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TOTALLY GEODESIC SUBMANIFOLDS OF THE EXCEPTIONAL RIEMANNIAN SYMMETRIC SPACES OF RANK 2

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Abstract

The present article is the final part of a series on the classification of the totally geodesic submanifolds of the irreducible Riemannian symmetric spaces of rank 2. After this problem has been solved for the 2-Grassmannians in my papers [7] and [8], and for the space $\mathrm{SU}(3)/\mathrm{SO}(3)$ in Section 6 of [9], we now solve the classification for the remaining irreducible Riemannian symmetric spaces of rank 2 and compact type: $\mathrm{SU}(6)/\mathrm{Sp}(3)$, $\mathrm{SO}(10)/\mathrm{U}(5)$, $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, E_6/F_4, $G_2/\mathrm{SO}(4)$, $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$ and G_2.

Similarly as for the spaces already investigated in the earlier papers, it turns out that for many of the spaces investigated here, the earlier classification of the maximal totally geodesic submanifolds of Riemannian symmetric spaces by Chen and Nagano ([5], §9) is incomplete. In particular, in the spaces $\mathrm{Sp}(2)$, $G_2/\mathrm{SO}(4)$ and G_2, there exist maximal totally geodesic submanifolds, isometric to 2- or 3-dimensional spheres, which have a "skew" position in the ambient space in the sense that their geodesic diameter is strictly larger than the geodesic diameter of the ambient space. They are all missing from [5].

1. Introduction

The classification of the totally geodesic submanifolds in Riemannian symmetric spaces is an interesting and significant problem of Riemannian geometry. Presently, I solve this problem for the irreducible Riemannian symmetric spaces of rank 2.

The totally geodesic submanifolds of the 2-Grassmannians $G_2^+(\mathbb{R}^n)$, $G_2(\mathbb{C}^n)$ and $G_2(\mathbb{H}^n)$ have already been classified in my papers [7] and [8]; moreover the totally geodesic submanifolds of $\mathrm{SU}(3)/\mathrm{SO}(3)$ have been classified in Section 6 of my paper [9]. In the present paper I complete the classification of the totally geodesic submanifolds in the irreducible Riemannian symmetric spaces of rank 2 (simply connected and of compact type) by considering the remaining spaces of this kind; they are the spaces of type I

$$\mathrm{SO}(10)/\mathrm{U}(5),\quad E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10)),\quad \mathrm{SU}(6)/\mathrm{Sp}(3),\quad E_6/F_4\quad\text{and}\quad G_2/\mathrm{SO}(4)$$

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as well as the spaces of Lie group type

$$\mathrm{SU}(3), \quad \mathrm{Sp}(2) \quad \text{and} \quad G_2;$$

herein G_2, F_4 and E_6 denote the exceptional, simply connected, compact, real Lie groups.

It should be mentioned that already Chen and Nagano gave what they claimed to be a complete classification of the isometry types of maximal totally geodesic submanifolds in all Riemannian symmetric spaces of rank 2 in §9 of their paper [5] based on their (M_+, M_-)-method. However, as it will turn out in the present paper, their classification is faulty also for several of the spaces under consideration here. In particular, in the spaces Sp(2), G_2 and $G_2/\mathrm{SO}(4)$, there exist maximal totally geodesic submanifolds, isometric to spheres of dimension 2 or 3, which have a "skew" position in the ambient space in the sense that their geodesic diameter is strictly larger than the geodesic diameter of the ambient space; these submanifolds are missing from Chen's and Nagano's classification. Also in the spaces SO(10)/U(5) and $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, such "skew" totally geodesic submanifolds exist, although they are not maximal. Moreover several other details of Chen's and Nagano's classification are incorrect. For a detailed discussion with respect to the individual spaces studied, see the following remarks of the present paper:

space	SO(10)/U(5)	$E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$	SU(6)/Sp(3)	E_6/F_4	$G_2/\mathrm{SO}(4)$	SU(3)	Sp(2)	G_2
Remark	3.11	3.6	4.5	4.3	5.5	4.7	3.9	5.3

Even apart from these problems, Chen's and Nagano's investigation is not satisfactory, as they name only the isometry type of the totally geodesic submanifolds, without giving any description of their position in the ambient space. (Such a description can, for example, be constituted by giving explicit totally geodesic, isometric embeddings for the various congruence classes of totally geodesic submanifolds, or at least by describing the tangent spaces of the totally geodesic submanifolds (i.e. the Lie triple systems) as subspaces of the tangent space of the ambient symmetric space in an explicit way.)

The usual strategy for the classification of totally geodesic submanifolds in a Riemannian symmetric space $M = G/K$, which is used also here, is as follows. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the decomposition of the Lie algebra of G induced by the symmetric structure of M. As it is well-known, the Lie triple systems $\mathfrak{m}'$ in $\mathfrak{m}$ (i.e. the linear subspaces $\mathfrak{m}' \subset \mathfrak{m}$ which satisfy $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$) are in one-to-one correspondence with the (connected, complete) totally geodesic submanifolds $M_{\mathfrak{m}'}$ of M running through the "origin point" $p_0 = eK \in M$, the correspondence being that $M_{\mathfrak{m}'}$ is characterized by $p_0 \in M_{\mathfrak{m}'}$ and $T_{p_0}M_{\mathfrak{m}'} = \tau(\mathfrak{m}')$, where $\tau\colon \mathfrak{m} \to T_{p_0}M$ is the canonical isomorphism.

Thus the task of classifying the totally geodesic submanifolds of M splits into two steps: (1) To classify the Lie triple systems in $\mathfrak{m}$, and (2) for each of the Lie triple systems $\mathfrak{m}'$ found in the first step, to construct a (connected, complete) totally geodesic submanifold $M_{\mathfrak{m}'}$ of M so that $p_0 \in M_{\mathfrak{m}'}$ and $\tau^{-1}(T_{p_0} M_{\mathfrak{m}'}) = \mathfrak{m}'$ holds.

Herein, step (1) is the one which generally poses the more significant difficulties. As an approach to accomplishing this step, we describe in Section 2 for an arbitrary Riemannian symmetric space M of compact type relations between the roots and root spaces of M and the roots resp. root spaces of its totally geodesic submanifolds (regarded as symmetric subspaces). These relations provide conditions which are necessary for a linear subspace $\mathfrak{m}'$ of $\mathfrak{m}$ to be a Lie triple system. However, these conditions are not generally sufficient, and therefore a specific investigation needs to be made to see which of the linear subspaces of $\mathfrak{m}$ satisfying the conditions are in fact Lie triple systems; this investigation is the laborious part of the proof of the classification theorems.

It should be emphasized that to carry out this investigation for a given Riemannian symmetric space M, it does not suffice to know the (restricted) root system (with multiplicities) of that space, or equivalently, the action of the Jacobi operators $R(\cdot, v)v$ on the various root spaces. Rather, a full description of the curvature tensor of M is needed. The well-known formula $R(u, v)w = -[[u, v], w]$ relating the curvature tensor R of M to the Lie bracket of the Lie algebra $\mathfrak{g}$ of the transvection group G of M lets one calculate R relatively easily if M is a classical symmetric space (then $\mathfrak{g}$ is a matrix Lie algebra, with the Lie bracket being simply the commutator of matrices), but not so easily if M is one of the exceptional symmetric spaces, because then the explicit description of the exceptional Lie algebra $\mathfrak{g}$ as a matrix algebra is too unwieldy to be useful.

In its place, we use the description of the curvature tensor based on the root space decomposition of $\mathfrak{g}$ which was described in [9], and which permits the reconstruction of R using only the Satake diagram of the Riemannian symmetric space M. To actually carry out the computations involved in the application of the results from [9], we use the example implementation of the algorithms for Maple also presented in that paper; this implementation is found on `http://satake.sourceforge.net`. Whenever in the present paper, a claim is made about the evaluation of the Lie bracket of a Lie algebra or the curvature tensor of a Riemannian symmetric space for specific input vectors, the result has been obtained in this way. Maple worksheets containing all the calculations can also be found on `http://satake.sourceforge.net`.

Certain of the spaces under investigation here are locally isometric to totally geodesic submanifolds of others; more specifically, we have the following inclusions of totally geodesic submanifolds:

$$\mathrm{Sp}(2)/\mathbb{Z}_2 \subset \mathrm{SO}(10)/\mathrm{U}(5) \subset E_6/(\mathrm{U}(1)\cdot \mathrm{Spin}(10)),$$
$$\mathrm{SU}(3) \subset \mathrm{SU}(6)/\mathrm{Sp}(3), \quad (\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3 \subset E_6/F_4, \quad G_2/\mathrm{SO}(4) \subset G_2.$$

If M is a Riemannian symmetric space and $M' \subset M$ a totally geodesic submanifold, then the totally geodesic submanifolds of M' are exactly those totally geodesic submanifolds of M which are contained in M'. For this reason, we can obtain a classification of the totally geodesic submanifolds of M' from a classification of the totally geodesic submanifolds of M: We just need to determine which of the totally geodesic submanifolds of M are contained in M'. Thus we do not need to carry out the classification of totally geodesic submanifolds for each space under investigation here individually by the approach described above. Rather it suffices to do the classification for the three spaces $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, E_6/F_4 and G_2, by virtue of the mentioned inclusions we then also obtain classifications for the remaining Riemannian symmetric spaces of rank 2.

The present paper is laid out as follows: Section 2 contains general facts on Lie triple systems, in particular on the relationship between their (restricted) roots resp. root spaces, and the roots resp. root spaces of the ambient space. Section 3 is concerned primarily with the investigation of the Riemannian symmetric space $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$: In Subsection 3.1 we make general observations about the geometry of this space; using these results we then classify the Lie triple systems of $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$ in Subsection 3.2, corresponding to step (1) of the classification as described above. In Subsection 3.3 we describe totally geodesic embeddings for each congruence class of Lie triple systems in $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, thereby completing the classification of totally geodesic submanifolds for that space. In Subsections 3.4 and 3.5, we use the inclusions of totally geodesic submanifolds $\mathrm{Sp}(2)\subset G_2(\mathbb{H}^4)$ resp. $\mathrm{SO}(10)/\mathrm{U}(5)\subset E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$ to derive the classification of totally geodesic submanifolds in $\mathrm{Sp}(2)$ resp. in $\mathrm{SO}(10)/\mathrm{U}(5)$ from previous results.

Section 4 covers the investigation of E_6/F_4 and is structured similarly: After the introduction of basic geometric facts on that space in Subsection 4.1, we classify its Lie triple systems in Subsection 4.2. As a consequence of the classification it turns out that in E_6/F_4, all maximal totally geodesic submanifolds are reflective. Thus we can learn the global isometry type of the corresponding totally geodesic submanifolds from the classification of reflective submanifolds in symmetric spaces by Leung, [13], as is described in Subsection 4.3, and do not need to construct totally geodesic embeddings in this case explicitly. In Subsections 4.4 resp. 4.5 we use the inclusion $(\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3\subset E_6/F_4$ resp. $\mathrm{SU}(3)\subset\mathrm{SU}(6)/\mathrm{Sp}(3)$ to derive the classification for the space $\mathrm{SU}(6)/\mathrm{Sp}(3)$ resp. $\mathrm{SU}(3)$. The space $\mathrm{SU}(3)/\mathrm{SO}(3)$, whose totally geodesic submanifolds have already been classified in Section 6 of [9], is contained in $\mathrm{SU}(3)$; therefore its Lie triple systems also occur in the present paper. Subsection 4.6 gives the relationship between the types of Lie triple systems of $\mathrm{SU}(3)/\mathrm{SO}(3)$ as defined in Section 6 of [9] and types of Lie triple systems defined here.

