Navarro, G.
Osaka J. Math.
48 (2011), 329–332

MODULARLY IRREDUCIBLE CHARACTERS AND NORMAL SUBGROUPS

GABRIEL NAVARRO

(Received September 18, 2009, revised November 18, 2009)

Abstract

Let G be a finite p-solvable group, where p is an odd prime. Suppose that $\chi \in \mathrm{Irr}(G)$ lifts an irreducible p-Brauer character. If G/N is a p-group, then we prove that the irreducible constituents of χ_N lift irreducible Brauer characters of N. This result was proven for $|G|$ odd by J.P. Cossey.

1. Introduction

Let G be a finite group and let p be a prime. Let $\mathrm{Irr}(G)$ be the set of the irreducible complex characters of G, and let $\mathrm{IBr}(G)$ be a set of irreducible Brauer characters of G. If $\chi \in \mathrm{Irr}(G)$, then the restriction χ^0 of χ to the p-regular elements of G,

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

decomposes as a sum of irreducible Brauer characters. Sometimes we have that $\chi^0 \in \mathrm{IBr}(G)$. When this occurs, J.P. Cossey has investigated when the same happens for the normal irreducible constituents of χ ([1]). Aside of the case where χ has defect zero (i.e., $\chi(1)_p = |G|_p$), it seems difficult to control the behavior of the normal constituents of χ. Somehow surprisingly, it is proven in [1] that normal irreducible constituents also lift modular characters, whenever G/N is a p-group and $|G|$ is odd. The proof of that result relies on non-trivial facts from [10] (among others).

Our aim in this note is to give an essentially self-contained proof of a slightly more general result.

Theorem A. *Let p be an odd prime and let G be a p-solvable group. Let $\chi \in \mathrm{Irr}(G)$ with $\chi^0 = \varphi \in \mathrm{IBr}(G)$. Suppose that G/N is a p-group. If $\theta \in \mathrm{Irr}(N)$ is under χ, then $\theta^0 \in \mathrm{IBr}(N)$.*

Theorem A is not true for $p = 2$, even for solvable groups. A counterexample is provided by $G = GL(2, 3)$ and $N = SL(2, 3)$, where here $\chi \in \mathrm{Irr}(G)$ is non-rational of

2000 Mathematics Subject Classification. 20C15, 20C20.

This research is partially supported by MTM2007-61161 and MTM2010-15296 of the Spanish Ministerio de Educación y Ciencia.

degree 2. On the other hand, it has not been easy at all to find a counterexample of Theorem A for non-p-solvable groups. Finally, T. Okuyama (and independently P.H. Tiep) found that if $p = 3$, $N = PSU_3(8)$ and $G = PGU_3(8)$, then G has irreducible characters $\chi \in \mathrm{Irr}(G)$ of degree $\chi(1) = 399$, which are modularly irreducible, and such that $\chi_N = \theta \in \mathrm{Irr}(N)$ does not lift an irreducible Brauer character of N.

Part of this work was done while I was visiting the University of Chiba in Japan. I would like to give my thanks to Shigeo Koshitani for the hospitality. Also, my thanks to N. Kunugi and T. Okuyama for helpful discussions on the subject.

2. Proofs

Our notation for characters follows [6] and [9]. For instance, if χ is a character, then $o(\chi)$ is the order of the linear character $\det(\mathcal{X})$, where $\mathcal{X}$ is any representation affording χ. We shall use Gajendragadkar π-special characters. These are the irreducible characters χ of π-separable groups such that for every subnormal irreducible constituent θ of χ we have that $o(\theta)\theta(1)$ are π-numbers ([3]). Every primitive character of a π-separable group factors as a product of a π-special and a π'-special characters (Corollary (4.7) of [4]). The reader is invited to read [3] and [4]. In the proof of the key lemma below we use a much deeper result: if $2 \notin \pi$, $\chi \in \mathrm{Irr}(G)$ is π-special and $\chi_H \in \mathrm{Irr}(H)$ for some subgroup H of G, then χ_H is π-special. (This is Theorem A of [5].)

The key result to prove Theorem A is the following.

Lemma 2.1. *Let G be p-solvable with $p > 2$. Let $\chi \in \mathrm{Irr}(G)$ be p-special. If $\chi(1) > 1$, then χ^0 is not in* $\mathrm{IBr}(G)$.

Our first proof of this lemma contained a mistake that was pointed out by the careful reading of the referee, and for what we thank him/her. But also, we are also very grateful to I.M. Isaacs for providing us with a more general result whose corollary gave a correct proof of our lemma. After reading Isaacs proof, we found the following one.

Proof of Lemma 2.1. Let $M = \ker(\chi)$. Then χ considered as an irreducible character of G/M is both p-special and p-Brauer irreducible. So arguing by induction on $|G|$, we may assume that χ is faithful. In particular, $\mathbf{O}_{p'}(G) = 1$ by Corollary (4.2) of [3]. Now, let $N = \mathbf{O}_p(G)$, $K/N = \mathbf{O}_{p'}(G/N)$, and let L be a p-complement of K. By the Frattini argument, we have that $G = N\mathbf{N}_G(L)$. Write $H = \mathbf{N}_G(L)$.

We claim now that $\chi_H \in \mathrm{Irr}(H)$. Write $\varphi = \chi^0 \in \mathrm{IBr}(G)$. Since $N \subseteq \ker(\varphi)$ and $NH = G$, then we see that φ_H is irreducible. (This follows from the fact that if $\mathcal{X}$ is any representation affording φ, then $\mathcal{X}(nh) = \mathcal{X}(h)$ for $n \in N$ and $h \in H$. Hence $\mathcal{X}(G) = \mathcal{X}(H)$ and $\mathcal{X}_H$ affords an irreducible representation.) Since $(\chi_H)^0 = \varphi_H$ is irreducible, we easily see that χ_H is irreducible too.

