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SIMPLE RIBBON MOVES FOR LINKS

Dedicated to Professor Fujitsugu HOSOKAWA on his 80th birthday

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Abstract

We introduce and study local moves for links, called simple ribbon moves. We also introduce a complexity of links, called the h-complexity, which coincides with the genus in the case of knots, and we show that simple ribbon moves never reduce the h-complexities of links.

1. Introduction

All links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in the oriented 3-sphere S^3. In this paper, we define and study local moves for links, called simple ribbon moves ([4]).

Let H be a 3-ball in S^3 and $\mathcal{D} = D_1 \cup \cdots \cup D_m$ (resp. $\mathcal{B} = B_1 \cup \cdots \cup B_m$) a union of mutually disjoint disks in $\operatorname{int} H$ (resp. H) satisfying the following:

(i) $B_i \cap \partial H = \partial B_i \cap \partial H$ is an arc;

(ii) $B_i \cap \partial\mathcal{D} = \partial B_i \cap \partial D_i$ is an arc; and

(iii) $B_i \cap \operatorname{int}\mathcal{D} = B_i \cap \operatorname{int} D_{\pi(i)}$ is a single arc of ribbon type (Fig. 1), where π is a certain permutation on $\{1, 2, \dots, m\}$.

Then we call $\bigcup_i (\partial(B_i \cup D_i) - \operatorname{int}(B_i \cap \partial H))$ an *SR*-tangle and denote it by $\mathcal{T}$, and we call each B_i a *band*.

Let l be a link in $S^3 - \operatorname{int} H$ such that $l \cap \partial H$ consists of arcs. Take an *SR*-tangle $\mathcal{T}$ such that $\mathcal{B} \cap \partial H = l \cap \partial H$. Then let L be the link obtained from l by substituting $\mathcal{T}$ for $l \cap \partial H$. We call the transformation either from l to L or from L to l a *simple ribbon-move* or an *SR-move*, and H (resp. $\mathcal{T}$) the *associated* 3-*ball* (resp. *tangle*) *of* the *SR*-move. The transformation from l to L (resp. from L to l) is called an *SR*$^+$-*move* (resp. *SR*$^-$-*move*) (see Fig. 2 for an example).

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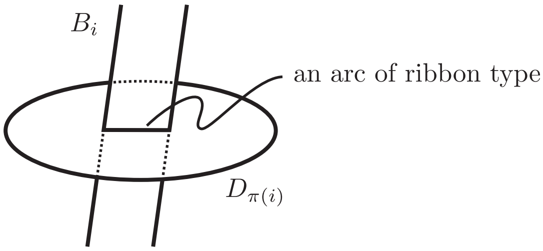


Fig. 1.

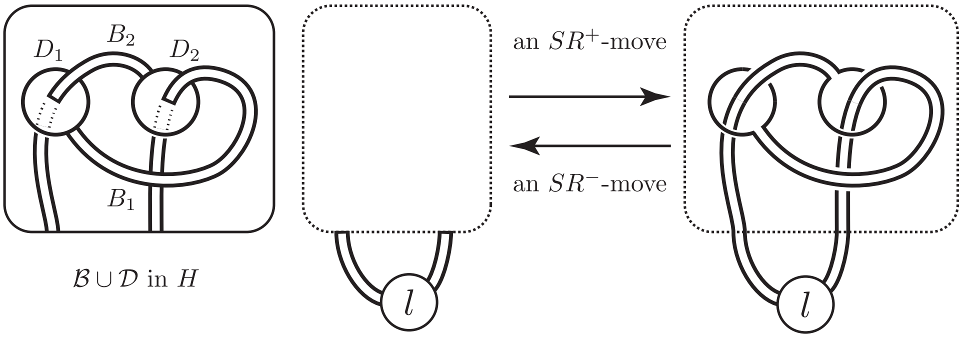


Fig. 2.

Since every permutation is a product of cyclic permutations, we rename the indices of the bands and disks as

$$\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}^k = \bigcup_{k=1}^{n} \left(\bigcup_{i=1}^{m_k} B_i^k \right) \quad \text{and} \quad \mathcal{D} = \bigcup_{k=1}^{n} \mathcal{D}^k = \bigcup_{k=1}^{n} \left(\bigcup_{i=1}^{m_k} D_i^k \right), \quad where$$

(1) $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$;
(2) $B_i^k \cap \partial \mathcal{D} = \partial B_i^k \cap \partial D_i^k$ is an arc; and
(3) $B_i^k \cap \operatorname{int} \mathcal{D} = B_i^k \cap \operatorname{int} D_{i+1}^k$ is a single arc of ribbon type.

In Condition (3), the lower indices are considered modulo m_k. For an *SR*-tangle $\mathcal{T}$, we call $\bigcup_{i=1}^{m_k} (\partial(B_i^k \cup D_i^k) - \operatorname{int}(B_i^k \cap \partial H))$ the (k-th) *component* of the *SR*-move or of the *SR*-tangle, denote it by $\mathcal{T}^k$, and call m_k the *index* of the component ($k = 1, 2, \ldots, n$). The *type* of the *SR*-move or of the *SR*-tangle is the ordered set $(m_1, m_2, \ldots, m_n)$ of the indices. If the index of each component is 1 (resp. no less than 2), then we say that the *SR*-move or the *SR*-tangle is of class I (resp. class II) (see Fig. 3 for examples).

Let $T_i^k = \partial(B_i^k \cup D_i^k) - \operatorname{int}(B_i^k \cap \partial H)$. We say that a string T_i^k of the *SR*-tangle is *trivial* if $T_i^k \cup (B_i^k \cap \partial H)$ bounds a non-singular disk in H whose interior is in $\operatorname{int} H$ and does not intersect with $\mathcal{T}$. We say that the k-th component $\mathcal{T}^k$ of the *SR*-tangle is *trivial* if the string T_i^k is trivial for any i. In fact, $\mathcal{T}^k$ is trivial if the string T_i^k is

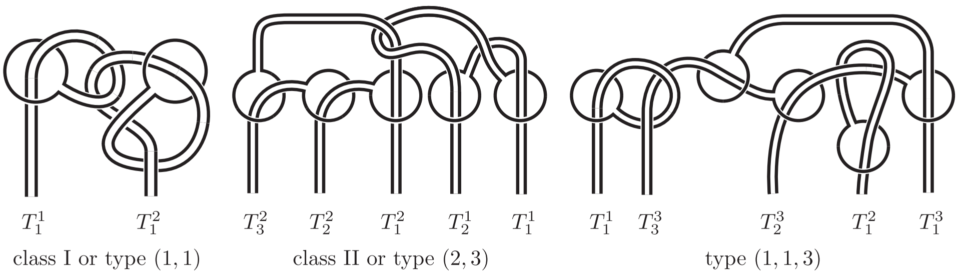


Fig. 3.

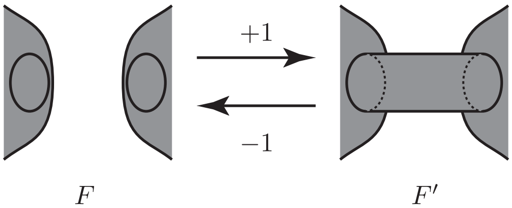


Fig. 4. Differences of the h-complexities.

trivial for some i, which is easy to see. We say that an *SR*-tangle is *reducible* if the string T_i^k is trivial for a certain pair of i and k. Otherwise we say that the *SR*-tangle is *irreducible*. We say that an *SR*-tangle is *trivial* if the string T_i^k is trivial for any i and k.

Consider an *SR*-move transforming l into L. We say that a string T_i^k of the *SR*-move is *trivial* if $T_i^k \cup (B_i^k \cap \partial H)$ bounds a non-singular disk in S^3 whose interior does not intersect with L. We say that the k-th component $\mathcal{T}^k$ of the *SR*-move is *trivial* if the string T_i^k is trivial for any i. We say that an *SR*-move is *reducible* if the string T_i^k is trivial for a pair of i and k. Otherwise we say that the *SR*-move is *irreducible*. We say that an *SR*-move is *trivial* if the string T_i^k is trivial for any i and k.

Let F be a surface (which is not necessary to be connected or to be orientable) with n boundary components. We define the *h-complexity* $h(F)$ of F as

$$h(F) = 1 - \frac{\chi(F) + n}{2}.$$

The following is a main property of the h-complexity, which is obtained by calculating the Euler characteristics (see Fig. 4 for an example).

Proposition 1.1. *Let F and F' be surfaces such that F' is obtained from F by deleting the interiors of two disks D_1 and D_2 on* $\operatorname{int} F$ *and identifying ∂D_1 and ∂D_2 by a homeomorphism $\varphi\colon \partial D_1 \to \partial D_2$. Then $h(F') = h(F) + 1$.*

REMARK 1.2. In the above statement, D_1 and D_2 may or may not belong to the same connected component of F.

Proposition 1.3. *If F is a surface with μ connected components, then $h(F) \geq 1 - \mu$. The equality holds if and only if each connected component of F is planar.*

Proof. Note that $\chi(F) + n$ is the Euler characteristic of the closed surface obtained from F by attaching n disks to n boundary components of F. Since $\chi(F) + n \leq 2\mu$, we have that $h(F) = 1 - (\chi(F) + n)/2 \geq 1 - 2\mu/2 = 1 - \mu$. Now the last statement is clear. □

Corollary 1.4. *If F is a connected surface, then $h(F) \geq 0$. The equality holds if and only if F is planar.*

Next we define the *h-complexity* of a link. In this paper, a *Seifert surface* for a link L is a compact oriented surface F embedded in S^3 such that $\partial F = L$ and F does not have any closed surface components (cf. [2]). Then we define the *h-complexity* $h(L)$ of L as the least h-complexity of all Seifert surfaces for L. From the definition, we have that if L and l are ambient isotopic, then $h(L) = h(l)$.

REMARK 1.5. The *genus* of a link is the least genus of all its connected Seifert surface (cf. [8]). Therefore if L is a link which admits only connected Seifert surface (for instance, if L is a knot, or a link with $\Delta_L(t) \neq 0$), then we have that $h(L) = g(L)$.

Proposition 1.6. *If L is a link with n components, then $h(L) \geq 1 - n$. The equality holds if and only if L is the n-component trivial link.*

Proof. Since any Seifert surface for L has at most n connected components, we have the inequality from Proposition 1.3. Moreover L is the n-component trivial link if and only if L has a Seifert surface with n disks, and thus the last statement is clear. □

A loop on a surface is called *essential* if it is not null-homotopic on the surface. Let F be a Seifert surface for l and $E(L)$ the exterior of L. A disk D in $E(L)$ is called a *compressing disk* for F in $E(L)$ if $D \cap F = \partial D$ and ∂D is essential on F. We say that F is *compressible* in $E(L)$ if there exists a compressing disk for F in $E(L)$. Otherwise, we say that F is *incompressible* in $E(L)$.

Proposition 1.7. *If F is a Seifert surface for a link L with $h(F) = h(L)$, then F is incompressible in $E(L)$.*

Proof. Suppose that F is compressible in $E(L)$. Let D be a compressing disk for F and F' the surface obtained from F by replacing a neighborhood of ∂D on F

with two parallel copies of D. Note that $\chi(F') = \chi(F) + 2$. Therefore if ∂D is a non-separating loop on F or a separating loop on F but F' has no closed components, then F' is another Seifert surface for L such that $h(F') = h(F) - 1$, which contradicts that $h(F) = h(L)$. If ∂D is a separating loop on F and $F' = F'_1 \cup F'_2$ with a closed component F'_2, then F'_1 is another Seifert surface for L such that $\chi(F'_1) = \chi(F') - \chi(F'_2) \geq \chi(F') + 0 = \chi(F) + 2$, since D is a compressing disk and thus $\chi(F'_2) \leq 0$. Therefore we have that $h(F'_1) \leq h(F) - 1$, which contradicts that $h(F) = h(L)$. □

Theorem 1.8. *Let L be a link obtained from a link l by a single SR^+-move. Then we have that $h(L) \geq h(l)$. Moreover, the following conditions are equivalent*:
(1) $h(L) = h(l)$;
(2) *L is ambient isotopic to l*; *and*
(3) *the SR^+-move is trivial.*

Corollary 1.9. *Let L be a link obtained from a link l by a single SR^+-move. If l is a non-trivial link, then L is a non-trivial link.*

REMARK 1.10. The first statement of Theorem 1.8 holds for the genus instead of the h-complexity. However, the last statement does not hold for the genus. Let l and L be the links as illustrated in the upper left and lower left of Fig. 5, respectively. Then L is obtained from l by an SR^+-move of class I and L is not ambient isotopic to l from Corollary 1.22. However both of l and L have Seifert surfaces of genus 2 as illustrated in the upper right and lower right of of Fig. 5, respectively. Since the signature of l is 4, we have $2g(l) \geq \sigma(l) - n + 1 = 4 - 2 + 1 = 3$ ([6], Theorem 9.1). Therefore we have that $g(L) = g(l) = 2$.