Section 5 then investigates the Lie group G_2 seen as a Riemannian symmetric space. In Subsection 5.1 we investigate the geometry of this space, then we proceed in Subsection 5.2 to the classification of its Lie triple systems, and describe embeddings for (most of) its totally geodesic submanifolds in Subsection 5.3. In Subsection 5.4

we use the inclusion $G_2/\mathrm{SO}(4) \subset G_2$ to derive a classification of the totally geodesic submanifolds of $G_2/\mathrm{SO}(4)$.

Finally, in Section 6 we give a table of the isometry types of the maximal totally geodesic submanifolds of all irreducible Riemannian symmetric spaces of rank 2 and compact type, thereby summarizing the results of my papers [7], [8], [9] (Section 6), as well as of the present paper.

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2. General facts on Lie triple systems

In this section we suppose that $M = G/K$ is any Riemannian symmetric space of compact type. We consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G induced by the symmetric structure of M. Because M is of compact type, the Killing form $\varkappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ is negative definite, and therefore $\langle \cdot\,, \cdot \rangle := -c \cdot \varkappa$ gives rise to a Riemannian metric on M for arbitrary $c \in \mathbb{R}_+$. In the sequel we suppose that M is equipped with such a Riemannian metric.[1]

Let us fix notations concerning flat subspaces, roots and root spaces of M (for the corresponding theory, see for example [14], Section V.2): A linear subspace $\mathfrak{a} \subset \mathfrak{m}$ is called *flat* if $[\mathfrak{a}, \mathfrak{a}] = \{0\}$ holds. The maximal flat subspaces of $\mathfrak{m}$ are all of the same dimension, called the *rank* of M (or $\mathfrak{m}$) and denoted by $\mathrm{rk}(M)$ or $\mathrm{rk}(\mathfrak{m})$; they are called the *Cartan subalgebras* of $\mathfrak{m}$. If a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$ is fixed, we put for any linear form $\lambda \in \mathfrak{a}^*$

$$\mathfrak{m}_\lambda := \{X \in \mathfrak{m} \mid \forall Z \in \mathfrak{a}\colon\ \mathrm{ad}(Z)^2 X = -\lambda(Z)^2 X\}$$

and consider the (*restricted*) *root system*

$$\Delta(\mathfrak{m}, \mathfrak{a}) := \{\lambda \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{m}_\lambda \neq \{0\}\}$$

of $\mathfrak{m}$ with respect to $\mathfrak{a}$. The elements of $\Delta(\mathfrak{m}, \mathfrak{a})$ are called (*restricted*) *roots* of $\mathfrak{m}$ with respect to $\mathfrak{a}$, for $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ the subspace $\mathfrak{m}_\lambda$ is called the *root space* corresponding to λ, and $n_\lambda := \dim(\mathfrak{m}_\lambda)$ is called the *multiplicity* of the root λ. If we fix a system of positive roots $\Delta_+ \subset \Delta(\mathfrak{m}, \mathfrak{a})$ (i.e. we have $\Delta_+ \,\dot\cup\, (-\Delta_+) = \Delta(\mathfrak{m}, \mathfrak{a})$), we obtain the

[1] The dependence of the sectional curvature of M on the choice of the Riemannian metric is as follows: If we multiply the Riemannian metric with some factor $c > 0$, then this causes the sectional curvature function to be multiplied with $1/c$.

(*restricted*) *root space decomposition* of $\mathfrak{m}$:

$$\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_+} \mathfrak{m}_\lambda. \tag{1}$$

The *Weyl group* $W(\mathfrak{m}, \mathfrak{a})$ is the transformation group on $\mathfrak{a}$ generated by the reflections in the hyperplanes $\{v \in \mathfrak{a} \mid \lambda(v) = 0\}$ (where λ runs through $\Delta(\mathfrak{m}, \mathfrak{a})$); it can be shown that the root system $\Delta(\mathfrak{m}, \mathfrak{a})$ is invariant under the action of $W(\mathfrak{m}, \mathfrak{a})$.

Let us now consider a Lie triple system $\mathfrak{m}' \subset \mathfrak{m}$, i.e. $\mathfrak{m}'$ is a linear subspace of $\mathfrak{m}$ so that $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$ holds. In spite of the fact that the symmetric space corresponding to $\mathfrak{m}'$ does not need to be of compact type (it can contain Euclidean factors), it is easily seen that the usual statements of the root space theory for symmetric spaces of compact type carry over to $\mathfrak{m}'$, see [7].

More specifically, the maximal flat subspaces of $\mathfrak{m}'$ are all of the same dimension (again called the *rank* of $\mathfrak{m}'$), and they are again called the *Cartan subalgebras* of $\mathfrak{m}'$. For any Cartan subalgebra $\mathfrak{a}'$ of $\mathfrak{m}'$, there exists a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ so that $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ holds. With respect to any Cartan subalgebra $\mathfrak{a}'$ of $\mathfrak{m}'$ we have a root system $\Delta(\mathfrak{m}', \mathfrak{a}')$ (defined analogously as for $\mathfrak{m}$) and the corresponding root space decomposition

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta_+(\mathfrak{m}', \mathfrak{a}')} \mathfrak{m}'_\alpha \tag{2}$$

(with a system of positive roots $\Delta_+(\mathfrak{m}', \mathfrak{a}') \subset \Delta(\mathfrak{m}', \mathfrak{a}')$); we also again call $n'_\alpha := \dim(\mathfrak{m}'_\alpha)$ the multiplicity of $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$. $\Delta(\mathfrak{m}', \mathfrak{a}')$ is again invariant under the action of the corresponding Weyl group $W(\mathfrak{m}', \mathfrak{a}')$. It should be noted, however, that in the case where a Euclidean factor is present in $\mathfrak{m}'$, $\Delta(\mathfrak{m}', \mathfrak{a}')$ does not span $(\mathfrak{a}')^*$.

The following proposition describes the relation between the root space decompositions (2) of $\mathfrak{m}'$ and (1) of $\mathfrak{m}$. In particular, it shows the extent to which the position of the individual root spaces $\mathfrak{m}'_\alpha$ of $\mathfrak{m}'$ is adapted to the root space decomposition (1) of the ambient space $\mathfrak{m}$. These relations will play a fundamental role in our classification of the Lie triple systems in the Riemannian symmetric spaces of rank 2.

Proposition 2.1. *Let $\mathfrak{a}'$ be a Cartan subalgebra of $\mathfrak{m}'$, and let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$ so that $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ holds.*

(a) *The roots resp. root spaces of $\mathfrak{m}'$ and of $\mathfrak{m}$ are related in the following way*:

$$\Delta(\mathfrak{m}', \mathfrak{a}') \subset \{\lambda|\mathfrak{a}' \mid \lambda \in \Delta(\mathfrak{m}, \mathfrak{a}),\ \lambda|\mathfrak{a}' \neq 0\}. \tag{3}$$

$$\forall \alpha \in \Delta(\mathfrak{m}', \mathfrak{a}'): \quad \mathfrak{m}'_\alpha = \Bigg(\bigoplus_{\substack{\lambda \in \Delta(\mathfrak{m}, \mathfrak{a}) \\ \lambda|\mathfrak{a}' = \alpha}} \mathfrak{m}_\lambda \Bigg) \cap \mathfrak{m}'. \tag{4}$$

In particular, if $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ satisfies $\lambda|\mathfrak{a}' = 0$, then $\mathfrak{m}'$ is orthogonal to $\mathfrak{m}_\lambda$.

(b) *We have* $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$ *if and only if* $\mathfrak{a}' = \mathfrak{a}$ *holds. If this is the case, then we have*

$$\Delta(\mathfrak{m}', \mathfrak{a}') \subset \Delta(\mathfrak{m}, \mathfrak{a}), \quad \forall \alpha \in \Delta(\mathfrak{m}', \mathfrak{a}') \colon \quad \mathfrak{m}'_\alpha = \mathfrak{m}_\alpha \cap \mathfrak{m}'. \tag{5}$$

Proof. See [7], the proof of Proposition 2.1. □

For the remainder of the section, we fix a Cartan subalgebra $\mathfrak{a}'$ of $\mathfrak{m}'$, and let $\mathfrak{a}$ be any Cartan subalgebra of $\mathfrak{m}$ so that $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ holds.

DEFINITION 2.2. Let $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ be given. Recall that by Proposition 2.1 (a) there exists at least one root $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha$. We call α
(a) *elementary*, if there exists only one root $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha$;
(b) *composite*, if there exist at least two different roots $\lambda, \mu \in \Delta(\mathfrak{m}, \mathfrak{a})$ with $\lambda|\mathfrak{a}' = \alpha = \mu|\mathfrak{a}'$.

Elementary roots play a special role: If $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ is elementary, then the root space $\mathfrak{m}'_\alpha$ is contained in the root space $\mathfrak{m}_\lambda$, where $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ is the unique root with $\lambda|\mathfrak{a}' = \alpha$. As we will see in Proposition 2.3 below, this property causes restrictions for the possible positions (in relation to $\mathfrak{a}'$) of λ. The exploitation of these restrictions will play an important role in the classification of the rank 1 Lie triple systems in the rank 2 spaces under investigation.

It should also be mentioned that in the case $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$ we have $\mathfrak{a}' = \mathfrak{a}$, and therefore in that case every $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ is elementary (compare Proposition 2.1 (b)).

For any linear form $\lambda \in \mathfrak{a}^*$ we now denote by $\lambda^\sharp$ the Riesz vector corresponding to λ, i.e. the vector $\lambda^\sharp \in \mathfrak{a}$ characterized by $\langle\,\cdot\,, \lambda^\sharp\rangle = \lambda$. Here $\langle\,\cdot\,,\,\cdot\,\rangle = -c \cdot \varkappa$ is again the inner product obtained from the Killing form $\varkappa$ of $\mathfrak{g}$.

Proposition 2.3. *Let* $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ *be given.*
(1) *If* α *is elementary and* $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$ *is the unique root with* $\lambda|\mathfrak{a}' = \alpha$, *then we have* $\lambda^\sharp \in \mathfrak{a}'$.
(2) *If* α *is composite and* $\lambda, \mu \in \Delta(\mathfrak{m}, \mathfrak{a})$ *are two different roots with* $\lambda|\mathfrak{a}' = \alpha = \mu|\mathfrak{a}'$, *then* $\lambda^\sharp - \mu^\sharp$ *is orthogonal to* $\mathfrak{a}'$.

Proof. For (a) see [7], the proof of Proposition 2.3 (a); (b) is obvious. □

Proposition 2.4. *Suppose that* $\alpha \in \Delta(\mathfrak{m}', \mathfrak{a}')$ *is a composite root such that there exist precisely two roots* $\lambda, \mu \in \Delta(\mathfrak{m}, \mathfrak{a})$ *with* $\lambda|\mathfrak{a}' = \alpha = \mu|\mathfrak{a}'$. *Because of* $\alpha \in (\mathfrak{a}')^*$, *we have* $\alpha^\sharp \in \mathfrak{a}'$; *we suppose that this element can be written as a linear combination* $\alpha^\sharp = a\lambda^\sharp + b\mu^\sharp$ *with non-zero* $a, b \in \mathbb{R}$.

Then we have $a, b > 0$, *and there exists a linear subspace* $\mathfrak{m}'_\lambda \subset \mathfrak{m}_\lambda$ *and an isometric linear map* $\Phi\colon \mathfrak{m}'_\lambda \to \mathfrak{m}_\mu$ *so that*

$$\mathfrak{m}'_\alpha = \left\{ x + \sqrt{\frac{b}{a}}\,\Phi(x) \;\middle|\; x \in \mathfrak{m}'_\lambda \right\} \tag{6}$$

holds. In particular we have $n'_\alpha \leq \min\{n_\lambda, n_\mu\}$.

