)} + O(\lambda^m)\right\}.$$

From (6.2) the right-hand side of (6.3) turns to be

$$\frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\left\{\frac{\partial}{\partial s}(e_0(\tilde{z}, \tilde{\omega}, s)\prod_{j=1}^{m}(is - \mu_j(\tilde{z}, \tilde{\omega})) + O(s^{m+1}))\Big|_{s=\lambda}\right\} \tag{6.4}$$

modulo $O(\lambda^m)\langle\xi\rangle_\gamma^{m-1}$ which proves the assertion. □

6.3. Microlocal energy estimates. To derive microlocal energy estimates we study $\mathsf{Im}\,(P_\psi\chi u, Q\chi u)$ where χ is a cutoff symbol supported in a conic neighborhood of ρ. Thus we are led to consider $\mathsf{Im}\,(P_\psi\bar{Q})$ in a conic neighborhood of ρ. Recall that $P_\psi = p_\psi + \sum_{j=0}^{m-1}(P_j)_\psi$ where $P_j(x, D)$ is the homogeneous part of degree j of P and $(P_j)_\psi \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^j)$ by Theorem 5.1. Take any small $0 < \epsilon^* \ll 1$ and we fix ϵ^* and put

$$\delta = (1 - \epsilon^*)/m, \quad \rho = (m - 1 + \epsilon^*)/m, \quad \kappa = \rho - \delta$$

where $\rho + \delta = 1$.

Lemma 6.6. *Let $S_0(z) = \mathsf{Im}(\tilde{p}(z - i\epsilon H_\psi)\overline{Q(z)})$. Then one can find a conic neighborhood V of ρ and $C > 0$ such that we have*

$$\epsilon\,\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2(z)/C \le S_0(z) \le C\epsilon\,\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2(z).$$

Proof. Write $e_0(\tilde{z}, \tilde{\omega}, \lambda) = e(\tilde{z}, \tilde{\omega}, 0) + i\lambda(\partial e/\partial\lambda)(\tilde{z}, \tilde{\omega}, 0) + O(\lambda^2)$ then it is clear $|e_0|^2 = |e(\tilde{z}, \tilde{\omega}, 0)|^2 + O(\lambda^2)$ so that $\mathsf{Re}\,(\partial\bar{e}_0/\partial\lambda)e_0 = 2^{-1}\partial|e_0|^2/\partial\lambda = O(\lambda)$. Thus from Lemma 6.5 and (6.2) it follows that

$$\mathsf{Im}(\tilde{p}(z - i\epsilon H_\psi)\overline{Q(z)}) = \langle\xi\rangle_\gamma^{2m-1}|e_0|^2\lambda\left\{\sum_{j=1}^{m}\prod_{k=1,k\ne j}^{m}(\lambda^2 + \mu_k^2)\left(1 + O\left(\lambda + \sum_{j=1}^{m}|\mu_j|\right)\right)\right\}.$$

Since $\mu_j(\rho, \tilde{\omega}) = 0$, $j = 1, 2, \ldots, m$ one obtains

$$C^{-1} \le \left\{\langle\xi\rangle_\gamma^{2m-1}\lambda\sum_{j=1}^{m}\prod_{k=1,k\ne j}^{m}(\lambda^2 + \mu_k(\tilde{z}, \tilde{\omega})^2)\right\}\Big/ S_0(z) \le C.$$

On the other hand noting $C^{-1} \le \lambda(z)/(\epsilon\,\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-1}) \le C$ and

$$C^{-1} \le \left\{ \langle\xi\rangle_\gamma^{2m-2} \sum_{j=1}^{m} \prod_{k=1,k\neq j}^{m} (\epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa-2} + \mu_k(\tilde z,(0,\theta))^2) \right\}$$
$$\times h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{-1} \le C$$

we conclude the assertion from Lemma 6.4 and Proposition 6.1. □

Lemma 6.7. *We have $Q \in \tilde S_{\rho,\delta}(h)$ and $S_0^{\pm 1} \in \tilde S_{\rho,\delta}((\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)^{\pm 1})$ in U for $\gamma \ge \gamma_0(\epsilon)$. Moreover $|Q| \le Ch$ with $C > 0$ independent of ϵ.*