The effect of an SR-move on a link type depends not only on its associated tangle but also on how we attach the tangle to l. In fact, for any SR-tangle, there is a trivial SR-move whose associated tangle is the SR-tangle. However, we have the following for non-split links.

Theorem 1.11. *An SR-move on a non-split link is reducible* (*resp. trivial*) *if and only if its associated tangle is reducible* (*resp. trivial*).

Corollary 1.12. *Let L be a link obtained from a non-split link l by a single SR^+-move. Then L is ambient isotopic to l if and only if its associated tangle is trivial.*

Theorem 1.13. *Let L be a link obtained from a non-split link l by a single SR^+-move. Then L is also non-split.*

For the effect of an SR-move on the Alexander polynomial $\Delta_l(t)$ of a link l, we have the following. Therefore if an SR-move is not of class I, i.e., if the SR-move has

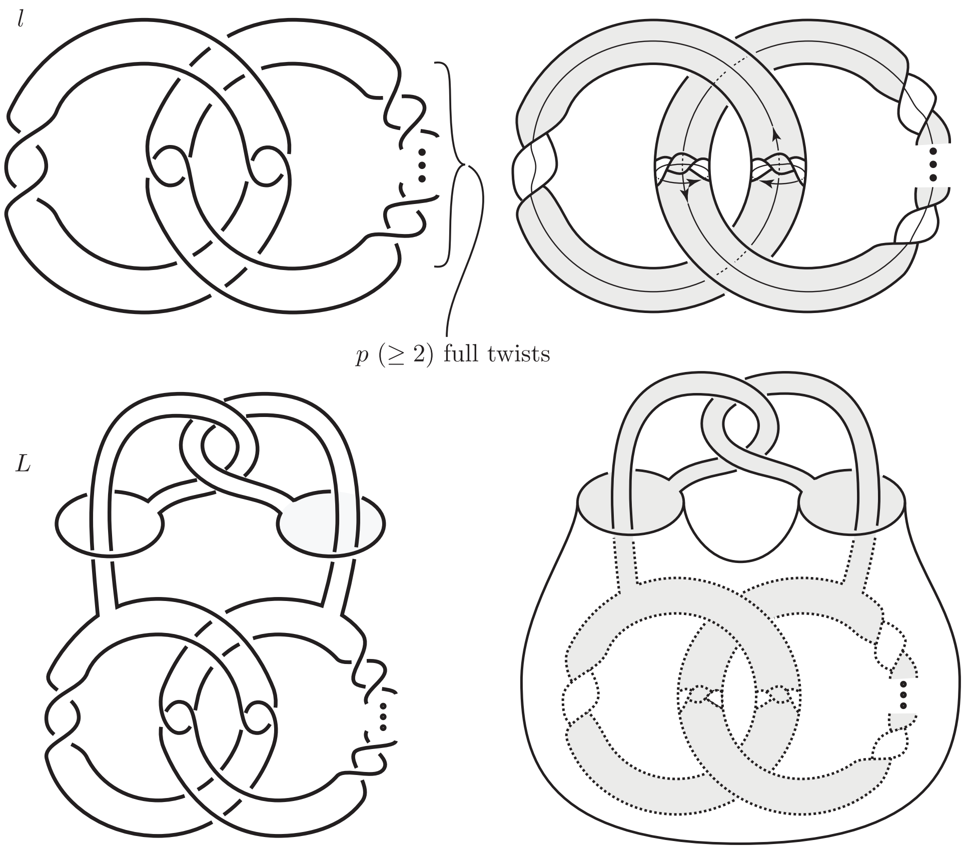


Fig. 5.

a component with index more than 1, then the *SR*-move on a knot changes the knot type, and thus its associated tangle is non-trivial.

Theorem 1.14 (cf. [1, Theorem 1]). *Let L be a link obtained from a link l by a single SR^+-move of type $(m_1, m_2, \dots, m_n)$ ($m_k = 1$ if $k \leq p-1$, $m_k \geq 2$ if $k \geq p$). Then we have the following, where q_k and r are integers with $0 \leq q_k \leq m_k/2$.*

$$\Delta_L(t) = \pm t^r \prod_{k=p}^{n} \{(1-t)^{m_k} - (-t)^{q_k}\}\{(1-t)^{m_k} - (-t)^{m_k - q_k}\}\Delta_l(t).$$

Especially if the SR^+-move is of class I, *then we have that $\Delta_L(t) = \pm t^r \Delta_l(t)$.*

Corollary 1.15. *An SR-tangle which is not of class* I *is non-trivial.*

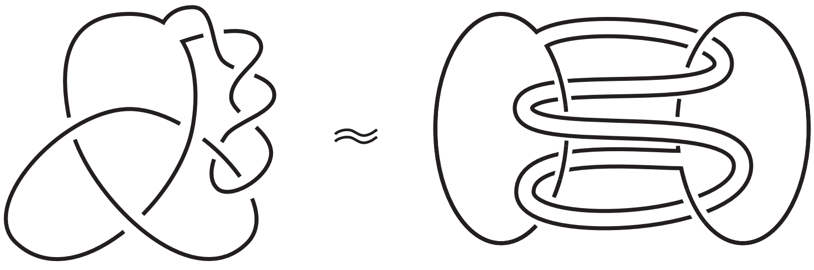


Fig. 6.

Corollary 1.16. *The k-th component of an SR-tangle with $m_k \geq 2$ is irreducible.*

Proof. The k-th component $\mathcal{T}^k$ of an SR-tangle with $m_k \geq 2$ is non-trivial from Theorem 1.14. Then we obtain the conclusion, since $\mathcal{T}^k$ is irreducible if and only if $\mathcal{T}^k$ is non-trivial. $\square$

REMARK 1.17. If $\mathcal{B}$ satisfies only Condition (i) in the definition of a simple ribbon move, then we call the transformation either from l to L or from L to l a *ribbon move*. It is easy to see that any ribbon link is obtained from a trivial link by a ribbon move. However, there is a ribbon link which is not obtained from a trivial link by SR^+-moves (see the following example).

EXAMPLE 1.18. The knot K $(= 8_8)$ illustrated in Fig. 6 is a ribbon knot which cannot be obtained from the trivial knot by a finite sequence of SR^+-moves.

Proof. Consider the degree $\deg \Delta_L(t)$ of a link L which is obtained from the trivial link by a finite sequence of SR^+-moves. Let $m_{i,k}$ be the index of the k-th component of the i-th SR^+-move. Then $\deg \Delta_L(t)$ is the sum of the degree $\deg f_{i,k}$ of a factor $\{(1-t)^{m_{i,k}} - (-t)^{q_{i,k}}\}\{(1-t)^{m_{i,k}} - (-t)^{m_{i,k}-q_{i,k}}\}$ from Theorem 1.14, where $q_{i,k}$ is an integer with $0 \leq q_{i,k} \leq m_{i,k}/2$. Note that $\deg f_{i,k}$ is $2m_{i,k} - 2$ if $q_{i,k} = 0$ and $2m_{i,k}$ if $q_{i,k} \neq 0$ and that $\deg \Delta_K(t)$ is 4, since $\Delta_K(t) = 2t^4 - 6t^3 + 9t^2 - 6t + 2$. Therefore if K is obtained from the trivial knot by a finite sequence of SR^+-moves, then $\Delta_K(t)$ is one of the following:

- $\pm t^r\{(1-t)^3 - (-t)^0\}\{(1-t)^3 - (-t)^3\}$;
- $\pm t^r\{(1-t)^2 - (-t)^1\}\{(1-t)^2 - (-t)^1\}$; and
- $\pm t^r\{(1-t)^2 - (-t)^0\}\{(1-t)^2 - (-t)^2\}\{(1-t)^2 - (-t)^0\}\{(1-t)^2 - (-t)^2\}$.

The coefficient of the lowest term of the above three cases are 3, 4, and 1, respectively. Thus we obtain a contradiction. $\square$

Take an SR-tangle $\mathcal{T}$ and let p (≥ 0) be the maximal number of mutually disjoint non-singular disks $F_1 \cup \cdots \cup F_p$ proper in $H - \mathcal{T}$ such that each component of

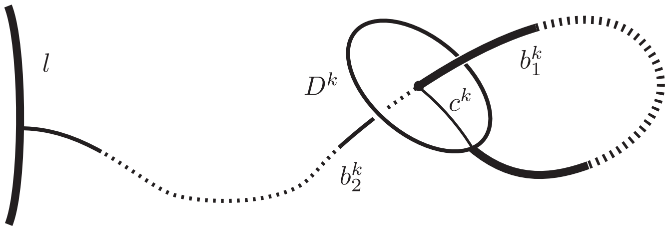


Fig. 7.

$H-(F_1\cup\cdots\cup F_p)$ contains a component of $\mathcal{T}$. Then we define the *number of non-separable components* $X(\mathcal{T})$ of $\mathcal{T}$ by $p+1$. An *SR*-tangle $\mathcal{T}$ is said to be *separable* if $X(\mathcal{T})\geq 2$. An *SR*-tangle with n components ($n\geq 2$) is said to be *completely separable* if $X(\mathcal{T})=n$.

Consider an *SR*-move of class I on a link l. Then each band $B^k=B_1^k$ can be regarded as $(b_1^k\cup b_2^k)\times[-1,1]$, where b_1^k (resp. b_2^k) is an arc with ends on $\operatorname{int} D^k$ and on ∂D^k (resp. l). Let c^k be an arc on D^k with $\partial c^k=\partial b_1^k$ (see Fig. 7). We call $\mathcal{J}=\bigcup J^k=\bigcup(b_1^k\cup c^k)$ the *attendant link* of the *SR*-move or of the *SR*-tangle. We say that $\mathcal{J}$ is *completely split* if there is a union $\mathcal{M}=M_1\cup\cdots\cup M_n$ ($n\geq 2$) of mutually disjoint non-singular 3-balls in H such that $M_k\cap\mathcal{J}=b_1^k\cup c_k$ for each k. It is easy to see that if an *SR*-tangle is completely separable, then the attendant link of the *SR*-tangle is completely split. Then we have the following.

Theorem 1.19. *Let $T=\partial(B_1^k\cup D_1^k)-\operatorname{int}(B_1^k\cap\partial H)$ be a string with $m_k=1$ of an SR-tangle $\mathcal{T}$. If T is trivial, then there is a non-singular disk proper in $H-(\mathcal{B}\cup\mathcal{D})$ which bounds a* 3*-ball N in H with a subdisk of ∂H such that $N\cap(\mathcal{B}\cup\mathcal{D})=B_1^k\cup D_1^k$.*

Corollary 1.20. *If an SR-tangle is reducible, then it is separable.*

Note that each component with index no less than 2 is irreducible from Corollary 1.16. For an *SR*-tangle of class I with no less than 2 components, we have the following. Here note that an *SR*-tangle of type (1) is trivial (see [3] for instance).