Proof. See [8], the proof of Proposition 2.4. □

We mention one important principle for the construction of Lie triple systems with only elementary roots.

DEFINITION 2.5. A subset $\Delta' \subset \Delta(\mathfrak{m}, \mathfrak{a})$ is called a *closed root subsystem* of $\Delta(\mathfrak{m}, \mathfrak{a})$ if for every $\lambda \in \Delta'$ we also have $-\lambda \in \Delta'$, and if for every $\lambda, \mu \in \Delta'$ with $\lambda + \mu \in \Delta(\mathfrak{m}, \mathfrak{a})$ we have $\lambda + \mu \in \Delta'$.

Proposition 2.6. *Let* Δ' *be a closed root subsystem of* $\Delta(\mathfrak{m}, \mathfrak{a})$, *and let* Δ'_+ *be a positive root system of* Δ'. *Then* $\mathfrak{m}' := \operatorname{span}_{\mathbb{R}}\{\lambda^\sharp \mid \lambda \in \Delta'\} \oplus \bigoplus_{\lambda \in \Delta'_+} \mathfrak{m}_\lambda$ *is a Lie triple system in* $\mathfrak{m}$. $\mathfrak{m}'$ *is called the* Lie triple system associated to Δ'.

Proof. This follows immediately from the fact that for any $\lambda, \mu \in \Delta(\mathfrak{m}, \mathfrak{a}) \cup \{0\}$ we have

$$[\mathfrak{m}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{k}_{\lambda+\mu} \oplus \mathfrak{k}_{\lambda-\mu} \quad \text{and} \quad [\mathfrak{k}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{m}_{\lambda+\mu} \oplus \mathfrak{m}_{\lambda-\mu},$$

see [14], Proposition VI.1.4c, p. 60. Here $\mathfrak{k}_\lambda$ denotes the root space of $\mathfrak{k}$ corresponding to $\lambda \in \Delta(\mathfrak{m}, \mathfrak{a})$. □

The isotropy group K of the symmetric space M acts on $\mathfrak{m}$ via the adjoint representation, i.e. by $K \times \mathfrak{m} \to \mathfrak{m}$, $(g, v) \mapsto \operatorname{Ad}(g)v$; this action is called the *isotropy action*. In the investigation of Riemannian symmetric spaces, the orbits of this action play an important role. In the case of spaces of rank 2, they form a 1-parameter family, which can be parametrized in the following way (generalizing the approach that was used for the 2-Grassmannians in [7] and [8]):

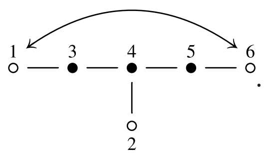
We suppose that M is of rank 2, and fix a Weyl chamber $\mathfrak{c}$ in $\mathfrak{a}$. We denote the two rays in $\mathfrak{a}$ delineating this Weyl chamber by R_1 and R_2; in the case where $\Delta(\mathfrak{m}, \mathfrak{a})$ contains roots of different length (i.e. the root system $\Delta(\mathfrak{m}, \mathfrak{a})$ is of one of the types B_2, BC_2 or G_2), we suppose that R_1 points into the direction of one of the shorter roots. Let $\varphi_{\max}$ be the angle between R_1 and R_2; $\varphi_{\max}$ equals $\pi/3$, $\pi/4$, $\pi/4$ or $\pi/6$, according to whether $\Delta(\mathfrak{m}, \mathfrak{a})$ is of type A_2, B_2, BC_2 or G_2, respectively.

Any given $v \in \mathfrak{m} \setminus \{0\}$ is congruent under the isotropy action to one and only one vector $v_0 \in \bar{\mathfrak{c}}$, and we denote the angle between R_1 and v_0 by $\varphi(v)$. In this way we obtain a continuous function $\varphi\colon \mathfrak{m} \setminus \{0\} \to [0, \varphi_{\max}]$. Two vectors $v_1, v_2 \in \mathfrak{m}$ with $\|v_1\| = \|v_2\| \neq 0$ are congruent under the isotropy action if and only if $\varphi(v_1) = \varphi(v_2)$ holds. We call the value $\varphi(v)$ the *isotropy angle* of a vector $v \in \mathfrak{m} \setminus \{0\}$.

Notice that if $\mathfrak{m}'$ is a Lie triple system of $\mathfrak{m}$ of rank 1, then φ is constant on $\mathfrak{m}' \setminus \{0\}$, and Proposition 2.3 shows that there are only finitely many $t \in [0, \varphi_{\max}]$ so that there exists a Lie triple system $\mathfrak{m}' \subset \mathfrak{m}$ of rank 1 with $\varphi|(\mathfrak{m}' \setminus \{0\}) = t$ and $\dim(\mathfrak{m}') \geq 2$. We will call the value t for such a Lie triple system $\mathfrak{m}'$ the *isotropy angle* of $\mathfrak{m}'$. On the other hand, if $\mathfrak{m}'$ is of rank 2, then we have $\varphi(\mathfrak{m}' \setminus \{0\}) = [0, \varphi_{\max}]$.

3. The symmetric spaces $E_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$, Sp(2) and SO(10)/U(5)

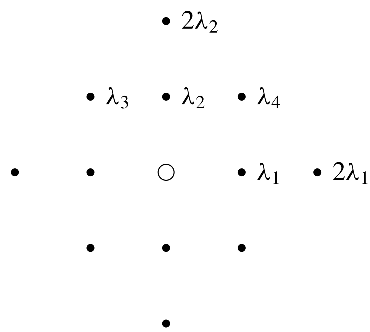
3.1. The geometry of $E_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$. In the present section we will study the Hermitian symmetric space EIII $:= E_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$, which has the Satake diagram



We consider the Lie algebra $\mathfrak{g} := \mathfrak{e}_6$ of the transvection group E_6 of EIII, and the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ induced by the symmetric structure of EIII. Here $\mathfrak{k} = \mathbb{R} \oplus \mathfrak{so}(10)$ is the Lie algebra of the isotropy group of EIII, and $\mathfrak{m}$ is isomorphic to the tangent space of EIII in the origin. The E_6-invariant Riemannian metric on EIII induces an $\mathrm{Ad}(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$-invariant Riemannian metric on $\mathfrak{m}$. As was explained in Section 2, this metric is only unique up to a factor; we choose the factor in such a way that the shortest restricted roots of EIII (see below) have length 1.

The root space decomposition. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ which is maximally non-compact, i.e. $\mathfrak{t}$ is chosen such that the flat subspace $\mathfrak{a} := \mathfrak{t} \cap \mathfrak{m}$ of $\mathfrak{m}$ is of the maximal dimension 2, and hence a Cartan subalgebra of $\mathfrak{m}$. Then we consider the root system $\Delta^{\mathfrak{g}} \subset \mathfrak{t}^*$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, as well as the restricted root system $\Delta \subset \mathfrak{a}^*$ of the symmetric space EIII with respect to $\mathfrak{a}$. EIII has the restricted Dynkin diagram with multiplicities $\bullet^{6} \Leftrightarrow \odot^{8[1]}$, in other words: its restricted root system Δ is of type BC_2, i.e. we have $\Delta = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4, \pm 2\lambda_1, \pm 2\lambda_2\}$, where (λ_1, λ_3) is a system of simple roots of Δ, these two roots are at an angle of $(3/4)\pi$ with λ_3 being the longer of the two, and we have $\lambda_2 = \lambda_1 + \lambda_3$, $\lambda_4 = 2\lambda_1 + \lambda_3$. Moreover, the restricted roots have the following multiplicities: $n_{\lambda_1} = n_{\lambda_2} = 8$, $n_{\lambda_3} = n_{\lambda_4} = 6$ and $n_{2\lambda_1} = n_{2\lambda_2} = 1$.

Δ has the following graphical representation:



To be able to apply the results from [9] and the corresponding computer package for the calculation of the curvature tensor of EIII, we need to describe the relationship between the restricted roots of the symmetric space EIII and the (non-restricted) roots of the Lie algebra $\mathfrak{e}_6$. For this purpose, we order the simple roots of $\mathfrak{e}_6$ as they are numbered in the Satake diagram of EIII given above. Then we label the 36 positive roots of $\mathfrak{e}_6$ by $\alpha_1, \dots, \alpha_{36}$ in the order in which they are produced by Algorithm (R) in Section 2 of [9] based on this ordering of the simple roots. It turns out that $\alpha_1, \dots, \alpha_{36}$ have the following coordinates with respect to the simple roots of $\mathfrak{e}_6$ ordered as before:

α_1	(1, 0, 0, 0, 0, 0)	α_{10}	(0, 0, 0, 1, 1, 0)	α_{19}	(0, 1, 1, 1, 1, 0)	α_{28}	(0, 1, 1, 2, 1, 1)
α_2	(0, 1, 0, 0, 0, 0)	α_{11}	(0, 0, 0, 0, 1, 1)	α_{20}	(0, 1, 0, 1, 1, 1)	α_{29}	(1, 1, 1, 2, 1, 1)
α_3	(0, 0, 1, 0, 0, 0)	α_{12}	(1, 0, 1, 1, 0, 0)	α_{21}	(0, 0, 1, 1, 1, 1)	α_{30}	(1, 1, 2, 2, 1, 0)
α_4	(0, 0, 0, 1, 0, 0)	α_{13}	(0, 1, 1, 1, 0, 0)	α_{22}	(1, 1, 1, 1, 1, 0)	α_{31}	(0, 1, 1, 2, 2, 1)
α_5	(0, 0, 0, 0, 1, 0)	α_{14}	(0, 1, 0, 1, 1, 0)	α_{23}	(1, 0, 1, 1, 1, 1)	α_{32}	(1, 1, 1, 2, 2, 1)
α_6	(0, 0, 0, 0, 0, 1)	α_{15}	(0, 0, 1, 1, 1, 0)	α_{24}	(0, 1, 1, 1, 1, 1)	α_{33}	(1, 1, 2, 2, 1, 1)
α_7	(1, 0, 1, 0, 0, 0)	α_{16}	(0, 0, 0, 1, 1, 1)	α_{25}	(0, 1, 1, 2, 1, 0)	α_{34}	(1, 1, 2, 2, 2, 1)
α_8	(0, 1, 0, 1, 0, 0)	α_{17}	(1, 1, 1, 1, 0, 0)	α_{26}	(1, 1, 1, 1, 1, 1)	α_{35}	(1, 1, 2, 3, 2, 1)
α_9	(0, 0, 1, 1, 0, 0)	α_{18}	(1, 0, 1, 1, 1, 0)	α_{27}	(1, 1, 1, 2, 1, 0)	α_{36}	(1, 2, 2, 3, 2, 1)

To find out which restricted root of EIII corresponds to each root of $\mathfrak{g} = \mathfrak{e}_6$, we consider the involutive automorphism σ on $\mathfrak{g}$ which describes the symmetric structure of EIII; we tabulate the orbits of the action of σ on the root system $\Delta^{\mathfrak{g}}$ and the restricted root of EIII corresponding to each orbit (compare Section 4 of [9]):

orbit	$\{\alpha_1, -\alpha_{21}\}$	$\{\alpha_6, -\alpha_{18}\}$	$\{\alpha_7, -\alpha_{16}\}$	$\{\alpha_{11}, -\alpha_{12}\}$	$\{\alpha_{23}, -\alpha_{23}\}$
corresp. restr. root	λ_1	λ_1	λ_1	λ_1	$2\lambda_1$

orbit	$\{\alpha_{17}, -\alpha_{31}\}$	$\{\alpha_{20}, -\alpha_{30}\}$	$\{\alpha_{22}, -\alpha_{28}\}$	$\{\alpha_{24}, -\alpha_{27}\}$	$\{\alpha_{36}, -\alpha_{36}\}$
corresp. restr. root	λ_2	λ_2	λ_2	λ_2	$2\lambda_2$

orbit	$\{\alpha_2, -\alpha_{25}\}$	$\{\alpha_8, -\alpha_{19}\}$	$\{\alpha_{13}, -\alpha_{14}\}$	$\{\alpha_{26}, -\alpha_{35}\}$	$\{\alpha_{29}, -\alpha_{34}\}$	$\{\alpha_{32}, -\alpha_{33}\}$
corresp. restr. root	λ_3	λ_3	λ_3	λ_4	λ_4	λ_4

Moreover, we have $\sigma(\alpha_k) = \alpha_k$ for $k \in \{3, 4, 5, 9, 10, 15\}$.

