Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 8 No. 1 Series A

Maximum principle for harmonic functions in Riemannian manifolds

By Hirosi YAMASUGE

(Received Nov. 9, 1956)

Let $\mathfrak{M}$ be an n-dimensional analytic Riemannian manifold with a positive-definite metric $ds^2 = g_{ik}dx^i dx^k$ where $g_{ik}(x_1, \cdots, x_n)$ are holomorphic functions of $x_1, \cdots, x_n$.

The "Laplacian" is defined by

$$\Delta = d\delta + \delta d.$$

THEOREM 1. *Let $r(x, \xi)$ be the geodesic distance between x and ξ, and ω_n the surface area of the n-dimensional unit sphere. Then for every point ξ_0 in $\mathfrak{M}$, the Laplace's eqation $\Delta\Xi = 0$ has an elementary solution*

$$\Xi(x, \xi) = \begin{cases} -\dfrac{1}{2\pi}\log r(x, \xi)\cdot u(x, \xi) + v(x, \xi), & (n = 2) \\ \dfrac{1}{(n-2)\omega_n}\cdot r^{2-n}(x, \xi)\cdot u(x, \xi) + \log r(x, \xi)\cdot v(x, \xi), & (n > 2) \end{cases}$$

defined for x, ξ in a certain neighbourhood of ξ_0, where r, u and v are holomorphic with respect to x, ξ, and $u(\xi, \xi) = 1$.

Proof. We shall expand u and v in Ξ in formal power series

$$u = m\{1 + \textstyle\sum_{\nu=1}^{\infty} r^{2\nu}u_\nu\}$$
$$v = m\textstyle\sum_{\nu=0}^{\infty} r^{2\nu}v_\nu$$

and dethrmine m, u and v so that $\Delta\Xi = 0$, then we can show that these series converge absolutely and uniformly in a certain domain. See [1].

LEMMA (GREEN'S FORMURA). *For C^2-functions φ, ψ*

$$(\varphi, \Delta\psi)_G - (\Delta\varphi, \psi)_G = \int_{BG} *(\psi d\varphi - \varphi d\psi)$$

where G is a subdomain of $\mathfrak{M}$ with the regular boundary BG and $*$ is the adjoint operator.

Proof.
$$(\varphi, \Delta\psi)_G - (\psi, \Delta\varphi)_G$$
$$= (\varphi, \delta d)_G - (\psi, \delta d\varphi)_G$$
$$= \int_G (\varphi\cdot * \delta d\psi - \psi\cdot * \delta d\varphi).$$

Using

(1) $$\delta\alpha = (-1)^{np+n+1} * d * \alpha \quad \text{and} \quad * * \alpha = (-1)^{pn+p}\alpha$$

for every *p-from* α, we have

$$(\varphi, \Delta\psi)_G - (\psi, \Delta\varphi)_G = \int_G (-\varphi\cdot d * d\psi + \psi\cdot d * d\varphi).$$

On the other hand we have

$$\int_{BG} *(\psi d\varphi - \varphi d\psi)$$
$$= \int_G d*(\psi d\varphi - \varphi d\psi)$$
$$= \int_G (\psi \cdot d * d\varphi - \varphi \cdot d * d\varphi).$$

Hence the Lemma is proved.

A function is said to be harmonic in G if it satisfies the Laplace's equation at all points of G.

THEOREM 2. *If φ is harmonic in G and its first derivatives are continuous in $\bar{G}=G+BG$ and $\Xi(x, \xi)$ is defined for x, ξ lying in G. Then for ξ in G $\varphi(\xi)$ can be represented as*

$$\varphi(\xi) = -\int_{BG} *(\varphi d\Xi - \Xi d\varphi).$$

Proof Let $S(C)$ be the set of all points satisfying the inequality $\Xi(x, \xi) \leqq C$, where C is a sufficiently large positive number.

Putting $G'=G-S(C)$ and using Green's formula we have

$$\int_{B(G-S(C))} *(\varphi d\Xi - \Xi d\varphi)$$
$$= (\Xi, \Delta\varphi)_{G'} - (\Delta\Xi, \varphi)_{G'}$$
$$= 0.$$

Hence

$$\int_{BG} *(\varphi d\Xi - \Xi d\varphi) = \int_{BS(C)} *(\varphi d\Xi - \Xi d\varphi). \tag{2}$$

Since Ξ is the constant C on $BS(C)$,

$$\int_{BS(C)} * \Xi d\varphi = C\int_{BS(C)} * d\varphi = C\int_{S(C)} d * d\varphi .$$

Using (1) we have $d * d\varphi = - * \delta d\varphi = - * \Delta\varphi = 0$.
Hence

$$\int_{BS(C)} * \Xi d\varphi = 0 . \tag{3}$$

Let G_δ be a geodesic shpere which has the radius δ and the center ξ. Taking δ so small that G_δ is cantained in $S(C)$, then we have

$$\int_{B(S(C)-G_\delta)} * d\Xi = \int_{S(C)-G_\delta} d * d\Xi = 0 . \tag{3}$$

Hence

$$\int_{BS(C)} * d\Xi = \int_{BG_\delta} * d\Xi . \tag{4}$$

We introduce on BG_δ a coordinate system (y).
Then an arbitrary point x in the neighbourhnnd of BG_δ is uniquely determined by $r(x, \xi)$ and the coordinates $(y_1, \cdots, y_{n-1})$ of the intersection of the geodesic between x and ξ with the surface BG_δ.