Corollary 1.21. *An SR-tangle of class* I *with n components* ($n\geq 2$) *is s trivial if and only if it is completely separable.*

Corollary 1.22. *Let L be a link obtained from a non-split link l by an SR^+-move of class* I *with n components* ($n\geq 2$). *If its attendant link is not completely split, then L is not ambient isotopic to l.*

REMARK 1.23. (1) The knot K_1 ($\approx 9_{27}$) illustrated in the leftside of Fig. 8 can be transformed into the trivial knot by an SR^--move of type (3). However since $\Delta_{K_1}(t)=$

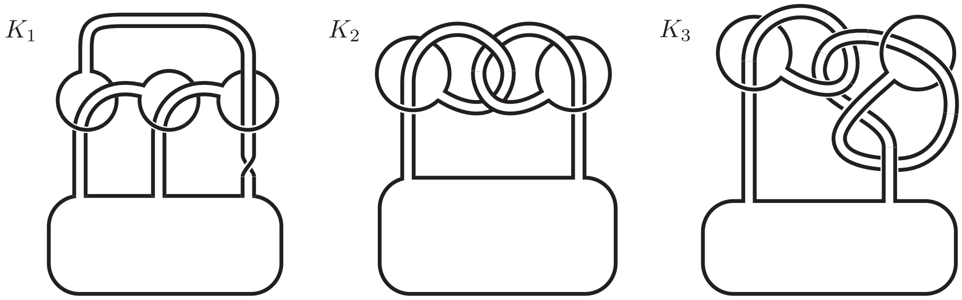


Fig. 8.

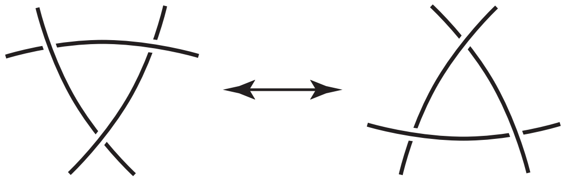


Fig. 9.

$t^6 - 5t^5 + 11t^4 - 15t^3 + 11t^2 - 5t + 1$, K_1 cannot be transformed into the trivial knot by a finite sequence of *SR*-moves of class I by Theorem 1.14.
(2) We can obtain a non-trivial knot whose Alexander polynomial is 1 by using Theorem 1.14 and Corollary 1.22 (see the knot K_2 illustrated in the middle of Fig. 8 for an example).
(3) There is an *SR*-move whose attendant link is completely split, but whose *SR*-tangle is not completely separable. The *SR*-move in the right-side of Fig. 8 illustrates such a case. The knot K_3 is not trivial, since the Jones polynomial of K_3 is not 1. Thus the *SR*-tangle is not completely separable by Corollary 1.21.

The move on a link as illustrated in Fig. 9 is called the Δ-*move*. If the three strands on the figure belong to the same component, then the move is called the *self* Δ-*move*. Two links are said to be *self* Δ-*equivalent* if one can be transformed into the other by a finite sequence of self Δ-moves and ambient isotopy.

We say that a component $\mathcal{T}^k$ of an *SR*-move on a link l is *distinct* if $\mathcal{B}^k \cap l = (B_1^k \cup \cdots \cup B_{m_k}^k) \cap l$ belong to distinct m_k components of l. Then we have the following.

Theorem 1.24. *If two links can be transformed one into the other by a finite sequence of SR-moves each of whose components has $m_k = 1$ or is not distinct, then the two links are self Δ-equivalent.*

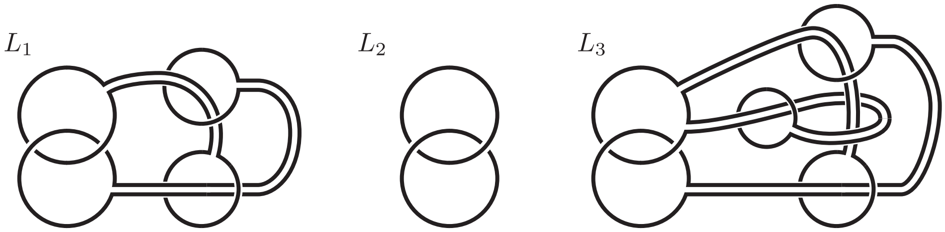


Fig. 10.

Corollary 1.25. *If two links can be transformed one into the other by a finite sequence of SR-moves of class* I, *then the two links are self* Δ*-equivalent.*

REMARK 1.26. There is a pair of links such that they can be transformed one into the other by a distinct *SR*-move and not self Δ-equivalent. For example, let L_1 and L_2 be two links as illustrated in Fig. 10. Then L_1 can be transformed into L_2 by a distinct SR^--move, but they are not self Δ-equivalent [7].

REMARK 1.27. Theorem 1.24 does not hold for an *SR*-move which is not distinct, where an *SR*-move on a link l is *distinct* if its *SR*-tangle $(\partial\mathcal{D}\oplus\partial\mathcal{B})\cap H$ satisfies that $\mathcal{B}\cap l$ belongs to distinct $\sum m_k$ components of l, where $\oplus$ means the homological addition. For example, the link L_3 as illustrated in Fig. 10 can be transformed into the Hopf link L_2 by an SR^--move which is not distinct (note that each of two components is distinct), but L_2 and L_3 are not self Δ-equivalent. This is because L_3 and L_1 are self Δ-equivalent, which is easy to see, and L_1 and L_2 are not self Δ-equivalent from Remark 1.26.

2. Simple ribbon moves and link types

Let L be a link obtained from a link l by an SR^+-move. Let E be a Seifert surface for L with $h(E)=h(L)$. In this section, we prove Theorem 1.8 and Theorem 1.13.

We analyze the intersections of E and $\mathcal{D}\cup\mathcal{B}$. We may assume that $\operatorname{int}E$ and $\operatorname{int}(\mathcal{D}\cup\mathcal{B})$ intersects transversely. Then the singular points of $\operatorname{int}E\cup\operatorname{int}\mathcal{D}\cup\operatorname{int}\mathcal{B}$ consists of double points of a pair among $\mathcal{B}$, $\mathcal{D}$, and E and triple points of $\mathcal{B}$, $\mathcal{D}$, and E, since each surface is non-singular. Note that $\mathcal{S}(\mathcal{D}\cup\mathcal{B})$ consists of mutually disjoint arcs $\bigcup_{i,k}\alpha_i^k$, where α_i^k is the singularity of $\mathcal{S}(D_i^k\cup B_{i-1}^k)$. Let $f_{\mathcal{C}}\colon \left(\bigcup_{i,k}D_i^{k*}\right)\cup\left(\bigcup_{i,k}B_i^{k*}\right)\to S^3$ be an immersion of a disk such that $f_{\mathcal{C}}(D_i^{k*})=D_i^k$ and $f_{\mathcal{C}}(B_i^{k*})=B_i^k$. We denote $\left(\bigcup_{i,k}D_i^{k*}\right)$ (resp. $\left(\bigcup_{i,k}B_i^{k*}\right)$) by $\mathcal{D}^*$ (resp. $\mathcal{B}^*$). We denote the pre-image of α_i^k on D_i^{k*} (resp. B_{i-1}^{k*}) by $\dot{\alpha}_i^{k*}$ (resp. $\ddot{\alpha}_i^{k*}$). Let $\mathcal{S}^*$ be the set of pre-images on $\mathcal{D}^*\cup\mathcal{B}^*$ of $\overline{\mathcal{S}(E\cup\mathcal{D}\cup\mathcal{B})-\bigcup_{i,k}\alpha_i^k}$. Then $\mathcal{S}^*$ is a set of mutually disjoint simple loops and simple arcs, and we denote an element of $\mathcal{S}^*$ by γ^*, for example.

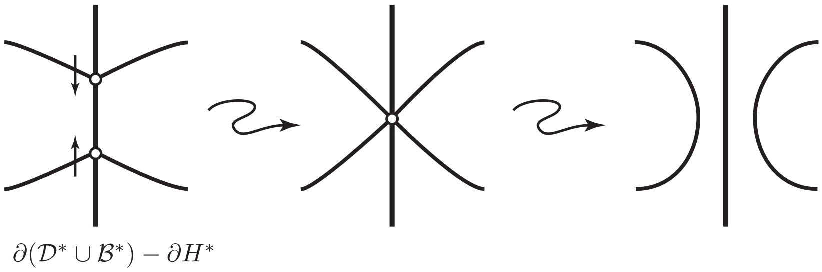


Fig. 11.

Define the *complexity* of E as the lexicographically ordered set (s, t, u), where s (resp. t) is the number of arcs (resp. loops) of $\mathcal{S}^*$ and u is the number of triple points in $\mathcal{S}(E \cup \mathcal{D} \cup \mathcal{B})$. In the following, we omit the index k unless we need to emphasize it. Let $B_{i,1}$ and $B_{i,2}$ be the disks such that $B_{i,1} \cup B_{i,2} = B_i$, $B_{i,1} \cap B_{i,2} = \alpha_{i+1}$, and an end of $B_{i,1}$ is on ∂D_i.

Lemma 2.1. *$\mathcal{S}^*$ does not have a loop which bounds a disk on $D_i^* \cup B_{i,1}^*$ containing exactly one end of $\dot{\alpha}_i^*$.*

Proof. Assume that there is such a loop γ^* in $\mathcal{S}^*$. Then $\gamma = f_C(\gamma^*)$ is a simple closed curve on $D_i \cup B_{i,1}$ which bounds a disk intersecting with L in one point, and thus $\mathrm{lk}(\gamma, L) = 1$. However since γ is also on $\mathrm{int}\, E$, γ^+ does not intersect with E, where γ^+ is γ pushed into the positive normal direction of E. Thus $\mathrm{lk}(\gamma, L) = \mathrm{lk}(\gamma^+, L) = 0$. This is a contradiction. □

Lemma 2.2. *Assume that E has the minimal complexity. Then, $\mathcal{S}^*$ does not have a loop which bounds a disk δ^* on $D_i^* \cup B_i^*$ with $\delta^* \cap \dot{\alpha}_i^* = \emptyset$ and $\delta^* \cap \ddot{\alpha}_{i+1}^* = \emptyset$.*

Proof. Assume that there is such a loop in $\mathcal{S}^*$ and take one γ^* which is innermost on $D_i^* \cup B_i^*$. Then γ bounds a disk on E, since $h(E) = h(L)$ and thus E is incompressible in $E(L)$. By replacing a neighborhood of γ on E with two parallel copies of δ, we obtain a sphere and another Seifert surface E' for L with $h(E') = h(L)$ whose complexity is less than that of E, which is a contradiction. □

An end of an arc γ^* of $\mathcal{S}^*$ on $\partial(\mathcal{D}^* \cup \mathcal{B}^*) - \partial H^*$ is a branch point p^*. Here we isotop E so that there exist no branch points on $\partial\ddot{\alpha}_{i+1}^* \times (-1, 1)$. Define the *orientation* of p^* as the orientation of γ^* around p^* induced by the orientation of E. We say that the orientations of two branch points which are adjacent on $\partial(\mathcal{D}^* \cup \mathcal{B}^*) - \partial H^*$ *match* if the same (positive or negative) sides face each other. If the orientations match, then we can isotop E to eliminate the branch points as illustrated in Fig. 11, where $\circ$ means a branch point.

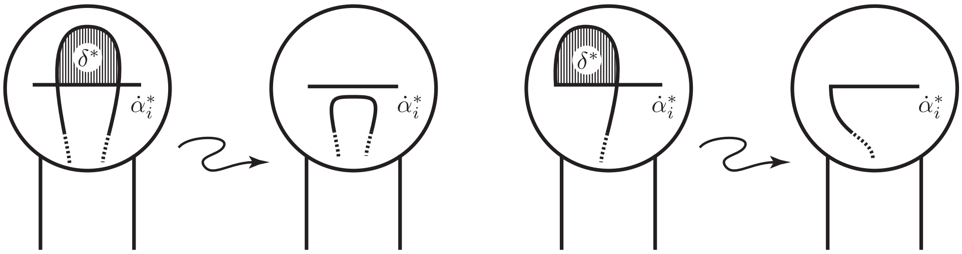


Fig. 12.