In the sequel of this paper, we need a parametrization of the restricted root spaces of EIII; in the calculations we will use this parametrization to pinpoint individual vectors in the root spaces of $\mathfrak{m}$. For this purpose we introduce the notations already used in [9]. First, we note that there exists a Chevalley basis $(X_\alpha)_{\alpha\in\Delta^{\mathfrak{g}}}$ for $\mathfrak{g}^{\mathbb{C}}$ with Chevalley constants $(c_{\alpha,\beta})_{\alpha,\beta\in\Delta^{\mathfrak{g}}}$, i.e. for any $\alpha, \beta \in \Delta^{\mathfrak{g}}$ we have $c_{\alpha,\beta} \in \mathbb{R}$,

$$[X_\alpha, X_\beta] = \begin{cases} c_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta^{\mathfrak{g}}, \\ \alpha^\sharp & \text{if } \alpha+\beta = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $c_{-\alpha,-\beta} = -c_{\alpha,\beta}$ (see, for example, [10], §VI.1); it has been shown in Section 3 of [9] that the Chevalley data can be chosen in such a way that $\overline{X_\alpha} = -X_{-\alpha}$ holds (where $\overline{X}$ denotes the conjugation of $X \in \mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$). For $\alpha \in \Delta^{\mathfrak{g}}$ and $z \in \mathbb{C}$, we have

$$V_\alpha(z) := \frac{1}{\sqrt{2}}(zX_\alpha - \overline{z}X_{-\alpha}) \in \mathfrak{g}, \tag{7}$$

and the root space $\mathfrak{g}_\alpha$ of the real Lie algebra $\mathfrak{g}$ corresponding to the root α is given by $\mathfrak{g}_\alpha = \{V_\alpha(z) \mid z \in \mathbb{C}\}$ (see [9], Proposition 3.3 (d)).

Like in [9], Proposition 5.2 (a) we now describe $\mathfrak{k}_\alpha := \mathfrak{g}_\alpha \cap \mathfrak{k}$ and $\mathfrak{m}_\alpha := \mathfrak{g}_\alpha \cap \mathfrak{m}$ for $\alpha \in \Delta^{\mathfrak{g}}$. In the case $\alpha \circ \sigma \neq \pm\sigma$, we put for $z \in \mathbb{C}$

$$K_\alpha(z) := \frac{1}{\sqrt{2}}(V_\alpha(z) + s_\alpha V_{\alpha\circ\sigma}(z))$$

and

$$M_\alpha(z) := \frac{1}{\sqrt{2}}(V_\alpha(z) - s_\alpha V_{\alpha\circ\sigma}(z)),$$

herein $s_\alpha \in \{\pm 1\}$ is characterized by the equation $\sigma(V_\alpha(z)) = s_\alpha V_{\alpha\circ\sigma}(z)$. In the case $\alpha \circ \sigma = \alpha$ we note $\mathfrak{k}_\alpha = \mathfrak{g}_\alpha = \{V_\alpha(z) \mid z \in \mathbb{C}\}$ and $\mathfrak{m}_\alpha = \{0\}$. In the case $\alpha \circ \sigma = -\alpha$ we put for $t \in \mathbb{R}$

$$\tilde{K}_\alpha(t) = \begin{cases} V_\alpha(it) & \text{if} \quad s_\alpha = 1, \\ V_\alpha(t) & \text{if} \quad s_\alpha = -1 \end{cases}$$

and

$$\tilde{M}_\alpha(t) = \begin{cases} V_\alpha(t) & \text{if} \quad s_\alpha = 1, \\ V_\alpha(it) & \text{if} \quad s_\alpha = -1. \end{cases}$$

Then we have $\mathfrak{k}_\alpha = \{\tilde{K}_\alpha(t) \mid t \in \mathbb{R}\}$ and $\mathfrak{m}_\alpha = \{\tilde{M}_\alpha(t) \mid t \in \mathbb{R}\}$.

We now apply this general parametrization of restricted root spaces to the symmetric space EIII using the description of the restricted roots and the orbits of σ for EIII given above. For $c_1, \dots, c_4 \in \mathbb{C}$ and $t \in \mathbb{R}$, and where A denotes either of the letters K and M, we put:

$$\begin{aligned}
A_{\lambda_1}(c_1, c_2, c_3, c_4) &:= A_{\alpha_1}(c_1) + A_{\alpha_6}(c_2) + A_{\alpha_7}(c_3) + A_{\alpha_{11}}(c_4),\\
A_{2\lambda_1}(t) &:= \tilde{A}_{\alpha_{23}}(t),\\
A_{\lambda_2}(c_1, c_2, c_3, c_4) &:= A_{\alpha_{17}}(c_1) + A_{\alpha_{20}}(c_2) + A_{\alpha_{22}}(c_3) + A_{\alpha_{24}}(c_4),\\
A_{2\lambda_2}(t) &:= \tilde{A}_{\alpha_{36}}(t),\\
A_{\lambda_3}(c_1, c_2, c_3) &:= A_{\alpha_2}(c_1) + A_{\alpha_8}(c_2) + A_{\alpha_{13}}(c_3),\\
A_{\lambda_4}(c_1, c_2, c_3) &:= A_{\alpha_{26}}(c_1) + A_{\alpha_{29}}(c_2) + A_{\alpha_{32}}(c_3).
\end{aligned}$$

Then we have $\mathfrak{m}_{\lambda_k} = M_{\lambda_k}(\mathbb{C}, \mathbb{C}, \mathbb{C}, \mathbb{C})$ and $\mathfrak{m}_{2\lambda_k} = M_{2\lambda_k}(\mathbb{R})$ for $k \in \{1, 2\}$, and $\mathfrak{m}_{\lambda_k} = M_{\lambda_k}(\mathbb{C}, \mathbb{C}, \mathbb{C})$ for $k \in \{3, 4\}$.

The action of the isotropy group. We next look at the isotropy action of EIII. Regarding it, we use the notations introduced at the end of Section 2, in particular we have the continuous function $\varphi\colon \mathfrak{m} \setminus \{0\} \to [0, \pi/4]$ parametrizing the orbits of the isotropy action. For the elements of the closure $\bar{\mathfrak{c}}$ of the positive Weyl chamber $\mathfrak{c} := \{v \in \mathfrak{a} \mid \lambda_1(v) \geq 0, \lambda_3(v) \geq 0\}$, we can explicitly describe the relation to their isotropy angle: $(\lambda_2^\sharp, \lambda_1^\sharp)$ is an orthonormal basis of $\mathfrak{a}$ so that with $v_t := \cos(t)\lambda_2^\sharp + \sin(t)\lambda_1^\sharp$ we have

$$\bar{\mathfrak{c}} = \left\{ s \cdot v_t \;\middle|\; t \in \left[0, \frac{\pi}{4}\right],\ s \in \mathbb{R}_{\geq 0} \right\}, \tag{8}$$

and because the Weyl chamber $\mathfrak{c}$ is bordered by the two vectors $v_0 = \lambda_2^\sharp$ with $\varphi(v_0) = 0$ and $v_{\pi/4} = (1/\sqrt{2})\lambda_4^\sharp$ with $\varphi(v_{\pi/4}) = \pi/4$, we have

$$\varphi(s \cdot v_t) = t \quad \text{for all} \quad t \in \left[0, \frac{\pi}{4}\right],\ s \in \mathbb{R}_+. \tag{9}$$

The action of the subgroup K_0 of K whose Lie algebra is the centralizer $\mathfrak{k}_0 := \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}$ of $\mathfrak{a}$ in $\mathfrak{k}$ leaves the restricted root spaces $\mathfrak{m}_\lambda$ invariant. The Dynkin diagram of $\mathfrak{k}_0$ is given by the black roots in the Satake diagram of EIII (see above), therefore we have $\mathfrak{k}_0 = (\mathfrak{t} \cap \mathfrak{k}) \oplus \bigoplus K_{\alpha_l}(\mathbb{C})$, where the sum runs over all those roots α_l of $\mathfrak{e}_6$ with $\sigma(\alpha_l) = \alpha_l$, i.e. $l \in \{3, 4, 5, 9, 10, 15\}$. Because of this and the fact that $\dim(\mathfrak{t} \cap \mathfrak{k}) = 4$ holds, it follows that $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(4)$, and hence K_0 is locally isomorphic to U(4).

By using the Maple implementation to look at the adjoint action of $\mathfrak{k}_0$ on the root spaces $\mathfrak{m}_\lambda$, we can describe the action of K_0 on the root spaces in more detail:

Proposition 3.1. *For $k \in \{1, 2\}$ the action of K_0 on $\mathfrak{m}_{\lambda_k}$ is locally equivalent to the vector representation of* U(4), *this means that if we denote by φ the linear isometry*

$$\varphi\colon \mathbb{C}^4 \to \mathfrak{m}_{\lambda_k}, \quad (c_1, c_2, c_3, c_4) \mapsto M_{\lambda_k}(c_1, c_2, c_3, c_4),$$

there exists a local isomorphism of Lie groups $\Phi\colon \mathrm{U}(4) \to K_0$ so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{U}(4) \times \mathbb{C}^4 \arrow[r, "\Phi\times\varphi"] \arrow[d] & K_0 \times \mathfrak{m}_{\lambda_k} \arrow[d, "\mathrm{Ad}"] \\ \mathbb{C}^4 \arrow[r, "\varphi"'] & \mathfrak{m}_{\lambda_k}, \end{tikzcd}$$

where the left vertical arrow represents the canonical action of U(4) *on* $\mathbb{C}^4$.

Moreover, if we fix $v \in \mathfrak{m}_{\lambda_k} \setminus \{0\}$, then the Lie subgroup $U' := \{B \in \mathrm{U}(4) \mid B(\varphi^{-1}v) = \varphi^{-1}v\}$ of U(4) *is isomorphic to* U(3), *and hence the Lie subgroup $K_0' := \{g \in K_0 \mid \mathrm{Ad}(g)v = v\}$ of K_0 is locally isomorphic to* U(3). *For $l \in \{3, 4\}$, the action of K_0' on $\mathfrak{m}_{\lambda_l}$ is locally equivalent to the vector representation of* U(3), *meaning that if we denote by ψ the linear isometry*

$$\psi\colon \mathbb{C}^3 \to \mathfrak{m}_l, \quad (c_1, c_2, c_3) \mapsto M_{\lambda_l}(c_1, c_2, c_3),$$

there exists a local isomorphism of Lie groups $\Psi\colon \mathrm{U}(3) \to K_0'$ so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{U}(3) \times \mathbb{C}^3 \arrow[r, "\Psi\times\psi"] \arrow[d] & K_0' \times \mathfrak{m}_{\lambda_l} \arrow[d, "\mathrm{Ad}"] \\ \mathbb{C}^3 \arrow[r, "\psi"'] & \mathfrak{m}_{\lambda_l}, \end{tikzcd}$$

where the left vertical arrow represents the canonical action of U(3) *on* $\mathbb{C}^3$.