Proof. From the definition one can see easily that Q is a sum of terms, up to constant factor;

$$\epsilon^{|\alpha+\beta|-1} p_{(\beta)}^{(\alpha)}(z)(\nabla_\xi\psi)^\beta(\nabla_x\psi)^\alpha / (\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{1/2} \tag{6.5}$$

with $1 \le |\alpha+\beta| \le m+1$. We also note that $\mathsf{Im}\, Q$ is a sum of such terms (6.5) with $2 \le |\alpha+\beta| \le m+1$. From Lemma 4.3 it follows

$$\begin{cases} \nabla_\xi\psi \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-1}), \\ \nabla_x\psi \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa}) \end{cases}$$

in U and hence $(\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{-1/2} \in S_{\rho,\delta}((\omega^{-1}\langle\xi\rangle_\gamma^\kappa)^{-1})$ then we have

$$V_\alpha^\beta = (\nabla_\xi\psi)^\beta(\nabla_x\psi)^\alpha(\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{-1/2} \in S_{\rho,\delta}(\omega\langle\xi\rangle_\gamma^{-\kappa}\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|}).$$

Noting $\omega^{-2}\langle\xi\rangle_\gamma^{\kappa-1} \le 1$ it suffices to repeat the proof of Lemma 6.3 to conclude $p_{(\beta)}^{(\alpha)}V_\alpha^\beta \in \tilde S_{\rho,\delta}(h)$ and $\epsilon^{|\alpha+\beta|-1}|p_{(\beta)}^{(\alpha)}V_\alpha^\beta| \le Ch$ with C independent of ϵ for $1 \le |\alpha+\beta| \le m+1$. From Lemma 6.3 it follows that $\tilde p(z - i\epsilon H_\psi) - p(z) \in \tilde S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$. Since from Lemma 6.2 one can check that $p(z)\mathsf{Im}\, Q \in \tilde S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)$ we get the assertion for $S_0(z)$. The assertion for $S_0^{-1}(z)$ follows from Lemma 6.6 and $S_0(z)S_0^{-1}(z) = 1$. □

From Theorem 5.1 and Lemma 6.3 one can write

$$p_\psi - \tilde p(z - i\epsilon H_\psi) = \sqrt{\epsilon}\, r + r_0$$

where $r \in \tilde S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ with $|r| \le C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ and $r_0 \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$. Thus $r\#\bar Q \in \tilde S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)$ and $|r\bar Q| \le C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2$. On the other hand Lemma 6.1 shows

$$\langle\xi\rangle_\gamma^{m-\delta(m+1)} = \langle\xi\rangle_\gamma^{m-1+\kappa-\delta(m-1)} \le C\epsilon^{1-m}\gamma^{-\delta}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$$

so that we see $r_0 \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ and $|r_0| \le C\epsilon^{3/2}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ for $\gamma \ge \gamma_0(\epsilon)$. Therefore in virtue of Lemma 6.7 there is $\tilde r \in \tilde S_{\rho,\delta}(1)$ with $|\tilde r| \le C\sqrt{\epsilon}$ such that

$$\mathsf{Im}\,(p_\psi\#\bar Q) = S_0(1-\tilde r)$$

in some conic neighborhood of ρ.

We turn to $R = \sum_{j=0}^{m-1}(P_j)_\psi \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-1})$. From Lemma 6.1 again we have

$$\begin{aligned}\langle\xi\rangle_\gamma^{m-1} &\le C\epsilon^{1-m}\omega^{-(m-1)}h = C\epsilon^{1-m}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h(\omega^{-(m-2)}\langle\xi\rangle_\gamma^{-\kappa}) \\ &\le C\epsilon^{1-m}\gamma^{-\kappa+\delta(m-2)}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h.\end{aligned}$$

Recalling that $\kappa - \delta(m-2) = \epsilon^* > 0$ and hence we can assume $C\epsilon^{1-m}\gamma^{-\epsilon^*} \le C\epsilon^{3/2}$ for

$\gamma \geq \gamma_0(\epsilon)$ so that there exists $\hat{r} \in \tilde{S}_{\rho,\delta}(1)$ with $|\hat{r}| \leq C\sqrt{\epsilon}$ such that $\mathsf{Im}\,(R\#\bar{Q}) = S_0(1-\hat{r})$ in a conic neighborhood of ρ. Thus we conclude

$$\begin{cases} \mathsf{Im}\,(P_\psi\#\bar{Q}) = E^2, \quad E \in \tilde{S}_{\rho,\delta}(\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h), \\ \epsilon^{1/2}\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h/C \leq |E| \leq C\epsilon^{1/2}\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h \end{cases}$$

in a conic neighborhood of ρ with C independent of ϵ. The rest of the proof of deriving microlocal energy estimates is just a repetition of the arguments in [13] and we conclude that the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(1/\kappa)}$ well-posed at the origin. Note that $1/\kappa = m/(m-2+2\epsilon^*)$ and $\epsilon^* > 0$ is arbitrarily small so that $1/\kappa$ is as close to $m/(m-2)$ as we please.

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