Lemma 2.3. *Assume that E has the minimal complexity. Then, there does not exist an arc of $\mathcal{S}^*$ on $D_i^* \cup B_{i,1}^*$ whose ends are on $\partial(D_i^* \cup B_{i,1}^*) - \ddot\alpha_{i+1}^*$ and which bounds a disk δ^* on $D_i^* \cup B_{i,1}^*$ with an arc on $\partial(D_i^* \cup B_{i,1}^*) - \ddot\alpha_{i+1}^*$ such that $\delta^* \cap \dot\alpha_i^* = \emptyset$.*

Proof. Assume that there is such an arc and take an innermost one γ^* on $D_i^* \cup B_{i,1}^*$, that is, there are no such arcs in δ^*. Then $\operatorname{int}\delta^*$ does not contain any loops of S^* from Lemma 2.2, and the ends of γ^* are adjacent on $\partial(D_i^* \cup B_{i,1}^*) - \ddot\alpha_{i+1}^*$ and the orientations of the two branch points match. Therefore eliminating the pair of branch points, we obtain a loop from γ^*, which contradicts that E has the minimal complexity. □

Proposition 2.4. *Assume that E has the minimal complexity and let γ^* be an arc in $\mathcal{S}^*$ such that $\partial\gamma^* \cap \partial\dot\alpha_i^* \neq \emptyset$. Then $\partial\gamma^* = \partial\dot\alpha_i^*$ and $\operatorname{int}\gamma^* \cap \operatorname{int}\dot\alpha_i^* = \emptyset$.*

Proof. Take a straight line β^* which is proper in D_i^* and contains $\dot\alpha_i^*$. Then γ^* does not have a subarc in the closure of a component of $D_i^* - \beta^*$ whose ends are on $\dot\alpha_i^*$ and at least one end is on $\operatorname{int}\dot\alpha_i^*$, and which bounds a disk δ^* on D_i^* with a subarc of $\dot\alpha_i^*$ such that $\operatorname{int}\delta^* \cap \mathcal{S}^* = \emptyset$. Assume otherwise. Then we can isotop E to reduce the complexity of E (see Fig. 12), which is a contradiction. Thus we have the following.

Claim 2.5. *γ^* does not have a subarc in the closure of a component of $D_i^* - \beta^*$ whose ends are on $\dot\alpha_i^*$ and at least one end is on $\operatorname{int}\dot\alpha_i^*$.*

We also have the following.

Claim 2.6. *$\mathcal{S}^*$ does not have an arc which has its ends on $\partial D_i^* - \partial B_i^*$ and intersects with $\dot\alpha_i^*$ once.*

Proof. If exists, then take an outermost one γ_1^*, that is, γ_1^* bounds with a subarc of $\partial D_i^* - \partial B_i^*$ a disk δ^* in whose interior there does not exist an arc of $\mathcal{S}^*$ intersecting with $\dot\alpha_i^*$. From Lemmas 2.1, 2.2, 2.3 and Claim 2.5, there exists only one element of $\mathcal{S}^*$ in δ^*, say γ_2^*, whose ends are on $\partial\dot\alpha_i^*$ and ∂D_i^*. Let $\partial\gamma_1^* = p_1^* \cup p_2^*$ and let $\gamma_2^* \cap \partial D_i^* = p_3^*$ (see Fig. 13). Then p_3^* is adjacent to both of p_1^* and p_2^* on $\partial D_i^* - \partial B_i^*$.

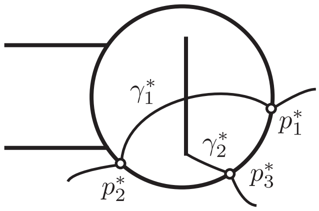


Fig. 13.

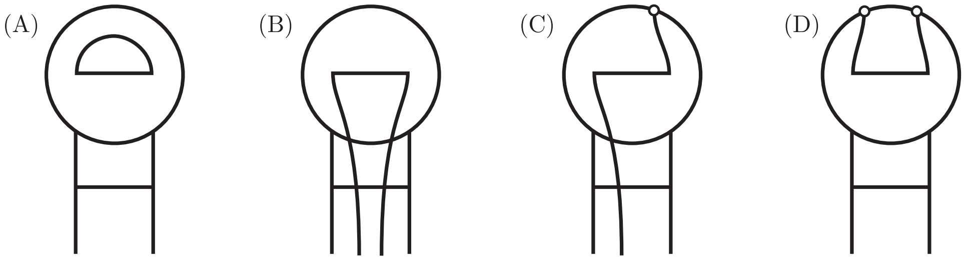


Fig. 14.

Thus the orientations of either p_3^* and p_1^*, or p_3^* and p_2^* match, and hence we can eliminate the pair of branch points whose orientations match to reduce the complexity of E, which is a contradiction. $\square$

Let P^* be a point of $\partial\gamma^* \cap \partial\dot{\alpha}_i^*$. Let β_1^* be the arc of $\overline{\beta^* - \dot{\alpha}_i^*}$ with $\beta_1^* \cap \dot{\alpha}_i^* = P^*$, β_2^* the other arc of $\overline{\beta^* - \dot{\alpha}_i^*}$, and $Q^* = \beta_2^* \cap \dot{\alpha}_i^*$. Let γ'^* be the arc of $\mathcal{S}^*$ with one of its ends on Q^*. Then we have the following.

Claim 2.7. *Rotating B_{i-1} around α_i properly, we may assume that γ^* (and γ'^*) is as illustrated in* Fig. 14 (A), (B), *or* (C).

Proof. Starting from P^* (resp. Q^*), we read the intersection data Δ_γ (resp. $\Delta_{\gamma'}$) of $\operatorname{int}\gamma^*$ (resp. $\operatorname{int}\gamma'^*$) with β^*, which is a sequence consisting of $\operatorname{int}\beta_1^*$, $\operatorname{int}\dot{\alpha}_i^*$, and $\operatorname{int}\beta_2^*$. Note that none of the three entries appears consecutively, since otherwise we can eliminate these intersections by isotoping E similarly to the proof of Claim 2.5.

If both of Δ_γ and $\Delta_{\gamma'}$ are empty, then clearly we can transform γ^* and γ'^* into the position of (A), (B), (C), or (D) by rotating B_{i-1} around α_i properly. Thus we may assume at least one of Δ_γ and $\Delta_{\gamma'}$ is not empty. In either case, we have a symmetric conclusion, which is resolved by rotating B_{i-1} around α_i properly. Hence we may assume that Δ_γ is not empty.

We know that the first entry of Δ_γ is not $\operatorname{int}\dot{\alpha}_i^*$ from Claim 2.5. If the first entry of Δ_γ is $\operatorname{int}\beta_1^*$, then we can eliminate the intersection by isotoping E similarly to the proof of Claim 2.5. Thus the first entry of Δ_γ is $\operatorname{int}\beta_2^*$. Then the second entry of Δ_γ is $\operatorname{int}\dot{\alpha}_i^*$, $\operatorname{int}\beta_1^*$ or empty. In the first case, tracing γ^* further similarly to the above, we know that Δ_γ is $\operatorname{int}\beta_2^*, \operatorname{int}\dot{\alpha}_i^*, \operatorname{int}\beta_2^*, \operatorname{int}\dot{\alpha}_i^*, \ldots$, and the other end of γ^* is Q^*. However this contradicts Claim 2.5. In the second case, also tracing γ^* further similarly to the above, we know that Δ_γ is $\operatorname{int}\beta_2^*$, $\operatorname{int}\beta_1^*$, $\operatorname{int}\beta_2^*$, $\operatorname{int}\beta_1^*, \ldots$ and the other end of γ^* is on ∂D_i^*. Then, similarly, we have that $\Delta_{\gamma'}$ is $\operatorname{int}\beta_1^*$, $\operatorname{int}\beta_2^*$, $\operatorname{int}\beta_1^*$, $\operatorname{int}\beta_2^*, \ldots$ and the other end of γ'^* is on ∂D_i^*. Thus rotating B_{i-1} around α_i properly, γ^* (and γ'^*) is as illustrated in Fig. 14 (A), (B), (C), or (D). The third case is similar to the second case, since in this case Δ_γ is $\operatorname{int}\beta_2^*$ and the other end of γ^* is on ∂D_i^*.

If γ^* (and γ'^*) is as illustrated in Fig. 14 (D), then let δ^* be the disk bounded by γ^*, $\dot{\alpha}_i^*$, γ'^*, and a subarc of $\partial D_i^* - \partial B_i^*$. Then the elements of $S^* \cap \delta^*$ are arcs each of which has ends on ∂D_i^* and on $\dot{\alpha}_i^*$ from Lemmas 2.2, 2.3 and Claim 2.5. Note that $S^* \cap \delta^*$ has γ^* and γ'^*, and thus $S^* \cap \delta^*$ is not empty. Then there exists at least one adjacent pair of branch points on $\partial\delta^* \cap \partial D_i^*$ whose orientation match. Hence we can eliminate the pair of branch points to reduce the complexity of E, which is a contradiction. We complete the proof of Claim 2.7. □

Proof of Proposition 2.4 (*continued*). Our goal is to show that γ^* $(= \gamma'^*)$ is as illustrated in Fig. 14 (A). We work on this task by dividing it into two cases: $m_k = 1$; and $m_k > 1$.

First consider the case when $m_k = 1$. Take a look at $\mathcal{S}^* \cap \dot{\alpha}_i^{k*}$. If γ^* and γ'^* are as illustrated in Fig. 14 (B), then let $\ddot{P}_1^* = \gamma^* \cap \ddot{\alpha}_1^{k*}$ (resp. $\ddot{Q}_1^* = \gamma'^* \cap \ddot{\alpha}_1^{k*}$). Thus there is $\dot{P}_1^*$ (resp. $\dot{Q}_1^*$) on $\dot{\alpha}_1^{k*}$ (see Fig. 15 (a)). Take a look at the subarc α' of α_1^k bounded by P_1 and Q_1, and let p be the number of intersections of $\operatorname{int}\alpha' \cap E$. Thus there are p arcs of $\mathcal{S}^*$ which intersect with $\operatorname{int}\dot{\alpha}'^* \subset \dot{\alpha}_1^{k*}$. From Claim 2.5, these p arcs also intersect with $\operatorname{int}\ddot{\alpha}'^*$. However, also from Claim 2.5, the arc which intersects with $\dot{\alpha}_1^{k*}$ in $\dot{P}_1^*$ (resp. $\dot{Q}_1^*$) intersects with $\ddot{\alpha}_1^{k*}$, in fact, intersects with $\operatorname{int}\ddot{\alpha}'^*$. This induces that the number of intersections on $\operatorname{int}\ddot{\alpha}'^*$ is no less than $p + 2$, which is a contradiction.

If γ^* and γ'^* are as illustrated in Fig. 14 (C), then we may assume that the orientations of $\dot{\alpha}_1^{k*}$ and $\ddot{\alpha}_1^{k*}$ coincide, i.e., $\dot{P}^*$ and $\ddot{P}^*$ (resp. $\dot{Q}^*$ and $\ddot{Q}^*$) are on the leftside (resp. the rightside) of $\dot{\alpha}_1^{k*}$ and $\ddot{\alpha}_1^{k*}$ in the figure, respectively. Let $\ddot{P}_1^* = \gamma^* \cap \ddot{\alpha}_1^{k*}$. Take a look at the subarc α' of α_1^k bounded by P and P_1, and let p be the number of intersections of $\operatorname{int}\alpha' \cap E$. The arc which is in the component of $(D_1^{k*} \cup B_1^{k*}) - (\gamma^* \cup \dot{\alpha}_1^{k*} \cup \gamma'^*)$ containing $\ddot{\alpha}'^*$ and intersects with $\dot{\alpha}_1^{k*}$ in $\dot{P}_1^*$ intersects with $\ddot{\alpha}'^*$ (Fig. 15 (b)) or $\partial D_1^{k*} - \partial B_1^{k*}$ (Fig. 15 (c)). In the former case, let p be the number of intersections of $\operatorname{int}\alpha' \cap E$. Then the p arcs intersect with $\dot{\alpha}'^*$ also intersect with $\ddot{\alpha}'^*$ from Claim 2.5, and thus the number of intersections on $\operatorname{int}\ddot{\alpha}'^*$ is no less than $p+1$, which is a contradiction. In the latter case, let α'' be the subarc of α_1^k bounded by P_1 and Q, and q the number of intersections of $\operatorname{int}\alpha'' \cap E$. However then, the arc which intersects with $\dot{\alpha}_1^{k*}$ in $\dot{P}_1^*$ and the q arcs which intersect with $\dot{\alpha}''^*$ intersect with $\operatorname{int}\ddot{\alpha}''^*$ from Claim 2.6, which induces a contradiction similarly to the former case.