In particular we see that $\mathrm{Ad}(K_0)$ *acts "jointly transitively" on the unit spheres in $\mathfrak{m}_{\lambda_k}$ and $\mathfrak{m}_{\lambda_l}$ in the sense that for any given $v_1, v_2 \in \mathfrak{m}_{\lambda_k}$ and $w_1, w_2 \in \mathfrak{m}_{\lambda_l}$ with $\|v_1\| = \|v_2\|$ and $\|w_1\| = \|w_2\|$ there exists $g \in K_0$ with $\mathrm{Ad}(g)v_1 = v_2$ and $\mathrm{Ad}(g)w_1 = w_2$.*

Finally, we note that the linear isometries

$$\mathfrak{m}_{\lambda_1} \to \mathfrak{m}_{\lambda_2}, \quad M_{\lambda_1}(c_1, c_2, c_3, c_4) \mapsto M_{\lambda_2}(c_2, c_1, c_4, c_3)$$

and

$$\mathfrak{m}_{\lambda_3} \to \mathfrak{m}_{\lambda_4}, \quad M_{\lambda_3}(c_1, c_2, c_3) \mapsto M_{\lambda_4}(c_1, c_2, c_3)$$

commute with the action of $\mathrm{Ad}(K_0)$ *on the respective root spaces.*

The complex structure of EIII. EIII is a Hermitian symmetric space; the action of its complex structure J on $\mathfrak{m}$ is given by $J|\mathfrak{m} = \mathrm{ad}(j)|\mathfrak{m}$, where j is a element of the center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ so that $(\mathrm{ad}(j)|\mathfrak{m})^2 = -\mathrm{id}_{\mathfrak{m}}$ holds. Because $\mathfrak{z}(\mathfrak{k})$ is one-dimensional, this condition already determines j up to sign; we find via computations with the Maple package for computation of the Lie bracket of $\mathfrak{e}_6$ that

$$j = \frac{2}{3}(\alpha_1^\sharp - \alpha_6^\sharp) + \frac{1}{3}(\alpha_3^\sharp - \alpha_5^\sharp) + K_{2\lambda_1}(1) - K_{2\lambda_2}(1)$$

is one of the two possible choices; here we again denote for $\alpha \in \mathfrak{t}^*$ by $\alpha^\sharp \in \mathfrak{t}$ the dual of α with respect to the Killing form $\varkappa$ of $\mathfrak{g}$, i.e. the vector so that $\varkappa(\alpha^\sharp, \cdot) = \alpha$ holds.

Using this presentation of j and the formula $Jv = \mathrm{ad}(j)v$ for $v \in \mathfrak{m}$, we can again use the Maple package to calculate the action of J on $\mathfrak{m}$. In this way, we obtain for $c_1, \dots, c_4 \in \mathbb{C}$ and $t, s \in \mathbb{R}$:

$$\begin{aligned} J(t\lambda_1^\sharp + s\lambda_2^\sharp) &= \frac{1}{2}(M_{2\lambda_1}(t) - M_{2\lambda_2}(s)), \\ J(M_{\lambda_1}(c_1, c_2, c_3, c_4)) &= M_{\lambda_1}(ic_1, -ic_2, ic_3, -ic_4), \\ J(M_{\lambda_2}(c_1, c_2, c_3, c_4)) &= M_{\lambda_2}(ic_1, -ic_2, ic_3, -ic_4), \\ J(M_{\lambda_3}(c_1, c_2, c_3)) &= M_{\lambda_4}(ic_1, -ic_2, -i\overline{c_3}), \\ J(M_{\lambda_4}(c_1, c_2, c_3)) &= M_{\lambda_3}(ic_1, -ic_2, i\overline{c_3}), \\ J(M_{2\lambda_1}(t)) &= -2t\lambda_1^\sharp, \\ J(M_{2\lambda_2}(s)) &= 2s\lambda_2^\sharp. \end{aligned}$$

In particular we see that $\mathfrak{m}_{\lambda_1}$ and $\mathfrak{m}_{\lambda_2}$ are complex linear subspaces of $\mathfrak{m}$, whereas $\mathfrak{a}$, $\mathfrak{m}_{2\lambda_1} \oplus \mathfrak{m}_{2\lambda_2}$, $\mathfrak{m}_{\lambda_3}$ and $\mathfrak{m}_{\lambda_4}$ are totally real linear subspaces with $J(\mathfrak{a}) = \mathfrak{m}_{2\lambda_1} \oplus \mathfrak{m}_{2\lambda_2}$ and $J(\mathfrak{m}_{\lambda_3}) = \mathfrak{m}_{\lambda_4}$.

3.2. Lie triple systems in $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$. We are now ready to describe the Lie triple systems in EIII.

DEFINITION 3.2. Let V be a unitary space. We say that an $\mathbb{R}$-linear subspace $U \subset V$ is
(a) of $\mathbb{C}$P-type $(\mathbb{C}, \dim_{\mathbb{C}}(U))$ if it is a complex subspace of V,
(b) of $\mathbb{C}$P-type $(\mathbb{R}, \dim_{\mathbb{R}}(U))$ if it is a totally real subspace of V.

Theorem 3.3. *The linear subspaces $\mathfrak{m}'$ of $\mathfrak{m}$ listed in the following are Lie triple systems, and every Lie triple system $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{m}$ is congruent under the isotropy action to one of them.*[2]

[2]Please read Remarks 3.5 and 3.6 below before you suspect that there might be Lie triple systems missing from the list.

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/4]$:
  $\mathfrak{m}' = \mathbb{R}(\cos(t)\lambda_2^\sharp + \sin(t)\lambda_1^\sharp)$ (*compare* Equation (8)).
- ($\mathbb{P}$, $\varphi = 0$, ($\mathbb{C}$, 5)):
  $\mathfrak{m}' = \mathbb{R}\lambda_2^\sharp \oplus \mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{2\lambda_2}$.
- ($\mathbb{P}$, $\varphi = \pi/4$, τ) *with* $\tau \in \{\mathbb{S}^5, \mathbb{S}^6, \mathbb{S}^7, \mathbb{S}^8, \mathbb{O}\mathrm{P}^2\}$:
  Put $H := \lambda_1^\sharp + \lambda_2^\sharp$ *and* $\tilde{H} := M_{2\lambda_1}(1) + M_{2\lambda_2}(1)$.
  For $\tau = \mathbb{S}^k$: $\mathfrak{m}'$ *is a k-dimensional linear subspace of* $\mathbb{R}H \oplus \mathfrak{m}_{\lambda_4} \oplus \mathbb{R}\tilde{H}$.
  For $\tau = \mathbb{O}\mathrm{P}^2$: $\mathfrak{m}' = \mathbb{R}H \oplus \{M_{\lambda_1}(c_1, c_2, c_3, c_4) + M_{\lambda_2}(c_2, c_1, -c_4, -c_3) \mid c_1, c_2, c_3, c_4 \in \mathbb{C}\} \oplus M_{\lambda_4}(\mathbb{C}, \mathbb{C}, \mathbb{C}) \oplus \mathbb{R}\tilde{H}$.
- ($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{K}_1$, l), $\mathbb{K}_2$) *with* $l \in \{4, 5\}$, $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$ *and* $(\mathbb{K}_1, \mathbb{K}_2) \neq (\mathbb{R}, \mathbb{R})$:
  We have $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1} \oplus \mathfrak{m}'_{2\lambda_1} \oplus \mathfrak{m}'_{2\lambda_2}$, *where* $\mathfrak{m}'_{\lambda_1}$ *is a subspace of* $\mathfrak{m}_{\lambda_1}$ *of* $\mathbb{C}\mathrm{P}$-*type* $(\mathbb{K}_1, l-1)$, *and where we put for* $k \in \{1, 2\}$ $\mathfrak{m}'_{2\lambda_k} := \mathfrak{m}_{2\lambda_k}$ *if* $\mathbb{K}_k = \mathbb{C}$, $\mathfrak{m}'_{2\lambda_k} := \{0\}$ *if* $\mathbb{K}_k = \mathbb{R}$.
- (Q):
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_3}(\mathbb{C}, \mathbb{C}, \mathbb{C}) \oplus M_{\lambda_4}(\mathbb{C}, \mathbb{C}, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$.
- (Q, τ) *where* τ *is one of the types listed in* [7], Theorem 4.1 *for* $m = 8$, *i.e.* τ *is one of* (G1, k) *with* $k \leq 8$, (G2, k_1, k_2) *with* $k_1 + k_2 \leq 8$, (G3), (P1, k) *with* $k \leq 8$, (P2), (A), (I1, k) *with* $k \leq 4$, *and* (I2, k) *with* $k \leq 4$:
  $\mathfrak{m}'$ *is contained in a Lie triple system* $\hat{\mathfrak{m}}'$ *of type* (Q), *corresponding to a complex quadric* Q^8, *and regarded as a Lie triple system of* $\hat{\mathfrak{m}}'$, $\mathfrak{m}'$ *is of type* τ *according to the classification in* [7], Theorem 4.1.
- ($G_2\mathbb{C}^6$):
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_3}(0, 0, \mathbb{C}) \oplus M_{\lambda_4}(0, 0, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$.
- ($G_2\mathbb{C}^6$, τ), *where* τ *is one of the following types listed in* [8], Theorem 7.1 *for* $n = 4$: $(\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{K}, k))$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k \leq 2$, $(\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 2))$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $(\mathrm{G}_2, (\mathbb{K}, k))$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k \in \{3, 4\}$, *and* $(\mathbb{P} \times \mathbb{P}, (\mathbb{K}, k), (\mathbb{K}', k'))$ *with* $\mathbb{K}, \mathbb{K}' \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k + k' \leq 4$:
  $\mathfrak{m}'$ *is contained in a Lie triple system* $\hat{\mathfrak{m}}'$ *of type* ($G_2\mathbb{C}^6$), *corresponding to a complex Grassmannian* $G_2(\mathbb{C}^6)$, *and regarded as a Lie triple system of* $\hat{\mathfrak{m}}'$, $\mathfrak{m}'$ *is of type* τ *according to the classification in* [8], Theorem 7.1.
- ($G_2\mathbb{H}^4$):
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}) \oplus M_{\lambda_2}(i\mathbb{R}, i\mathbb{R}, i\mathbb{R}, i\mathbb{R}) \oplus M_{\lambda_3}(\mathbb{R}, \mathbb{R}, \mathbb{R}) \oplus M_{\lambda_4}(\mathbb{R}, \mathbb{R}, \mathbb{R})$.
- ($G_2\mathbb{H}^4$, τ), *where* τ *is one of the following types listed in* [8], Theorem 5.3 *for* $n = 2$: $(\mathbb{P}, \varphi = 0, (\mathbb{K}, 2))$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $(\mathbb{S}, \varphi = \arctan(1/3), 3)$, $(\mathbb{P}, \varphi = \pi/4, (\mathbb{S}^3))$, $(\mathbb{P}, \varphi = \pi/4, (\mathbb{H}, 1))$, $(\mathbb{S}^5, \varphi = \pi/4)$, $(\mathrm{G}_2, (\mathbb{H}, 1))$, $(\mathbb{S}^1 \times \mathbb{S}^5, k)$ *with* $3 \leq k \leq 5$, *and* (Sp_2):
  $\mathfrak{m}'$ *is contained in a Lie triple system* $\hat{\mathfrak{m}}'$ *of type* ($G_2\mathbb{H}^4$), *corresponding locally to a quaternionic Grassmannian* $G_2(\mathbb{H}^4)$, *and regarded as a Lie triple system of* $\hat{\mathfrak{m}}'$, $\mathfrak{m}'$ *is of type* τ *according to the classification in* [8], Theorem 5.3.
- (DIII):
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_2}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_3}(0, \mathbb{C}, \mathbb{C}) \oplus M_{\lambda_4}(0, \mathbb{C}, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$.