We use therefore

$$x_1=r,\ x_2=y_1,\cdots,x_n=y_{n-1}$$

as a coordinate system in the neighbourhood of BG_δ. Then for arbitrary function f we have

$$*\,df=\frac{\partial f}{\partial r}*dr+\frac{\partial f}{\partial y_k}*dy^k$$

Since $dr=0$ on BG_δ,

We have $\quad *dy^k=0 \quad$ on BG_δ.

Hence we have $*df=\dfrac{\partial f}{\partial r}*dr$ on BG.

By Theorem 1 an elementary caluculation shows that

$$(5)\qquad *d\Xi=\begin{cases} -\dfrac{1}{\omega_n}r^{1-n}u*dr+hr^{2-n}*dr, & (n>2)\\ -\dfrac{1}{2\pi}\dfrac{u}{r}*dr+\left(-\dfrac{1}{2\pi}\log r\dfrac{\partial u}{\partial r}+\dfrac{\partial v}{\partial r}\right)*dr, & (n=2)\end{cases}\quad \text{on } BG_\delta$$

where $h(x)$ is continuous on BG_δ.

For an arbitrary point P on BG_δ we can choose coordinates (y) so that $r, y_1,\cdots, y_{n-1}$ may be a ortho-normal coordinate system at point P. Then we have

$$*\,dr=dy^1\cdots dy^{n-1}=dS$$

where dS is the surface element of BG_δ.

Using $\lim\limits_{\delta\to 0}\dfrac{1}{\delta^{n-1}}\displaystyle\int_{BG_\delta}dS=\omega_n$, $\lim\limits_{x\to\xi}u(x,\xi)=1$ and (5)

we have easily

$$(6)\qquad \lim_{\delta\to 0}\int_{BG_\delta}*d\Xi=-1.$$

By (2), (3) we have

$$\int_{BG}*(\varphi d\Xi-\Xi d\varphi)=\int_{BS(C)}*\varphi d\Xi,$$

and by (4), (6)

$$\begin{aligned}\lim_{C\to\infty}\int_{BS(C)}*\varphi d\Xi&=\varphi(\xi)\lim_{C\to\infty}\int_{BS(C)}*d\Xi\\&=\varphi(\xi)\lim_{\delta\to 0}\int_{BG_\delta}*d\Xi\\&=-\varphi(\xi),\end{aligned}$$

q. e. d.

Now we shall put $G=S(C)$ in Theorem 2, then by (3) we have

COROLLARY. *If φ is harmonic in $S(C)$ and continuous in $\bar{S}=S+BS$, then $\varphi(\xi)$ can be writen as*

$$\varphi(\xi)=-\int_{BS(C)}\varphi(x)*d\Xi(x,\xi).$$

Let (z) be a coordinate system on $BS(C)$. Then we may use $-\Xi, z_1,\cdots,z_{n-1}$ as a coordinate system in the neighbourhood of $BS(C)$. Then we have

$$*\,d\Xi=-dz^1.\cdots.dz^{n-1}=-a\,dS,\quad a>0$$

where dS is the surface element of $BS(C)$.

Using (4) and (6) we see that

$$\int_{BS(C)} a dS = 1$$

and hence by the corollary of Theorem 2 we have

THEOREM 3. *If φ is harmonic in $S(C)$, then $\varphi(\xi)$ can be written as*

$$\varphi(\xi) = \int_{BS(C)} a\varphi dS$$

where $a > 0$ *and* $\int_{BS(C)} a dS = 1$.

THEOREM 4 (MAXIMUM PRINCIPL). *If a function φ is not constant and harmonic in a bounded domain G and continuous in $\bar{G} = G + BG$* where *BG is the boundary of G. Then the maxismum and minimum of φ in $\bar{G}$ are attained at points of BG.*

Proof. Since $\bar{G}$ is compact and f is continuous in $\bar{G}$, there exists a point ξ at which f takes its maximum. Suppose that ξ is in G. By [1] φ is holomorphic in G, hence if φ is a constant in a neighbourhood of ξ, then φ would be identically the constant in G, contrary to our assumption. Therefore in an arbitrary neigbourhood of ξ, exists a point x_0 such that $\varphi(x_0) < \varphi(\xi)$, hence in a sufficiently small neighbourhnnd of x_0, $\varphi(x) < \varphi(\xi)$. Hence putting $\Xi(x_0, \xi) = C$ and using Theorem 3 for $\varphi(\xi)$, we have

$$\varphi(\xi) = \int_{BS(C)} a\varphi dS < \int_{BS(C)} a\varphi(\xi) dS = \varphi(\xi) \int_{BS(C)} a dS = \varphi(\xi).$$

This is contradictory. Hence ξ is on BG. Similary the points at which φ takes its minimum are on BG. q. e. d.

COROLLARY. If a function is harmonic in the whole of a compact manifold $\mathfrak{M}$, then it must be a constant.

References

[1] K. Kodaira, *Harmonic fields in Riemanniam manifolds*, Annals of Mathematics, vol. 50 (1949).

[2] G. de Rham and K. Kodaira, *Harmonic Integrals*, Mimeographed Notes, Institute for Advanced Study, 1950.