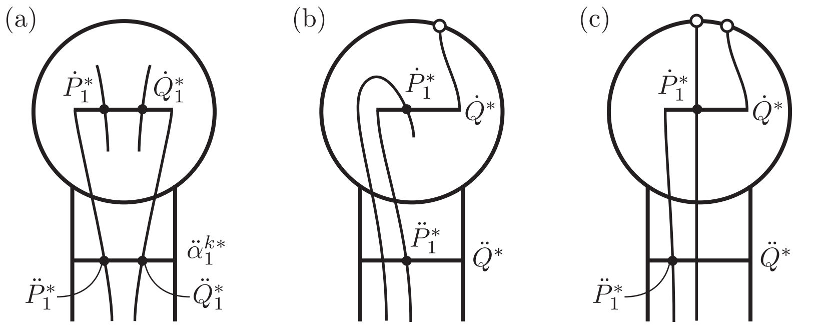


Fig. 15.

Next consider the case when $m_k > 1$. Let γ_i^{k*} and ρ_i^{k*} be the arcs of $\mathcal{S}^*$ with an end on $\partial\dot\alpha_1^{k*}$ $(1 \le i \le m_k)$. If γ_i^{k*} and ρ_i^{k*} are as illustrated in Fig. 14 (B) or (C), then either γ_i^{k*} or ρ_i^{k*} intersects with $\ddot\alpha_{i+1}^{k*}$. Thus γ_{i+1}^{k*} and ρ_{i+1}^{k*} are as illustrated in Fig. 14 (B) or (C), since $\mathcal{S}^* \cap \dot\alpha_{i+1}^{k*} = \emptyset$ from Claim 2.5 in the case of Fig. 14 (A). Therefore, γ_i^{k*} and ρ_i^{k*} are as illustrated in Fig. 14 (B) or (C) for each i $(1 \le i \le m_k)$. Then let p be the number of intersections on $\operatorname{int}\dot\alpha_1^{k*}$ with $\mathcal{S}^*$. Since an arc that intersects with $\operatorname{int}\dot\alpha_1^{k*}$ intersects with $\ddot\alpha_2^{k*}$ and either γ_1^{k*} or ρ_1^{k*} intersects with $\ddot\alpha_2^{k*}$, there are no less than $p+1$ intersections on $\operatorname{int}\dot\alpha_2^{k*}$ with $\mathcal{S}^*$. Then inductively we have that there are no less than $p+m_k$ intersections on $\operatorname{int}\dot\alpha_{m_k+1}^{k*} = \operatorname{int}\dot\alpha_1^{k*}$ with $\mathcal{S}^*$, which is a contradiction. □

Proof of Theorem 1.8. Let E be a Seifert surface for L with $h(E) = h(L)$ and with minimal complexity. From Proposition 2.4, each arc γ_i^{k*} of $\mathcal{S}^*$ with an end on $\partial\dot\alpha_i^{k*}$ satisfies that $\partial\gamma_i^{k*} = \partial\dot\alpha_i^{k*}$ and $\operatorname{int}\gamma_i^{k*} \cap \operatorname{int}\dot\alpha_i^{k*} = \emptyset$. Therefore $\operatorname{int}\gamma_i^{k*}$ and $\dot\gamma_i^{k*}$ are as illustrated in Fig. 16, where $\dot\gamma_i^{k*}$ is the arc on E^* such that $f_C(\dot\gamma_i^{k*}) = \gamma$. Let ξ_i^{k*} be $\partial(D_i^{k*} \cup B_{i,1}^{k*}) - \operatorname{int}\ddot\alpha_{i+1}^{k*}$. Note that ξ_i^k is a subarc of L.

Note that $\mathcal{S}^*$ may have arcs and loops on $D_i^{k*} \cup B_{i,1}^{k*}$. From Lemma 2.3 and Proposition 2.4, such an arc on $D_i^{k*} \cup B_{i,1}^{k*}$ which is not γ_i^{k*} has its ends on ξ_i^{k*} and bounds a disk ε^* on $D_i^{k*} \cup B_{i,1}^{k*}$ with a subarc of ξ_i^{k*} such that ε^* contains $\dot\alpha_i^{k*}$. If there exists such an arc, then we can transform it into a loop by eliminating an innermost pair of branch points as the proof of Lemma 2.3. Then we obtain a contradiction that E has the minimal complexity. From Lemma 2.2, each loop of $\mathcal{S}^*$ on $D_i^{k*} \cup B_{i,1}^{k*}$ bounds a disk ε^* containing $\dot\alpha_i^{k*}$. In fact, we may assume that such a loop is on D_i^{k*} by ambient isotopy.

We construct a Seifert surface F for l from E. Note that $\mathcal{S}^*$ now consists of arcs $\bigcup_{i,k}\gamma_i^{k*}$ $(1 \le i \le m_k,\ 1 \le k \le n)$ and loops, say ρ_j^* $(1 \le j \le r)$, and that each γ_i^{k*} bounds a disk δ_i^{k*} on D_i^{k*} with $\dot\alpha_i^{k*}$ and each ρ_j^* bounds a disk ε_j^* on a disk of $\mathcal{D}^*$. Replace a neighborhood of γ_i^k on E with two parallel copies of δ_i^k, and replace

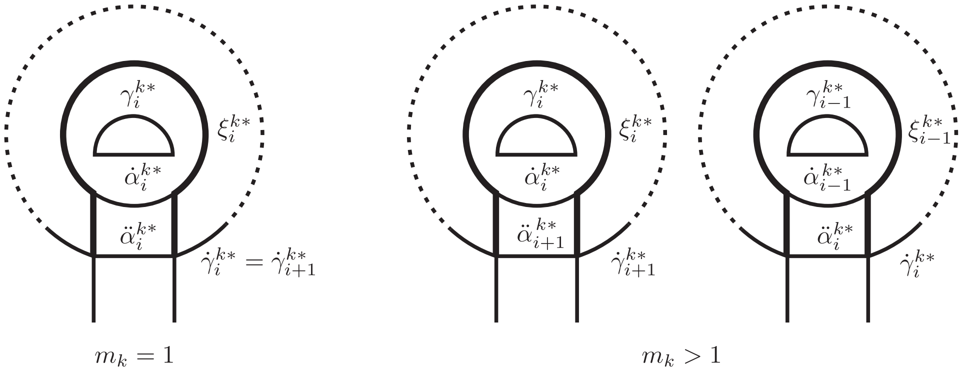


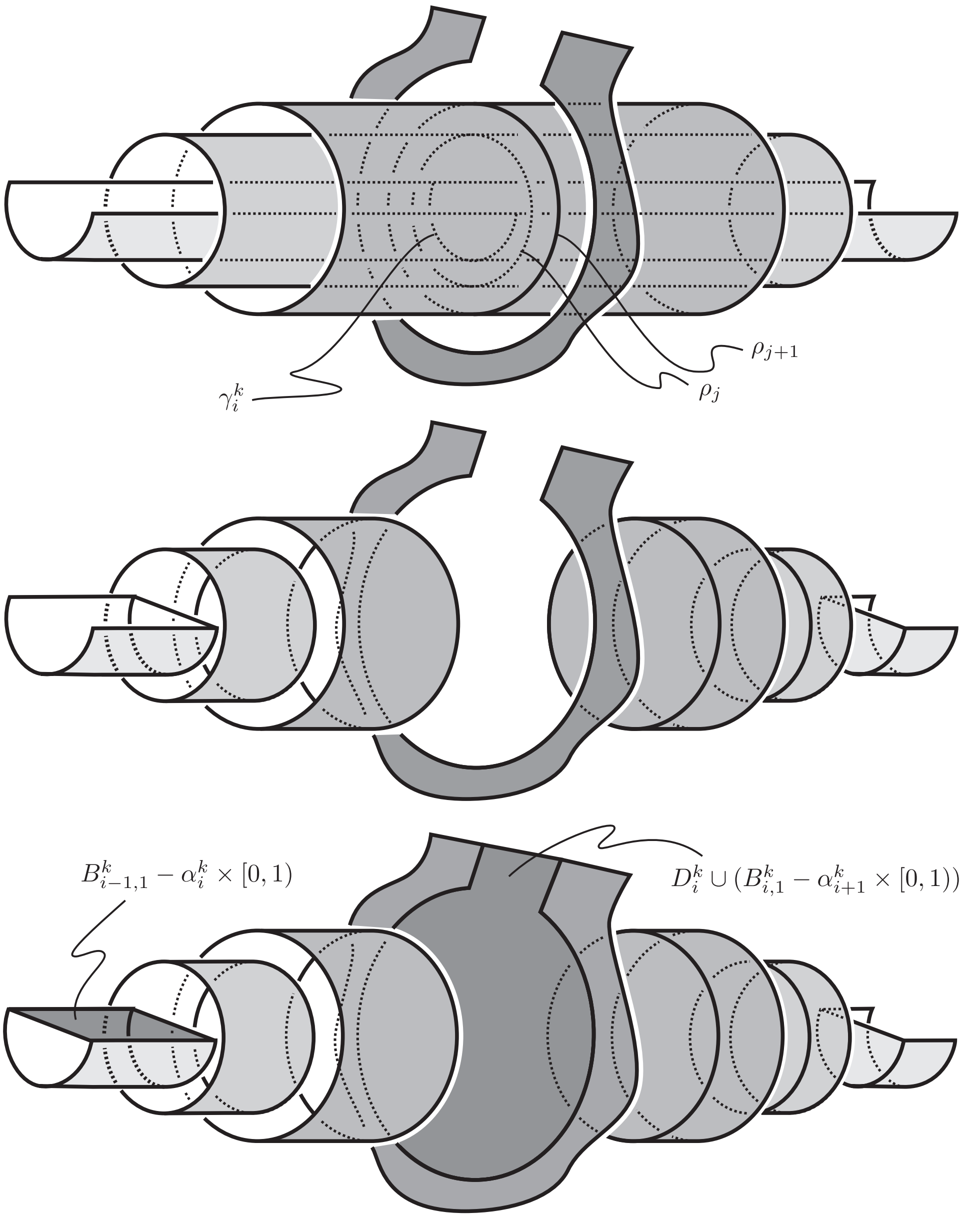
Fig. 16.

a neighborhood of ρ_j on E with two parallel copies of ε_j, where we operate the replacement and these cancellations on each disk D_i^k from the innermost one on the disk so to have a non-singular surface (see Fig. 17). Let the result surface be E' and let F be the result obtained from $E' \cup \bigcup_{i,k}(D_i^k \cup (B_{i,1}^k - \alpha_{i+1}^k \times [0, 1)))$ by removing the closed components.

Since $\chi(E') = \chi(E) + \sum_k m_k + 2r$, we have that $\chi(F) = \chi(E) + 2(\sum_k m_k + r) - \sum_t \chi(F_t)$, where F_t is a closed component which was removed above. Since $\chi(F_t) \leq 2$ and $\sum_t 1 \leq \sum_k m_k + r$, we have that $\chi(F) \geq \chi(E)$, and thus $h(l) \leq h(L)$. The equality $h(l) = h(L)$ holds only when $\sum_k m_k + r$ spheres are removed when we construct F from E', which implies that $\xi_i^k \cup \gamma_{i+1}^k$ bounds a disk on E for any pair of i and k. Thus our *SR*-move is trivial. For if $\xi_i^k \cup \gamma_{i+1}^k$ bounds a disk on E, then ξ_i^k is ambient isotopic to γ_{i+1}^k along the disk. Since γ_{i+1}^k and α_{i+1}^k bounds a subdisk of D_{i+1}^k, γ_{i+1}^k is ambient isotopic to α_{i+1}^k. This implies that T_i^k of the *SR*-move is trivial. Therefore we can conclude that condition (1) implies condition (3), which is sufficient to show the last part of the statement, and thus we complete the proof. □

Proof of Theorem 1.13. Suppose L is split. Namely, there is a 2-sphere Σ in S^3 such that $\Sigma \cap L = \emptyset$ and each connected component of $S^3 - \Sigma$ contains a component of L. Here let Σ split L into L_1 and L_2. Let $\mathcal{S}^*$ be the set of pre-images on $\mathcal{D}^* \cup \mathcal{B}^*$ of $\overline{\mathcal{S}(\Sigma \cup \mathcal{D} \cup \mathcal{B}) - \bigcup_{i,k} \alpha_i^k}$. Then $\mathcal{S}^*$ is a set of mutually disjoint simple loops and simple arcs, where note that each arc has its ends on $\mathcal{B}^* \cap \partial H^*$. Define the *complexity* of Σ as the lexicographically ordered set (s, t, u), where s (resp. t) is the number of arcs (resp. loops) of $\mathcal{S}^*$ and u is the number of triple points in $\mathcal{S}(\Sigma \cup \mathcal{D} \cup \mathcal{B})$. Here we may assume that Σ has the minimal complexity.