We call the full name (Geo, $\varphi = t$), ($\mathbb{P}$, $\varphi = 0$, ($\mathbb{C}$, 5)) *etc. given in the above table the type of the Lie triple systems which are isotropy-congruent to the space given in that entry. Then every Lie triple system of* $\mathfrak{m}$ *is of exactly one type.*

In the type names of Lie triple systems of rank 1, *the value given in the form* $\varphi = t$ *is the isotropy angle* (*see the end of* Section 2) *of the Lie triple systems of that type.*

The Lie triple systems $\mathfrak{m}'$ *of the various types have the properties given in the following table. The column "isometry type" gives the isometry type of the totally geodesic submanifolds corresponding to the Lie triple systems of the respective type in abbreviated form* (*without specification of the scaling factors of the Riemannian metrics*), *for the details see* Section 3.3.

type of $\mathfrak{m}'$	$\dim(\mathfrak{m}')$	$\mathrm{rk}(\mathfrak{m}')$	$\mathfrak{m}'$ *complex or totally real*?	$\mathfrak{m}'$ *maximal*	*isometry type*
(Geo, $\varphi = t$)	1	1	*totally real*	*no*	$\mathbb{R}$ *or* $\mathbb{S}^1$
($\mathbb{P}$, $\varphi = 0$, ($\mathbb{C}$, 5))	10	1	*complex*	*no*	$\mathbb{C}\mathrm{P}^5$
($\mathbb{P}$, $\varphi = \pi/4$, $\mathbb{S}^l$)	l	1	*totally real*	*no*	$\mathbb{S}^l$
($\mathbb{P}$, $\varphi = \pi/4$, $\mathbb{O}\mathrm{P}^2$)	16	1	*totally real*	*yes*	$\mathbb{O}\mathrm{P}^2$
($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{R}$, l), $\mathbb{C}$)	$l + 2$	2	*neither*	*no*	$\mathbb{R}\mathrm{P}^l \times \mathbb{C}\mathrm{P}^1$
($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{C}$, l), $\mathbb{R}$)	$2l + 1$	2	*neither*	*no*	$\mathbb{C}\mathrm{P}^l \times \mathbb{R}\mathrm{P}^1$
($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{C}$, l), $\mathbb{C}$)	$2l + 2$	2	*complex*	*for* $l = 5$	$\mathbb{C}\mathrm{P}^l \times \mathbb{C}\mathrm{P}^1$
(Q)	16	2	*complex*	*yes*	Q^8
(Q, τ)	*see* [7], Theorem 4.1			*no*	
($G_2\mathbb{C}^6$)	16	2	*complex*	*yes*	$G_2(\mathbb{C}^6)$
($G_2\mathbb{C}^6$, τ)	*see* [8], Theorem 7.1			*no*	
($G_2\mathbb{H}^4$)	16	2	*totally real*	*yes*	$G_2(\mathbb{H}^4)/\mathbb{Z}_2$
($G_2\mathbb{H}^4$, τ)	*see* [8], Theorem 5.3		*totally real*	*no*	
(DIII)	20	2	*complex*	*yes*	SO(10)/U(5)

REMARK 3.4. The Lie triple systems of type (Q, τ), ($G_2\mathbb{C}^6$, τ) and ($G_2\mathbb{H}^4$, τ) are contained in Lie triple systems of type (Q) (corresponding to a complex quadric Q^8), ($G_2\mathbb{C}^6$) (corresponding to $G_2(\mathbb{C}^6)$) and ($G_2\mathbb{H}^4$) (corresponding to $G_2(\mathbb{H}^4)/\mathbb{Z}_2$), respectively. To obtain explicit descriptions of these types, one needs to apply the results in [7] and [8] on the classification of Lie triple systems in these spaces.

To be able to do so, it is important to know how the root systems of the Lie triple systems of type (Q), ($G_2\mathbb{C}^6$) and ($G_2\mathbb{H}^4$) are embedded in the root system of EIII, and also how the function φ parametrizing the orbits of the isotropy action defined for Q^m and $G_2(\mathbb{K}^n)$ in [7] resp. in [8] relates to the corresponding function φ defined for EIII in the present paper.

Because the Lie triple systems of type (Q), ($G_2\mathbb{C}^6$) and ($G_2\mathbb{H}^4$) have maximal rank in EIII, their respective root systems $\Delta_{(Q)}$, $\Delta_{(G_2\mathbb{C}^6)}$ and $\Delta_{(G_2\mathbb{H}^4)}$ are simply subsets of the root system Δ of EIII (see Proposition 2.1 (b), and also see the proof of

Theorem 3.3 below). In fact, from the definition of these types in Theorem 3.3 it follows immediately that we have

$$\Delta_{(Q)} = \{\pm\lambda_3, \pm\lambda_4, \pm 2\lambda_1, \pm 2\lambda_2\},$$
$$\Delta_{(G_2\mathbb{C}^6)} = \Delta,$$
$$\Delta_{(G_2\mathbb{H}^4)} = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}.$$

For each of the types $\hat{\tau} \in \{(Q), (G_2\mathbb{C}^6), (G_2\mathbb{H}^4)\}$ we now let $\mathfrak{m}_{\hat{\tau}}$ be a Lie triple system of EIII of type $\hat{\tau}$, and let $\varphi_{\hat{\tau}} \colon \mathfrak{m}_{\hat{\tau}} \setminus \{0\} \to [0, \pi/4]$ be the function parametrizing the orbits of the isotropy action of the symmetric space corresponding to $\mathfrak{m}_{\hat{\tau}}$ (i.e. Q^8, $G_2(\mathbb{C}^6)$ or $G_2(\mathbb{H}^4)/\mathbb{Z}_2$) as introduced in [7] at the beginning of Section 4.2 resp. in [8], Section 4. Note that in these cases, we always measured the angle $\varphi(v)$ from the vector corresponding to the shortest root present in Q^n resp. $G_2(\mathbb{C}^n)$ for large n, even if this root vanishes for certain small values of n (as happens for $G_2(\mathbb{H}^4)$). Keeping this in mind, and considering the root systems $\Delta_{\hat{\tau}}$ as given above, we see that the functions $\varphi_{\hat{\tau}}$ is related to the function $\varphi\colon \mathfrak{m} \setminus \{0\} \to [0, \pi/4]$ parametrizing the isotropy orbits of EIII by

$$\varphi_{(Q)}(v) = \frac{\pi}{4} - \varphi(v) \quad \text{for} \quad v \in \mathfrak{m}_{(Q)} \setminus \{0\},$$
$$\varphi_{(G_2\mathbb{C}^6)}(v) = \varphi(v) \quad \text{for} \quad v \in \mathfrak{m}_{(G_2\mathbb{C}^6)} \setminus \{0\},$$
$$\varphi_{(G_2\mathbb{H}^4)}(v) = \frac{\pi}{4} - \varphi(v) \quad \text{for} \quad v \in \mathfrak{m}_{(G_2\mathbb{H}^4)} \setminus \{0\}.$$

REMARK 3.5. We now introduce alternative definitions for some types of Lie triple systems, to make it more intuitive that indeed all congruence classes of Lie triple systems are covered in Theorem 3.3, and also to simplify the notations in what follows.

First, we consider the types $(G_2\mathbb{C}^6, \tau)$ resp. $(G_2\mathbb{H}^4, \tau)$ also for those types τ listed in [8], Theorem 7.1 for $n = 4$ resp. in [8], Theorem 5.3 for $n = 2$ which have not been mentioned in Theorem 3.3. Then a Lie triple system of EIII is contained in a Lie triple system of type $(G_2\mathbb{C}^6)$ resp. $(G_2\mathbb{H}^4)$ if and only if it is of type $(G_2\mathbb{C}^6, \tau)$ resp. of type $(G_2\mathbb{H}^4, \tau)$ with some τ.

Moreover, we define the types $(\mathbb{P}, \varphi = 0, (\mathbb{K}, l))$ for any $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $l \leq 5$: We say that a linear subspace of $\mathfrak{m}$ is of that type if and only if it is isotropy-congruent to $\mathfrak{m}' = \mathbb{R}\lambda_2^\sharp \oplus \mathfrak{m}'_{\lambda_2} \oplus \mathfrak{m}'_{2\lambda_2}$, where $\mathfrak{m}'_{\lambda_2} \subset \mathfrak{m}_{\lambda_2}$ is a linear subspace of $\mathbb{C}$P-type $(\mathbb{K}, l-1)$ and we put $\mathfrak{m}'_{2\lambda_2} := \mathfrak{m}_{2\lambda_2}$ if $\mathbb{K} = \mathbb{C}$, $\mathfrak{m}'_{2\lambda_2} := \{0\}$ if $\mathbb{K} = \mathbb{R}$. Any such space is a Lie triple system of $\mathfrak{m}$, and the Lie triple systems of these types are exactly those which are contained in a Lie triple system of type $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 5))$.

Likewise, we can define the type $(\mathbb{P}, \varphi = \pi/4, \tau)$ also for $\tau = \mathbb{S}^l$ with $l \leq 4$ and for $\tau = \mathbb{K}\mathrm{P}^2$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ in the following way: We put $H := \lambda_1^\sharp + \lambda_2^\sharp$ and $\tilde{H} := M_{2\lambda_1}(1) + M_{2\lambda_2}(1)$. Then a Lie triple system is of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{S}^l)$ if it is isotropy-congruent to a l-dimensional linear subspace of $\mathbb{R}H \oplus \mathfrak{m}_{\lambda_4} \oplus \mathbb{R}\tilde{H}$. A Lie triple system is of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{K}\mathrm{P}^2)$ if it is congruent to the Lie triple system $\mathfrak{m}'$, where we have

- for $\mathbb{K} = \mathbb{R}$: $\mathfrak{m}' = \mathbb{R}H \oplus \{M_{\lambda_1}(t, 0, 0, 0) + M_{\lambda_2}(0, t, 0, 0) \mid t \in \mathbb{R}\}$,
- for $\mathbb{K} = \mathbb{C}$: $\mathfrak{m}' = \mathbb{R}H \oplus \{M_{\lambda_1}(c, 0, 0, 0) + M_{\lambda_2}(0, c, 0, 0) \mid c \in \mathbb{C}\} \oplus \mathbb{R}\tilde{H}$,
- for $\mathbb{K} = \mathbb{H}$: $\mathfrak{m}' = \mathbb{R}H \oplus \{M_{\lambda_1}(c_1, c_2, 0, 0) + M_{\lambda_2}(c_2, c_1, 0, 0) \mid c_1, c_2 \in \mathbb{C}\} \oplus M_{\lambda_4}(0, 0, \mathbb{C}) \oplus \mathbb{R}\tilde{H}$.

Then the Lie triple systems of EIII which are contained in a Lie triple system of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$ are exactly those which are of a type of the form $(\mathbb{P}, \varphi = \pi/4, \tau)$.

Finally, the type $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}_1, l), \mathbb{K}_2)$ can be defined also for $l \leq 3$, and also for $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{R}$ by applying the same definition as in the Theorem. Then the Lie triple systems of EIII which are contained in $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{C}, 5), \mathbb{C})$ are exactly those which are of the type $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}_1, l), \mathbb{K}_2)$ with some $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$ and $l \leq 5$.