But now, by Theorem A of [5], we have that χ_H is p-special. Therefore, $L \subseteq \ker \chi_H \subseteq \ker(\chi) = 1$. We conclude that $N = G$. In this case, the principal Brauer character of G is the only Brauer irreducible Brauer character of G, and the proof of the lemma is complete. □

In the proof of Theorem A we shall use vertices of Brauer characters and a result of A. Watanabe ([11]). This result of Watanabe is proven in a different form in [8]. Even more recently, another proof is presented in [2], which uses techniques closer to the ones used in this paper.

In p-solvable groups, vertices of Brauer characters are particularly easy to understand: If $\varphi \in \mathrm{IBr}(G)$, then φ is induced from some $\mu \in \mathrm{IBr}(U)$ of p'-degree (see Huppert's Theorem (10.11) of [9]), and the Sylow p-subgroups of U are uniquely determined by φ up to G-conjugacy (see [7]). These are the *vertices* of φ. If φ has vertex Q, then we have that $\varphi(1)_p = |G|_p/|Q|$.

Lemma 2.2 (Watanabe). *Suppose that G is p-solvable, and $\varphi \in \mathrm{IBr}(G)$. Let $N \lhd G$ and $\theta \in \mathrm{IBr}(N)$ be under φ. Then there exists a vertex Q of φ such that $Q \cap N$ is a vertex of θ.*

Now we prove Theorem A. The final assertion on the stabilizers was also noticed in [1].

Theorem 2.3. *Suppose that G is p-solvable, where p is odd. Let $\chi \in \mathrm{Irr}(G)$ with $\chi^0 = \varphi \in \mathrm{IBr}(G)$. Suppose that G/N is a p-group. If $\theta \in \mathrm{Irr}(N)$ is under χ, then $\theta^0 \in \mathrm{IBr}(N)$. Furthermore the stabilizers $I_G(\theta) = I_G(\theta^0)$ coincide.*

Proof. We argue by induction on $|G : N|$ that $\theta^0 \in \mathrm{IBr}(N)$. Suppose that $N < M \lhd G$, with $|G : M| = p$, and let $\psi \in \mathrm{Irr}(M)$ be between χ and θ. By induction we have that $\psi^0 \in \mathrm{IBr}(M)$ and also, by induction, we have that $\theta^0 \in \mathrm{IBr}(N)$. Hence we may assume that $|G : N| = p$.

First suppose that $\theta^G = \chi$. Then $(\theta^0)^G = \varphi$ is irreducible and necessarily $\theta^0 = \eta \in \mathrm{IBr}(N)$ is irreducible too. Hence, we may assume that $\chi_N = \theta$.

Now, let $\gamma \in \mathrm{Irr}(W)$ be a primitive character inducing χ. Then $\gamma = \alpha\beta$, where α is p'-special and β is p-special. Since $\gamma^0 \in \mathrm{IBr}(W)$, it follows that $\beta^0 \in \mathrm{IBr}(W)$. By Lemma 2.1, we deduce that γ has p'-degree. Thus $\chi(1)_p = |G|_p/|Q|$, where $Q \in \mathrm{Syl}_p(W)$. Also, since $WN = G$ by Mackey (Problem (5.7) of [6]), it follows that $NQ = G$. Notice now that $\varphi = (\gamma^0)^G$ and Q is a vertex for φ.

Now, let $\tau \in \mathrm{IBr}(N)$ a Brauer constituent of θ^0, which therefore lies under φ. By Lemma 2.2, there exists a vertex Q_1 of φ such that $Q_1 \cap N$ is a vertex for τ. Now, $Q_1 = Q^n$ for some $n \in N$ (because $QN = G$), and hence we may assume that $Q_1 = Q$. Thus $Q \cap N$ is a vertex for τ. Therefore $\tau(1)_p = |N|_p/|Q \cap N|$. But then

$$\varphi(1)_p = \tau(1)_p.$$

Since G/N is a p-group, we also deduce that $\varphi(1)_{p'} = \tau(1)_{p'}$ (using Theorem (8.30) of [9]), and therefore $\varphi(1) = \tau(1)$. Hence $\varphi_N = \tau$. Now $\varphi(1) = \chi(1) \geq \theta(1) \geq \tau(1) = \varphi(1)$, and hence $\theta(1) = \tau(1)$, and $\theta^0 = \tau$.

Finally, we prove that G_θ, the stabilizer of θ in G equals G_{θ^0}, the stabilizer of θ^0 in G. Write again $\tau = \theta^0 \in \mathrm{IBr}(N)$. Of course, we have that $G_\theta \subseteq G_\tau$. By Green's theorem and the Clifford's correspondence, notice that $\varphi(1) = |G : G_\tau|\tau(1)$. Now, using $\varphi(1) = \chi(1) \geq |G : G_\theta|\theta(1) \geq |G : G_\tau|\tau(1) = \varphi(1)$, and the proof of the theorem is complete. □

We should perhaps mention that Theorem A holds in general if χ has p'-degree. Indeed, in this case $\chi_N = \theta$ has p'-degree and therefore there exists $\tau \in \mathrm{IBr}(N)$ in the decomposition of θ^0 which is P-invariant, where $P \in \mathrm{Syl}_p(G)$. Now by Green's Theorem (8.11) of [9], we have that there is a unique Brauer character of G over τ, which necessarily is φ. In this case, $\varphi_N = \tau$ and by degrees, $\theta^0 = \tau$.

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Departament d'Àlgebra
Facultat de Matemàtiques
Universitat de València
46100 Burjassot. València
Spain
e-mail: gabriel.navarro@uv.es