Note that the numbers of components of L and l coincide, since an *SR*-move does not change the number of components. If $\Sigma \cap l = \emptyset$, then l is entirely in a component of $S^3 - \Sigma$, since l is non-split. However then, it contradicts that the numbers of components of L and l coincide. Hence $\Sigma \cap l \neq \emptyset$, and thus $\Sigma \cap (l - L) \neq \emptyset$, since



ρ_{j+1}
ρ_j
γ_i^k
$B_{i-1,1}^k - \alpha_i^k \times [0,1)$
$D_i^k \cup (B_{i,1}^k - \alpha_{i+1}^k \times [0,1))$

Fig. 17.

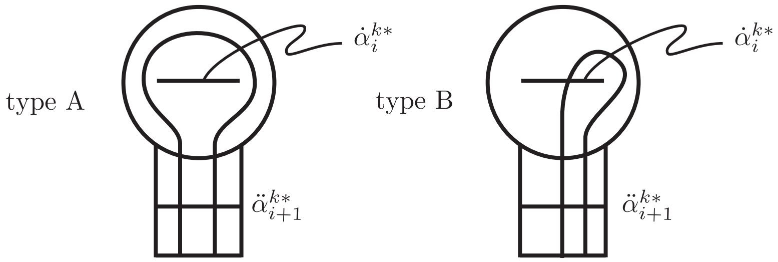


Fig. 18.

$\Sigma \cap L = \emptyset$. Therefore we obtain that $\Sigma \cap \mathcal{B} \neq \emptyset$. We have the following similarly to Lemma 2.1 and Claim 2.5.

Claim 2.8. *$\mathcal{S}^*$ does not have a loop which bounds a disk on $D_i^{k*} \cup B_{i,1}^{k*}$ containing exactly one end of $\dot{\alpha}_i^{k*}$.*

Claim 2.9. *Every element of S^* does not have a subarc which bounds with a subarc of $\dot{\alpha}_i^{k*}$ or $\ddot{\alpha}_{i+1}^{k*}$ a disk on $D_i^{k*} \cup B_i^{k*}$ whose interior does not intersect with $\dot{\alpha}_i^{k*}$ or $\ddot{\alpha}_{i+1}^{k*}$.*

Using the above claims, we have the following.

Claim 2.10. *$\mathcal{S}^*$ does not have a loop which intersects with $\dot{\alpha}_i^{k*}$ or $\ddot{\alpha}_{i+1}^{k*}$.*

Every arc of $\mathcal{S}^*$ on $D_i^{k*} \cup B_i^{k*}$ intersects with $\ddot{\alpha}_{i+1}^{k*}$. Otherwise, we can eliminate it by ambient isotopy. Then the arc is either an arc which intersects with $\ddot{\alpha}_{i+1}^{k*}$ twice and bounds with a subarc of $\ddot{\alpha}_{i+1}^{k*}$ a disk on $D_i^{k*} \cup B_i^{k*}$ which contains $\dot{\alpha}_i^{k*}$ (type A) or an arc which intersects with $\ddot{\alpha}_{i+1}^{k*}$ twice and intersects with $\dot{\alpha}_i^{k*}$ once (type B) (see Fig. 18). Here note that $\#(\dot{\alpha}_i^{k*} \cap \mathcal{S}^*) = \#(\ddot{\alpha}_i^{k*} \cap \mathcal{S}^*) = \#(\ddot{\alpha}_{i+m_k}^{k*} \cap \mathcal{S}^*)$. Then we obtain a contradiction, since we have Claim 2.10 and that $\#(\dot{\alpha}_i^{k*} \cap \gamma_i^{k*}) < \#(\ddot{\alpha}_{i+1}^{k*} \cap \gamma_i^{k*})$ in both cases of type A and type B for every arc γ_i^{k*} of $\mathcal{S}^*$ on $D_i^{k*} \cup B_i^{k*}$. Therefore, L is non-split. □

3. Reducibility of Simple ribbon moves

Consider an SR^+-move on a link l such that a string T_i^k is trivial. Let E be a non-singular disk in S^3 whose boundary is $T_i^k \cup (B_i^k \cap \partial H)$ such that $\operatorname{int} E$ does not intersect with the resultant link L. In this section, we prove Theorems 1.11 and 1.19.

Similarly to the previous section, let $\mathcal{S}^*$ be the set of pre-images on $\mathcal{D}^* \cup \mathcal{B}^*$ of $\overline{\mathcal{S}(E \cup \mathcal{D} \cup \mathcal{B}) - \bigcup_{i,k} \alpha_i^k}$ and define the complexity of E as the lexicographically ordered set (s, t, u, v), where s (resp. t) is the number of arcs (resp. loops) of $\mathcal{S}^*$, u is the number of triple points in $\mathcal{S}(E \cup \mathcal{D} \cup \mathcal{B})$, and v is the number of intersections of $E \cap \partial H$. Then we can obtain Lemmas 2.1, 2.2, 2.3, and Proposition 2.4. Moreover, we have the following.

Lemma 3.1. *If E has the minimal complexity, then* $\operatorname{int} E$ *does not intersect with* $\bigcup_{i,k} B^k_{i,2}$.

Proof. Assume that E intersects with $\bigcup_{i,k} B^k_{i,2}$. Then the intersections consists of arcs from Lemma 2.2. Moreover from Proposition 2.4, each of these arcs has its ends on $\partial B^k_{i,2} - \alpha^k_{i+1}$. Then we can transform an innermost arc into a loop by isotoping E as illustrated in Fig. 11, which contradicts that E has the minimal complexity. $\square$

Proof of Theorem 1.11. It is sufficient to show that if an *SR*-move on a non-split link is reducible, then its associated tangle is reducible. Assume that there exists a counter example, and take such an *SR*-move on a non-split link l, that is, it has a string T^k_i such that $T^k_i \cup (B^k_i \cap \partial H)$ bounds a non-singular disk E in S^3 whose interior does not intersect with the resultant link L but does not bound a non-singular disk in H whose interior does not intersect with the resultant link L.

We may assume that E has the minimal complexity. Then from Lemma 3.1, $\operatorname{int} E \cap \partial H$ consists of mutually disjoint simple loops each of which does not intersect with $\mathcal{B} \cap \partial H$, and thus $\operatorname{int} E$ does not intersect with l. From the assumption, we have that $\operatorname{int} E \cap \partial H \neq \emptyset$. Thus take a loop λ of $\operatorname{int} E \cap \partial H$ which is innermost on $\operatorname{int} E$, that is, λ bounds a disk E_λ on $\operatorname{int} E$ such that $\operatorname{int} E_\lambda \cap \partial H = \emptyset$.

Claim 3.2. *Each component of $\partial H - \lambda$ intersects with $\mathcal{B}$.*

Proof. If a component δ of $\partial H - \lambda$ does not intersect with $\mathcal{B}$, then replacing a neighborhood of λ on E with two parallel copies of the disk δ, we obtain a sphere and a non-singular disk E' whose boundary is $T^k_i \cup (B^k_i \cap \partial H)$ such that $\operatorname{int} E'$ does not intersect with L and the complexity of E' is less than that of E, which contradicts that E has the minimal complexity. $\square$

Assume that E_λ is in $S^3 - \operatorname{int} H$. From Claim 3.2, each component of $\partial H - \lambda$ intersects with $\mathcal{B}$. Thus each component of $S^3 - \operatorname{int} H - E_\lambda$ contains a component of l, since E_λ does not intersect with l. However this contradicts that l is non-split.

Next assume that E_λ is in H. From Claim 3.2, each component of $\partial H - \lambda$ intersects with $\mathcal{B}$. Thus each component of $H - E_\lambda$ contains a component of $\mathcal{T}$, since E_λ does not intersect with L. We show that we have a contradiction by an induction on the number of connected components $X(\mathcal{T})$. In the case when $X(\mathcal{T}) = 1$, that is, $\mathcal{T}$ is non-separable, we have a contradiction, since E_λ separates $\mathcal{T}$. Assuming that we have a contradiction in the case when $X(\mathcal{T}) \leq t - 1$, consider the case when $X(\mathcal{T}) = t$. Let H_1 be the closure of the component of $H - E_\lambda$ which contains T^k_i, and let $\mathcal{T}_1$ be

$H_1 \cap \mathcal{T}$. Let H_2 be the closure of $H - H_1$ and $\mathcal{T}_2$ be $H_2 \cap \mathcal{T}$. Let L' be the link obtained from l by the SR^+-move whose associated tangle is $\mathcal{T}_2$ and whose associated 3-ball is H_2. Since l is non-split, L' is non-split from Theorem 1.13. Thus $\mathcal{T}_1$ is the associated tangle of the SR-move on a non-split link L' such that $X(\mathcal{T}_1) \leq t-1$. Thus $T_i^k \cup (B_i^k \cap \partial H_1)$ bounds a non-singular disk E_1 in H_1 whose interior does not intersect with $\mathcal{T}_1$. This implies that $T_i^k \cup (B_i^k \cap \partial H)$ bounds a non-singular disk E_1 in H whose interior does not intersect with $\mathcal{T}_1$, since $H_1 \subset H$. This is a contradiction. $\square$

Proof of Theorem 1.19. Let E be a non-singular disk in H whose boundary is $T \cup (B_1^k \cap \partial H)$ and whose interior does not intersect with $\mathcal{T}$. Assume that E has the minimal complexity. By a similar argument to the end of the proof of Theorem 1.8 and by Lemma 3.1, we have that $\mathcal{S}^*$ consists of the arc γ^{k*} and loops $\bigcup_s \rho_s^*$, and that γ^{k*} bounds a disk δ_k^* on D^{k*} with $\dot{\alpha}^{k*}$ and each ρ_s^* bounds a disk ε_s^* on a disk of $D^{k(s)*}$ containing $\dot{\alpha}^{k(s)*}$.

If $\mathcal{S}^*$ does not have loops, then let E_1 and E_2 be the disks such that $E_1 \cup E_2 = E$, $E_1 \cap E_2 = \gamma^k$, and $\partial E_1 \cap \partial D_1^k \neq \emptyset$. Then $F = E \cup D_1^k \cup B_1^k$ consists of a torus $F_1 = (D_1^k - \delta_k) \cup E_1 \cup B_{1,1}^k$ and a sphere $F_2 = \delta_k \cup E_2 \cup B_{1,2}^k$, where we recall that $B_{1,1}^k$ and $B_{1,2}^k$ are the disks such that $B_{1,1}^k \cup B_{1,2}^k = B_1^k$, $B_{1,1}^k \cap B_{1,2}^k = \alpha^k$, and an end of $B_{1,1}^k$ is on ∂D_1^k. Let N_1 and N_2 be 3-manifolds in H bounded by F_1 and F_2, respectively. Then a neighborhood N on H of the union of N_1 and N_2 is a 3-ball such that $\operatorname{int} N \cap (\mathcal{B} \cup \mathcal{D}) = B_1^k \cup D_1^k$, since $N_1 \cap N_2$ is a meridian of F_1. Thus the boundary of N in H is a required non-singular disk.