These "newly defined" types are identical, however, to types of the form (Q, τ) or $(G_2\mathbb{C}^6, \tau)$ defined in Theorem 3.3. This is detailed in the following table:

The type … defined here	is identical to the type … from Theorem 3.3.
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{R}, k)))$	$(Q, (\mathrm{I2}, k))$
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{C}, k)))$	$(Q, (\mathrm{I1}, k))$
$(G_2\mathbb{C}^6, (\mathbb{S}, \varphi = \arctan(1/3), 2))$	$(Q, (\mathrm{A}))$
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{R}, 1)))$	$(Q, (\mathrm{P1}, 1))$
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{C}, 1)))$	$(Q, (\mathrm{P1}, 2))$
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{S}^3)))$	$(Q, (\mathrm{P1}, 3))$
$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{H}, 1)))$	$(Q, (\mathrm{P1}, 4))$
$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{R}, 1)))$	$(Q, (\mathrm{I2}, 2))$
$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 1)))$	$(Q, (\mathrm{I1}, 2))$
$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{R}, 2)))$	$(Q, (\mathrm{P1}, 4))$
$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 2)))$	$(Q, (\mathrm{G1}, 4))$
$(G_2\mathbb{C}^6, (\mathbb{S}^1 \times \mathbb{S}^5, k))$	$(Q, (\mathrm{P2}, 1, k))$
$(G_2\mathbb{C}^6, (Q_3))$	$(Q, (\mathrm{G2}, 3))$
$(G_2\mathbb{H}^4, (\mathbb{P}, \varphi = 0, \tau'))$ with $\dim(\tau') = 1$	$(Q, (\mathrm{P1}, w(\tau')))$
$(G_2\mathbb{H}^4, (\mathbb{P}, \varphi = \arctan(1/3), 2))$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{R}, 2)))$
$(G_2\mathbb{H}^4, (\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 1)))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{K}, 1)))$
$(G_2\mathbb{H}^4, (\mathrm{G}_2, (\mathbb{K}, 1)))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 2)))$
$(G_2\mathbb{H}^4, (\mathrm{G}_2, (\mathbb{R}, 2)))$	$(G_2\mathbb{C}^6, (\mathbb{P} \times \mathbb{P}, (\mathbb{R}, 1), (\mathbb{R}, 1)))$
$(G_2\mathbb{H}^4, (\mathrm{G}_2, (\mathbb{C}, 2)))$	$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{R}, 4)))$
$(G_2\mathbb{H}^4, (\mathbb{P} \times \mathbb{P}, \tau', \tau''))$ with $\dim(\tau') = \dim(\tau'') = 1$	$(Q, (\mathrm{G2}, w(\tau'), w(\tau'')))$
$(G_2\mathbb{H}^4, (\mathbb{S}^1 \times \mathbb{S}^5, 1))$	$(G_2\mathbb{C}^6, (\mathbb{P} \times \mathbb{P}, (\mathbb{R}, 1), (\mathbb{R}, 1)))$
$(G_2\mathbb{H}^4, (\mathbb{S}^1 \times \mathbb{S}^5, 2))$	$(G_2\mathbb{C}^6, (\mathbb{P} \times \mathbb{P}, (\mathbb{C}, 1), (\mathbb{R}, 1)))$
$(G_2\mathbb{H}^4, (Q_3))$	$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{R}, 3)))$
$(\mathbb{P}, \varphi = 0, (\mathbb{R}, 1))$	$(\mathrm{Geo}, \varphi = 0)$
$(\mathbb{P}, \varphi = 0, (\mathbb{R}, 2))$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{C}, 1)))$
$(\mathbb{P}, \varphi = 0, (\mathbb{R}, 3))$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{S}^3)))$
$(\mathbb{P}, \varphi = 0, (\mathbb{R}, 4))$	$(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = 0, (\mathbb{H}, 1)))$
$(\mathbb{P}, \varphi = 0, (\mathbb{R}, 5))$	$(G_2\mathbb{H}^4, (\mathbb{S}^5, \varphi = \pi/4))$
$(\mathbb{P}, \varphi = 0, (\mathbb{C}, l))$ with $l \leq 4$	$(Q, (\mathrm{I1}, l))$

The type … defined here	is identical to the type … from Theorem 3.3.
$(\mathbb{P}, \varphi = \pi/4, \mathbb{S}^l)$ with $l \leq 4$ $(\mathbb{P}, \varphi = \pi/4, \mathbb{K}\mathrm{P}^2)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$	$(Q, (\mathrm{P1}, l))$ $(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 2)))$
$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}_1, l), \mathbb{K}_2)$ with $l \leq 3$ $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{R}, l), \mathbb{R})$ with $l \leq 5$	$(G_2\mathbb{C}^6, (\mathbb{P} \times \mathbb{P}, (\mathbb{K}_1, l), (\mathbb{K}_2, 1)))$ $(G_2\mathbb{H}^4, (\mathbb{S}^1 \times \mathbb{S}^5, l))$

As an example for proving these identities, we consider the type $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 4))$. To prove that this type is identical to the type $(Q, (\mathrm{I1}, 4))$, it suffices to show that the space

$$\mathfrak{m}' := \mathbb{R}\lambda_2^\sharp \oplus M_{\lambda_2}(\mathbb{C}, \mathbb{C}, \mathbb{C}, 0) \oplus M_{2\lambda_2}(\mathbb{R})$$

of type $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 4))$ is isotropy-congruent to a space $\mathrm{Ad}(g)\mathfrak{m}'$ contained in the Lie triple system

$$\hat{\mathfrak{m}}' := \mathfrak{a} \oplus M_{\lambda_3}(\mathbb{C}, \mathbb{C}, \mathbb{C}) \oplus M_{\lambda_4}(\mathbb{C}, \mathbb{C}, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$$

of type (Q). Because $\mathrm{Ad}(g)\mathfrak{m}'$ has the isotropy angle 0 with respect to EIII, it has the isotropy angle $\pi/4$ with respect to Q^8 (see Remark 3.4); because it is also a complex subspace, it then must be of type $(Q, (\mathrm{I1}, 4))$ by the classification of Lie triple systems of the complex quadric given in [7], Theorem 4.1. —To show that such an isotropy-congruence indeed holds, notice that with $Z := K_{\alpha_7}(\sqrt{8}) \in \mathfrak{k}$ we have $\mathrm{ad}(Z)\lambda_2^\sharp = \mathrm{ad}(Z)M_{2\lambda_2}(t) = 0$ and

$$\mathrm{ad}(Z)M_{\lambda_2}(c_1, c_2, c_3, 0) = M_{\lambda_3}(c_2, c_1, -\overline{c_3}) + M_{\lambda_4}(-c_1, -c_2, c_3)$$

for any $t \in \mathbb{R}$, $c_1, c_2, c_3 \in \mathbb{C}$. This shows that with $g := \exp(\pi/2Z) \in K$ we have

$$\begin{aligned}&\mathrm{Ad}(g)\mathfrak{m}'\\ &= \mathbb{R}\lambda_2^\sharp \oplus \{M_{\lambda_3}(c_2, c_1, -\overline{c_3}) + M_{\lambda_4}(-c_1, -c_2, c_3) \mid c_1, c_2, c_3 \in \mathbb{C}\} \oplus M_{2\lambda_2}(\mathbb{R})\\ &\subset \hat{\mathfrak{m}}'.\end{aligned}$$

REMARK 3.6. For the space EIII, Table VIII of [5] correctly lists the *local* isometry types of the *maximal* totally geodesic submanifolds. However, the totally geodesic submanifolds corresponding to the types $(G_2\mathbb{H}^4)$ and (Q) are of isometry type $G_2(\mathbb{H}^4)/\mathbb{Z}_2$ resp. $G_2^+(\mathbb{R}^{10}) \cong Q^8$ (see Section 3.3), and not of isometry type $G_2(\mathbb{H}^4)$ resp. $G_2(\mathbb{R}^{10})$ (as [5] claims).

It should be noted that EIII contains spaces of rank 1 as totally geodesic submanifolds in a “skew” position in the sense that their geodesic diameter is strictly larger than the geodesic diameter of the ambient space EIII. However, none of them is maximal in EIII. The “skew” totally geodesic submanifolds which are maximal among the totally geodesic submanifolds of EIII of rank 1 are those of the types $(G_2\mathbb{H}^4, (\mathbb{P}, \varphi = \arctan(1/3), 3))$ (isometric to an $\mathbb{R}\mathrm{P}^3$ of sectional curvature $2/5$), $(Q, (\mathrm{A}))$ (isometric to a 2-sphere of radius $(1/2)\sqrt{10}$) and $(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{C}, 2))$ (isometric to a $\mathbb{C}\mathrm{P}^2$ of holomorphic sectional curvature $4/5$). The existence of these "skew" totally geodesic submanifolds cannot be inferred from Table VIII of [5]. For explicit constructions of these "skew" totally geodesic submanifolds in $G_2(\mathbb{H}^4)$, Q^3 resp. $G_2(\mathbb{C}^6)$, see [8], Sections 6 and 7.[3]

The remainder of the present section is concerned with the proof of Theorem 3.3.

We first mention that it is easily checked using the Maple implementation of the algorithms for the computation of the curvature tensor that the spaces defined in the theorem, and therefore also the linear subspaces $\mathfrak{m}' \subset \mathfrak{m}$ which are congruent to one of them, are Lie triple systems. It is also easily seen that the information in the table concerning the dimension, the rank, and the question if $\mathfrak{m}'$ is complex or totally real is correct (for the latter, use the description of the complex structure of EIII given in Section 3.1). The information on the isometry type of the corresponding totally geodesic submanifolds will be proved in Section 3.3.

Note that the information on the individual types of Lie triple systems given in the table in the Theorem is invariant under congruence transformation, as is, in the case of Lie triple systems of rank 1, the isotropy angle of the Lie triple system (which is given in the form $\varphi = \cdots$ in the names of the types of rank 1 Lie triple systems). Considering the information given in the table, and in the corresponding tables in the classification theorems cited from [7] and [8] for the types (Q, τ), $(G_2\mathbb{C}^6, \tau)$ and $(G_2\mathbb{H}^4, \tau)$ (in view of the isotropy angle, take Remark 3.4 into account), we see that no two of the types of Lie triple systems given in the Theorem coincide in all the mentioned characteristics. Therefore no Lie triple system is of more than one type.

We next show that the information on the maximality of the Lie triple systems given in the table is correct. For this purpose, we presume that the list of Lie triple systems given in the theorem is in fact complete; this will be proved in the remainder of the present section.