If $\mathcal{S}^*$ has a loop, then let $\lambda \in \mathcal{S}$ be an innermost loop on E, and E_λ the innermost subdisk of E bounded by λ, that is, E_λ does not intersect with $\mathcal{D} \cup \mathcal{B}$. Assume that λ is on D_i^l. Here note that T_i^l of $\mathcal{T}$ is trivial through the disk which is a union of E_λ, B_i^l and the annulus in D_i^l bounded by λ and ∂D_i^l. Thus we have that $m_l = 1$ from Corollary 1.16. Now assume that $(D_1^l \times \{1\}) \cap B_{1,1}^l \neq \emptyset$ and $(D_1^l \times \{-1\}) \cap B_{1,2}^l \neq \emptyset$. If $E_\lambda \cap (D_1^l \times \{1\}) \neq \emptyset$, then $\lambda \times \{1\}$ bounds a disk (a subdisk of E_λ) in $S^3 - (D_1^l \cup B_{1,1}^l)$, which is impossible. Thus we have that $E_\lambda \cap (D_1^l \times \{-1\}) \neq \emptyset$. Let δ_λ be the subdisk of D_1^l which is bounded by λ. Then $E_\lambda \cup \delta_\lambda$ bounds a 3-ball N_λ in H which contains $B_{1,1}^l$. Since $\operatorname{int} E_\lambda$ does not intersect with $\mathcal{D} \cup \mathcal{B}$ and δ_λ intersects with $\mathcal{B}$ in α^l, we have that $N_\lambda \cap (\mathcal{D} \cup \mathcal{B}) = D_1^l \cup B_{1,1}^l$.

If $l = k$, then a neighborhood $N \subset H$ of the union of N_λ and $B_{1,2}^l$ satisfies that $\operatorname{int} N \cap (\mathcal{B} \cup \mathcal{D}) = B_1^k \cup D_1^k$. Thus the closure of $\partial N - \partial H$ is a required disk.

If $l \neq k$, then let Δ be a disk properly embedded in N_λ such that $\partial\Delta$ is a union of an arc η_E in E_λ and an arc η_D in δ_λ such that $\alpha^l \subset \eta_D$ and $\partial\eta_E = \partial\eta_D$ is on λ. Take a look at $E \cap \Delta - \{\eta_E\}$. Since $l \neq k$ and $N_\lambda \cap (\mathcal{D} \cup \mathcal{B}) = D_1^l \cup B_{1,1}^l$, $\partial E = T \cup (B_1^k \cap \partial H)$ is in $H - N_\lambda$, and thus $E \cap \Delta - \{\eta_E\}$ consists of properly embedded arcs and loops. Moreover since each loop of $E \cap D_1^l$ bounds a subdisk of D_1^l containing α^l and there exists no subdisks of E in N_λ shown as above, the ends of each arc of $E \cap \Delta - \{\eta_E\}$ are on the same component of $\eta_D - \alpha^l$. Let ξ be one of the two arcs on Δ one of

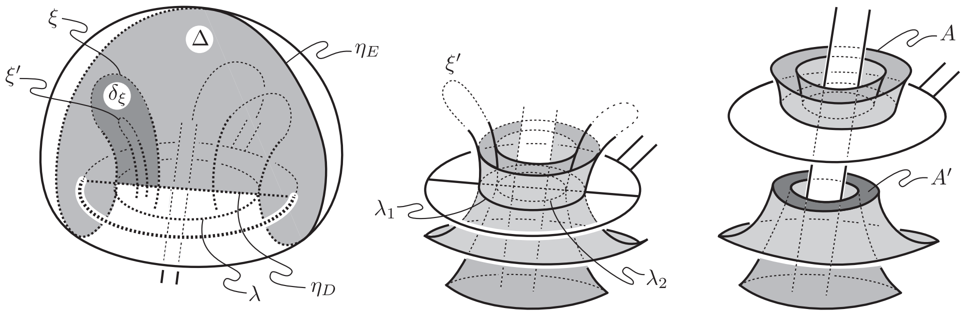


Fig. 19.

whose ends is on $\partial\eta_E = \partial\eta_D$ and let δ_ξ be the subdisk of Δ bounded by ξ and an arc on $\partial\eta_D$. Then take an outermost arc ξ' of $E \cap \Delta - \{\eta_E\}$ on δ_ξ. Let $ł_1$ and $ł_2$ be the loops of $E \cap D_1^l$ on which the ends of ξ' are. Then $ł_1 \times \{-1\}$ and $ł_2 \times \{-1\}$ bounds an annulus A on E and A' on $D_1^l \times \{-1\}$. Since A' does not intersect with E, substituting A' for A, we obtain have another non-singular disk in H whose boundary is $T \cup (B \cap \partial H)$ and which does not intersect with $\mathcal{T}$ with less complexity than that of E, which is a contradiction. □

4. Proof of Theorem 1.14

Proof of Theorem 1.14. Let $\mathcal{D} \cup \mathcal{B}$ be the union of disks and bands which gives an SR^+-move transforming l into L. Let E' be the orientable surface obtained from $\mathcal{D} \cup \mathcal{B}$ by performing an orientation preserving cut along the intersection of each pair of a disk D_{i+1}^k and a band B_i^k (see Fig. 20). Let F be a connected Seifert surface for l. If E' intersects with F, then it consists of arcs of ribbon type of $B_i^k \cap F$. Performing the orientation preserving cut along these arcs, we obtain a connected Seifert surface E for L.

Take a set of basis of the first homology $H_1(E)$ of E including a_i^k and b_i^k as illustrated in Fig. 20. Let M be a Seifert matrix of L, where a_i^{k+} and b_i^{k+} mean the lift

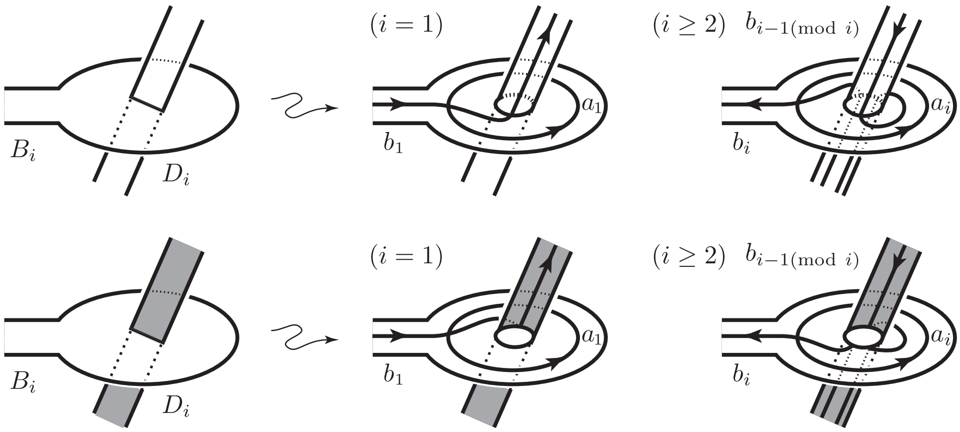


Fig. 20.

of a_i^k and b_i^k over the positive side of E, respectively. Then we have

$$\Delta_L(t) = |M - tM^t|$$

$$= \begin{array}{c} \\ a_1^1 \\ \vdots \\ a_{m_1}^1 \\ b_1^1 \\ \vdots \\ b_{m_1}^1 \\ \vdots \\ a_1^n \\ \vdots \\ a_{m_1}^n \\ b_1^n \\ \vdots \\ b_{m_1}^n \\ \\ \end{array}
\begin{array}{c}
a_1^{1+} \cdots a_{m_1}^{1+} \; b_1^{1+} \cdots b_{m_1}^{1+} \cdots a_1^{n+} \cdots a_{m_n}^{n+} \; b_1^{n+} \cdots b_{m_n}^{n+} \\
\left|\begin{array}{cc|c|cc|c}
O & A_1 & O & O & O & O \\
B_1 & * & * & O & * & * \\ \hline
O & * & \ddots & O & * & * \\ \hline
O & O & O & O & A_n & O \\
O & * & * & B_n & * & * \\ \hline
O & * & * & O & * & \Delta_l(t)
\end{array}\right|
\end{array},$$

where $A_1 = \varepsilon_1 t^{\delta_1}$ and $B_1 = -\varepsilon_1 t^{1-\delta_1}$,

$$A_i = \begin{pmatrix} \varepsilon_1 t^{\delta_1} & & & t-1 \\ t-1 & \ddots & O & \\ & \ddots & \ddots & t-1 \\ O & & t-1 & \varepsilon_{m_i} t^{\delta_{m_i}} \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} -\varepsilon_1 t^{1-\delta_1} & t-1 & & O \\ & \ddots & \ddots & \\ & O & \ddots & t-1 \\ t-1 & & & -\varepsilon_{m_i} t^{1-\delta_{m_i}} \end{pmatrix}$$

if $m_i \geq 2$, and $\delta_k = 0$, $\varepsilon_k = 1$ or $\delta_k = 1$, $\varepsilon_k = -1$. Denoting $q_k = \delta_1 + \cdots + \delta_{m_k}$, we have that $\varepsilon_1 \cdots \varepsilon_{m_k} = (-1)q_k$ and $(-1)^{m_k} \varepsilon_1 \cdots \varepsilon_{m_k} = (-1)^{m_k - q_k}$. Therefore,

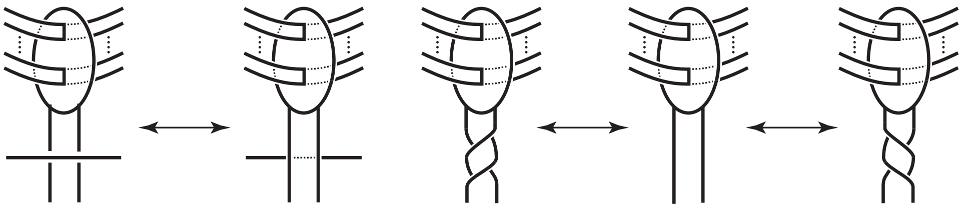
$$\begin{aligned}
\Delta_L(t) &= \pm t^r \prod_{k=p}^{n} |A_k||B_k| \Delta_l(t) \\
&= \pm t^r \prod_{k=p}^{n} \{(-1)^{m_k-1}(t-1)^{m_k} + (-t)^{q_k}\}\{(-1)^{m_k-1}(t-1)^{m_k} + (-t)^{m_k-q_k}\} \Delta_l(t) \\
&= \pm t^r \prod_{k=p}^{n} \{(1-t)^{m_k} - (-t)^{q_k}\}\{(1-t)^{m_k} - (-t)^{m_k-q_k}\} \Delta_l(t). \qquad \square
\end{aligned}$$

5. Simple ribbon moves and the self delta-equivalence

In this section, we prove Theorem 1.24. Consider an SR^+-move on a link l each index of whose component is $m_k = 1$ or is not distinct, and let L be the resultant link. We show that the SR^+-tangle can be transformed into trivial by self Δ-moves on L and ambient isotopy in H. To do this, we use $\mathcal{D} \cup \mathcal{B}$ of the SR^+-tangle. On the process, we apply self Δ-moves (resp. isotopies) on links such that we can naturally consider a substitution for $\mathcal{D} \cup \mathcal{B}$ after the moves (resp. isotopies). Thus for convenience, in such a situation, we just say, for instance, that we apply self Δ-moves and isotopies on $\mathcal{D} \cup \mathcal{B}$.

As we defined in Section 2, each band B_i^k is divided by D_{i+1}^k into two sub-bands, $B_{i,1}^k$ and $B_{i,2}^k$, where $B_{i,1}^k$ has an end on ∂D_i^k. Let $B = B_{i,1}^k$ or $B_{i,2}^k$. On the process of the proof, B may intersect with a disk D_j^l ($l < k$). Let α and α' be singularities of $B \cap D_j^l$, and B' be the sub-band of B whose ends are α and α'. We call α and α' an *innermost intersection pair* if $B' \cap \mathcal{D} = \{\alpha, \alpha'\}$, and $B' \cap (D_j^l \times (0, 1]) = \emptyset$ or $B' \cap (D_j^l \times (0, -1]) = \emptyset$. Consider the sequence of the singularities of $B \cap \mathcal{D}$ on B by reading singularities from $B \cap D_{i+1}^k = \{\alpha_{i+1}^k\}$ to the other end of B. We say that B is *well-situated with respect to* $\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^l$ if $B \cap (\mathcal{D}^{l+1} \cup \cdots \cup \mathcal{D}^n) = \{\alpha_{i+1}^k\}$ and we can reduce the sequence to $\{\alpha_{i+1}^k\}$ by removing innermost intersection pairs one by one. The following lemma has been shown in [12].

Lemma 5.1 (Lemma 2.2, [12]). *The transformations as illustrated in* Fig. 21 *are realized by* Δ*-moves.*



(a) (b)

Fig. 21.

Proof of Theorem 1.24. Let $\mathcal{D} \cup \mathcal{B}$ be the union of disks and bands for an SR^+-move on a link l each index of whose component is $m_k = 1$ or is not distinct, and let L be the resultant link. Let p be the number such that each of the k-th components ($k \leq p$) has index $m_k = 1$, and each of the k-th components ($k \geq p + 1$) has index $m_k \geq 2$. First we transform each component with index $m_k = 1$ into trivial. If $p = 0$, then go to Step 2. In the following, we denote B_1^k and D_1^k simply by B^k and D^k when $m_k = 1$.

STEP 1a: Take the 1-st component. Since the Δ-move is an unknotting operation [5], $B_{1,1}^1$ can be transformed into unknotted by using the transformations as illustrated in Fig. 21 if $B_{1,1}^1$ is knotted. Here a sub-band B whose ends are on a disk D is *unknotted* if there is a disk δ such that $\delta \cap B = \partial\delta \cap B$ and $\delta \cap D = \partial\delta \cap D$ are complementary two arcs of $\partial\delta$. If $B_{1,2}^1$ intersects with δ for $B_{1,1}^1$, then remove the intersections by the transformations as illustrated in Fig. 21. If the other sub-bands intersect with the disk δ for $B_{1,1}^1$, then isotop these sub-bands out of δ and remove the intersection of B^1 and D^1 as illustrated in Fig. 22. Then shrink B^1 so that $D^1 \cap B^1$ $(= \partial D^1 \cap \partial B^1)$ is an arc on ∂H. Note that the sub-bands of k (> 1)-th components are all well-situated with respect to $\mathcal{D}^1$.

STEP 1b: Let l be the number satisfying $2 \leq l \leq p$. Assuming that $B^k \cap D^k = \partial B^k \cap \partial D^k$ is an arc on ∂H $(k = 1, \ldots, l-1)$ and that each sub-band of a component with index no less than l is well-situated with respect to $\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{l-1}$, transform $\mathcal{B} \cup \mathcal{D}$ so that $B^l \cap D^l$ $(= \partial B^l \cap \partial D^l)$ is an arc on ∂H and that each sub-band of a component with index no less than $l + 1$ is well-situated with respect to $\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^l$. This can be done similarly to Step 1a unless we remove all the singularities of $B_{1,1}^l \cap (\mathcal{D}^1 \cup \cdots \cup \mathcal{D}^{l-1})$ before transforming $B_{1,1}^l$ into unknotted. Note that each sub-band B bounded by an innermost intersection pair of $B_{1,1}^l \cap D^k$ is unknotted from the construction, and thus there is a disk δ such that $\delta \cap B = \partial\delta \cap B$ and $\delta \cap D^k = \partial\delta \cap D^k$ are complementary two arcs of $\partial\delta$. If $B_{1,1}^l$ itself intersects with δ for B, then remove the intersections by the transformations as illustrated in Fig. 21 (see Fig. 23). Then eliminate the innermost intersection pair by isotoping $B_{1,1}^l$ along δ out of D^k. Note that this isotopy may create new innermost intersection pairs for a sub-band which is $B_{1,2}^l$ or belongs to $B^{l+1} \cup \cdots \cup B^n$.

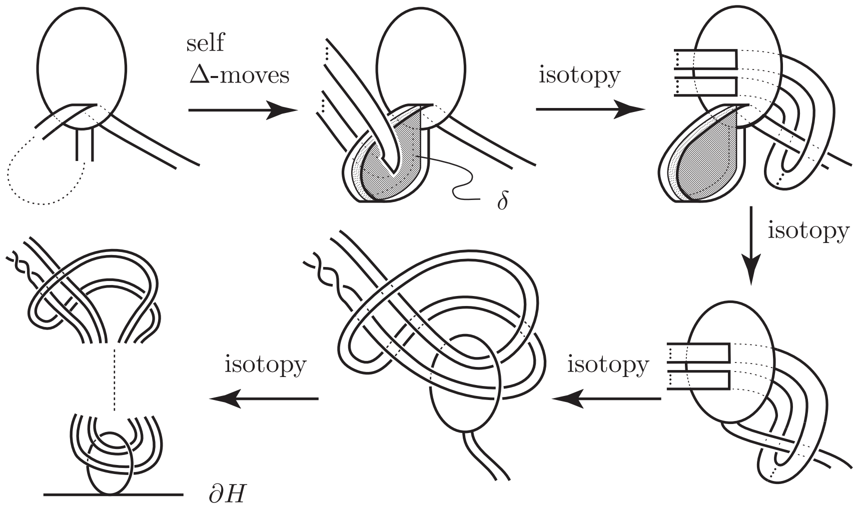


Fig. 22.

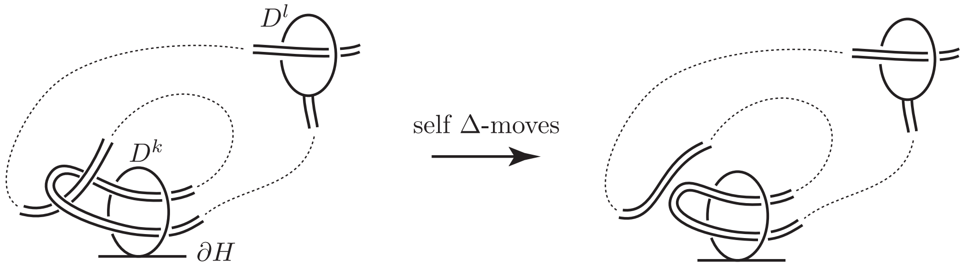
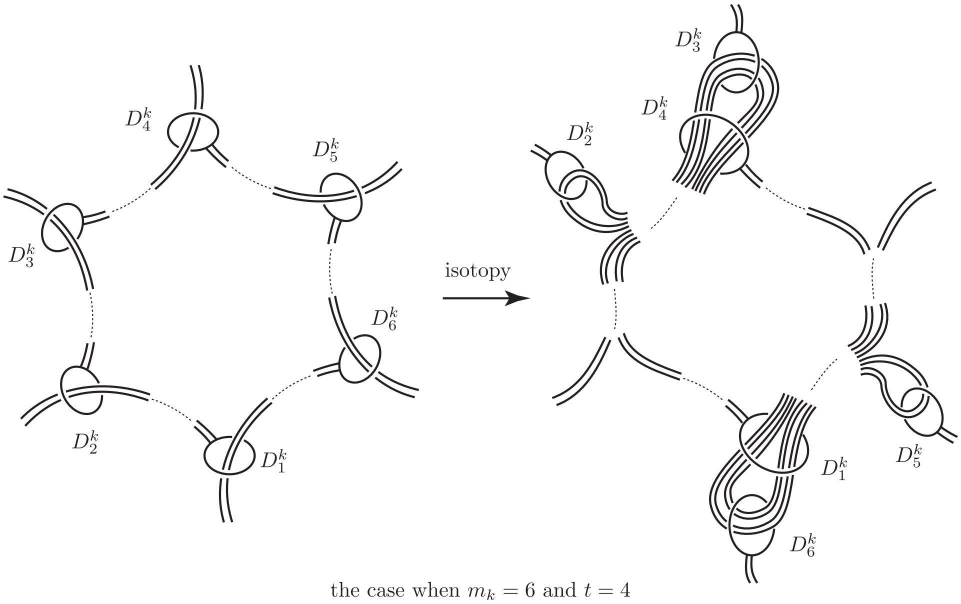


Fig. 23.

STEP 2: Now we have that $B^k \cap D^k$ $(= \partial B^k \cap \partial D^k)$ is an arc on ∂H $(k = 1, \ldots, p)$. In this step, we transform each component with index $m_k \geq 2$ into trivial. Take an k-th component $(k = p+1, \ldots, n)$. Since this component is not distinct from the assumption, there is a pair of bands, say B_1^k and B_t^k, which have ends on the same component of l.

STEP 2a: First we claim that we can deform $\mathcal{D} \cup \mathcal{B}$ so that $\operatorname{int} D_t^k \cap \mathcal{B} = \operatorname{int} D_t^k \cap B_1^k$. By the definition of the *SR*-tangle, we have that $\operatorname{int} D_j^k \cap \mathcal{B} = \operatorname{int} D_j^k \cap B_{j-1}^k$. Thus the claim is true when $t = 2$. If $t > 2$, then shrink B_2^k so that D_2^k pass through D_3^k and we have that $\operatorname{int} D_3^k \cap \mathcal{B} = \operatorname{int} D_3^k \cap B_1^k$ (see Fig. 24). Then inductively we shrink B_{i-1}^k so that D_{i-1}^k pass through D_i^k and we have that $\operatorname{int} D_i^k \cap \mathcal{B} = \operatorname{int} D_i^k \cap B_1^k$ for $i = 3, \ldots, t$. Similarly we can deform $(D_{t+1}^k \cup B_{t+1}^k) \cup \cdots \cup (D_{m_k}^k \cup B_{m_k}^k)$ so that $\operatorname{int} D_1^k \cap \mathcal{B} = \operatorname{int} D_1^k \cap B_t^k$. Then remove the intersections of $\operatorname{int} D_1^k \cap B_t^k$ by the transformations as illustrated in Fig. 21, and shrink B_1^k so that $D_1^k \cap B_1^k = \partial D_1^k \cap \partial B_1^k$ is an arc on ∂H.



the case when $m_k = 6$ and $t = 4$

Fig. 24.

STEP 2b: By applying Step 2a to all the components $\mathcal{T}^p, \mathcal{T}^{p+1}, \ldots,$ and $\mathcal{T}^n$, we have that $D_i^k \cap B_i^k$ $(= \partial D_i^k \cap \partial B_i^k)$ is an arc on ∂H for any k and i $(1 \leq k \leq n$ and $1 \leq i \leq m_k)$, i.e., L is self Δ-equivalent to l. $\square$

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References

[1] T. Ishikawa, K. Kobayashi and T. Shibuya: *On Milnor moves and Alexander polynomials of knots*, Osaka J. Math. **40** (2003), 845–855.
[2] A. Kawauchi: A Survey of Knot Theory, Birkhäuser, Basel, 1996.
[3] A. Kawauchi and K. Yoshida: *Topology of prion proteins*, Journal of Mathematics and System Science, **2** (2012), 237–248.
[4] K. Kobayashi: *On ribbon moves*, unpublished manuscript in Japanese, (2001).
[5] H. Murakami and Y. Nakanishi: *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89.
[6] K. Murasugi: *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.
[7] Y. Nakanishi and T. Shibuya: *Link homotopy and quasi self delta-equivalence for links*, J. Knot Theory Ramifications **9** (2000), 683–691.

[8] D. Rolfsen: Knots and Links, Publish or Perish, Berkeley, CA, 1976.

[9] T. Shibuya: *Self Δ-equivalence of ribbon links*, Osaka J. Math. **33** (1996), 751–760.

[10] T. Shibuya: *Milnor move and self delta-equivalence for links*, Hakone Seminar Notes, (2008).

[11] T. Shibuya and T. Tsukamoto: *Self delta-equivalence of links obtained by product fusions*, Mem. Osaka Inst. Tech. Ser. A **54** (2009), 1–8.

[12] T. Shibuya and A. Yasuhara: *Boundary links are self delta-equivalent to trivial links*, Math. Proc. Cambridge Philos. Soc. **143** (2007), 449–458.

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