That the Lie triple systems which are claimed to be maximal in the table indeed are: This is clear for the type (DIII), because it has the maximal dimension among all the Lie triple systems of EIII. The Lie triple systems of the types $(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$, (Q), $(G_2\mathbb{C}^6)$ and $(G_2\mathbb{H}^4)$ all are of dimension 16, therefore if they were not maximal, they could only be contained in a Lie triple system of type (DIII), because these are the only ones of greater dimension. The spaces of the types $(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$ and $(G_2\mathbb{H}^4)$ are real forms of EIII, and therefore cannot be contained in a (complex) Lie

[3] The most general of these constructions in [8] is the construction of a "skew" $\mathbb{H}\mathrm{P}^2$ (of type $(\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{H}, 2))$) in $G_2(\mathbb{H}^7)$ described in Section 6 of [8]. It is based on the fundamental 14-dimensional representation with quaternionic structure of Sp(3), which is realized as a subrepresentation of the representation of Sp(3) on $\bigwedge^3 \mathbb{C}^6$, see also [4], p. 269ff. I would like to remark that this representation is not equivalent to the representation of Sp(3) on $\mathfrak{J}(3, \mathbb{H})^{\mathbb{C}}$ involved in Cartan's construction of isoparametric hypersurfaces in the sphere. This is easily seen, because the latter representation, although it is also 14-dimensional and irreducible, admits a real structure, and thus cannot admit a quaternionic structure, see [4], Proposition II.6.5, p. 98.

triple systems of type (DIII). The restricted Dynkin diagrams with multiplicities of the Lie triple systems of type (Q) and (DIII) are $\bullet^1 \Rightarrow \bullet^6$ and $\bullet^4 \Rightarrow \odot^{4[1]}$, respectively. Thus the short roots in a Lie triple system of type (Q) have greater multiplicity than all the roots in a Lie triple system of type (DIII), and hence a Lie triple system of type (Q) cannot be contained in any Lie triple system of type (DIII) either. Assume that the Lie triple system $\mathfrak{m}' := \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_3}(0, 0, \mathbb{C}) \oplus M_{\lambda_4}(0, 0, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$ of type $(G_2\mathbb{C}^6)$ were contained in a Lie triple system of type (DIII) i.e. in a space isotropy-congruent to $\hat{\mathfrak{m}}' := \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_2}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_3}(0, \mathbb{C}, \mathbb{C}) \oplus M_{\lambda_4}(0, \mathbb{C}, \mathbb{C}) \oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})$. Then there would exist $g \in K$ so that $\mathrm{Ad}(g)$ maps $M_{\lambda_k}(\mathbb{C}, \mathbb{C}, 0, 0)$ onto $M_{\lambda_k}(\mathbb{C}, 0, \mathbb{C}, 0)$ for $k \in \{1, 2\}$. But this is a contradiction to the fact that the action of $\mathrm{Ad}(g)$ commutes with the map $M_{\lambda_1}(c_1, c_2, c_3, c_4) \mapsto M_{\lambda_2}(c_2, c_1, c_4, c_3)$ (Proposition 3.1), so also the Lie triple systems of type $(G_2\mathbb{C}^6)$ cannot be contained in a Lie triple system of type (DIII). Finally, we note that the Lie triple systems of type $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{C}, 5), \mathbb{C})$ are of rank 2 and have the Dynkin diagram $\odot^{8[1]}\ \bullet^1$. They have a restricted root of multiplicity 8, which is greater than the multiplicity of any root in any other Lie triple system of EIII of rank 2. Therefore also this type is maximal.

That no Lie triple systems are maximal besides those mentioned in the theorem follows from the following table:

Every Lie triple system of type...	is contained in a Lie triple system of type...
(Geo, $\varphi = t$)	$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{R}, 1), \mathbb{R})$
$(\mathbb{P}, \varphi = \pi/4, \mathbb{S}^k)$	$(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$
$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}, l), \mathbb{R})$	$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}, l), \mathbb{C})$
$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}, l), \mathbb{C})$ with $(\mathbb{K}, l) \neq (\mathbb{C}, 5)$	$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{C}, 5), \mathbb{C})$
(Q, τ)	(Q)
$(G_2\mathbb{C}^6, \tau)$	$(G_2\mathbb{C}^6)$
$(G_2\mathbb{H}^4, \tau)$	$(G_2\mathbb{H}^4)$

We now turn to the proof that the list of Lie triple systems of EIII given in Theorem 3.3 is indeed complete. For this purpose, we let an arbitrary Lie triple system $\mathfrak{m}'$ of $\mathfrak{m}$, $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{m}$, be given. In the sequel, we will also use the additional names for types of Lie triple systems introduced in Remark 3.5; it has been shown there that these types are equivalent to other types defined in Theorem 3.3.

Because the symmetric space EIII is of rank 2, the rank of $\mathfrak{m}'$ is either 1 or 2. We will handle these two cases separately in the sequel.

We first suppose that $\mathfrak{m}'$ is a Lie triple system of rank 2. Let us fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}'$; because of $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$, $\mathfrak{a}$ is then also a Cartan subalgebra of $\mathfrak{m}$. In relation to this situation, we use the notations introduced in Sections 2 and 3.1. In particular, we consider the positive root system $\Delta_+ := \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, 2\lambda_1, 2\lambda_2\}$ of the root system $\Delta := \Delta(\mathfrak{m}, \mathfrak{a})$ of $\mathfrak{m}$, and also the root system $\Delta' := \Delta(\mathfrak{m}', \mathfrak{a})$ of $\mathfrak{m}'$.

By Proposition 2.1 (b), Δ' is a root subsystem of Δ, and therefore $\Delta'_+ := \Delta' \cap \Delta_+$ is a positive system of roots for Δ'. Moreover, in the root space decompositions of $\mathfrak{m}$ and $\mathfrak{m}'$

$$\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta_+} \mathfrak{m}_\lambda \quad \text{and} \quad \mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta'_+} \mathfrak{m}'_\lambda \tag{10}$$

the root space $\mathfrak{m}'_\lambda$ of $\mathfrak{m}'$ with respect to $\lambda \in \Delta'_+$ is related to the corresponding root space $\mathfrak{m}_\lambda$ of $\mathfrak{m}$ by $\mathfrak{m}'_\lambda = \mathfrak{m}_\lambda \cap \mathfrak{m}'$.

As was noted in Section 3.1, $\mathfrak{m}_{\lambda_k}$ is a complex subspace of $\mathfrak{m}$ for $k \in \{1, 2\}$. The following proposition describes how the position of $\mathfrak{m}'_{\lambda_k}$ in $\mathfrak{m}_{\lambda_k}$ with respect to the complex structure is controlled by the presence of the root $2\lambda_k$ in Δ'.

Proposition 3.7. *For $k \in \{1,2\}$, $\mathfrak{m}'_{\lambda_k}$ is either a complex or a totally real subspace of $\mathfrak{m}_{\lambda_k}$; it is a complex subspace if and only if $2\lambda_k \in \Delta'$ holds.*

Proof. First suppose $2\lambda_k \in \Delta'$. Because of $n_{2\lambda_k} = 1$ we then have $\mathfrak{m}'_{2\lambda_k} = \mathfrak{m}_{2\lambda_k} = M_{2\lambda_k}(\mathbb{R})$. For any given $v \in \mathfrak{m}'_{\lambda_1}$ we have[4] $R(\lambda_k^\sharp, v)M_{2\lambda_k}(1) = -(1/8)Jv$, and this vector is a member of $\mathfrak{m}'$ by the fact that $\mathfrak{m}'$ is a Lie triple system. Thus $Jv \in \mathfrak{m}_{\lambda_k} \cap \mathfrak{m}' = \mathfrak{m}'_{\lambda_k}$ holds, and hence $\mathfrak{m}'_{\lambda_k}$ is a complex subspace of $\mathfrak{m}_{\lambda_k}$.

Now suppose $2\lambda_k \notin \Delta'$. For any given $v, w \in \mathfrak{m}'_{\lambda_k}$ we have $\mathfrak{m}' \ni R(\lambda_k^\sharp, v)w = (1/8)\langle v, w\rangle\lambda_1^\sharp + (1/8)\langle v, Jw\rangle M_{2\lambda_k}(1)$; because of $2\lambda_k \notin \Delta'$ it follows that $\langle v, Jw\rangle = 0$ holds. Hence $\mathfrak{m}'_{\lambda_k}$ is a totally real subspace of $\mathfrak{m}_{\lambda_k}$. □

We now distinguish three cases depending on the structure of Δ', which we will treat separately in the sequel:

(a) $\lambda_3, \lambda_4 \in \Delta'$,

(b) either, but not both, of λ_3 and λ_4 are members of Δ',

(c) $\lambda_3, \lambda_4 \notin \Delta'$.

CASE (a). Because of $\lambda_3 \in \Delta'$, there exists $v \in \mathfrak{m}'_{\lambda_3}$ with $\|v\| = 1$. By Proposition 3.1, there exists $g \in K_0 \subset K$ so that $\mathrm{Ad}(g)$ maps v into $M_{\lambda_3}(0,0,1)$, and therefore $\mathfrak{m}'$ into another Lie triple system $\mathfrak{m}'' := \mathrm{Ad}(g)\mathfrak{m}'$, so that we have $M_{\lambda_3}(0, 0, 1) \in \mathfrak{m}''_{\lambda_3}$. This argument shows that we may suppose without loss of generality that $M_{\lambda_3}(0,0,1) \in \mathfrak{m}'_{\lambda_3}$ holds.

We have for any $v = M_{\lambda_1}(c_1, c_2, c_3, c_4)$:

$$R(\lambda_1^\sharp, v)M_{\lambda_3}(0, 0, 1) = \frac{\sqrt{2}}{16} M_{\lambda_2}(c_1 i, c_2 i, -c_3 i, -c_4 i),$$

[4]The evaluation of R is done here, as in all the following situations, using the Maple package described in [9], as explained in the Introduction.

and for any $v = M_{\lambda_2}(c_1, c_2, c_3, c_4)$:

$$R(\lambda_2^\sharp, v)M_{\lambda_3}(0, 0, 1) = -\frac{\sqrt{2}}{16}M_{\lambda_1}(c_1 i, c_2 i, -c_3 i, -c_4 i).$$

Because of the fact that $\mathfrak{m}'$ is a Lie triple system, it follows that we have for any $c_1, \ldots, c_4 \in \mathbb{C}$

$$M_{\lambda_1}(c_1, c_2, c_3, c_4) \in \mathfrak{m}'_{\lambda_1} \iff M_{\lambda_2}(c_1 i, c_2 i, -c_3 i, -c_4 i) \in \mathfrak{m}'_{\lambda_2}. \tag{11}$$

This equivalence in particular implies $(\lambda_1 \in \Delta' \Leftrightarrow \lambda_2 \in \Delta')$ and $n'_{\lambda_1} = n'_{\lambda_2}$.

Because Δ' is invariant under the Weyl transformation given by the reflection in $(\lambda_3^\sharp)^{\perp,\mathfrak{a}}$ (note $\lambda_3 \in \Delta'$), we also have $(2\lambda_1 \in \Delta' \Leftrightarrow 2\lambda_2 \in \Delta')$.

Let us first suppose $2\lambda_1, 2\lambda_2 \in \Delta'$. Then $\mathfrak{m}'_{\lambda_k}$ is a complex subspace of $\mathfrak{m}_{\lambda_k}$ for $k \in \{1, 2\}$ by Proposition 3.7. Hence $n := n'_{\lambda_1} = n'_{\lambda_2}$ is an even number, and we consider the possible values 0, 2, 4, 6, 8 for n individually in the sequel.

If $n = 0$ holds, we have $\Delta' = \{\pm\lambda_3, \pm\lambda_4, \pm 2\lambda_1, \pm 2\lambda_2\}$; this is a closed root subsystem of Δ. Therefore the maximal linear subspace $\hat{\mathfrak{m}}' := \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta'} \mathfrak{m}_\lambda$ of $\mathfrak{m}$ corresponding to Δ' is a Lie triple system (see Proposition 2.6); its corresponding Dynkin diagram with multiplicities is $\bullet^1 \Rightarrow \bullet^6$. Therefore the totally geodesic submanifold corresponding to $\hat{\mathfrak{m}}'$ is locally isometric to the complex quadric Q^8. $\mathfrak{m}'$ is also regarded as a subspace of $\hat{\mathfrak{m}}'$ a Lie triple system; therefore $\mathfrak{m}'$ is of one of the types described in the classification of the Lie triple systems of Q^m in [7]. It follows that if $\mathfrak{m}' = \hat{\mathfrak{m}}'$ holds, then $\mathfrak{m}'$ is of type (Q); otherwise it is of type (Q, τ), where τ is one of the types of Lie triple systems of $\hat{\mathfrak{m}}'$ as described in [7], Theorem 4.1 for $m = 8$.

For $n \neq 0$, an argument based on Proposition 3.1 similar to the previous one shows that we may suppose without loss of generality besides the earlier condition $M_{\lambda_3}(0, 0, 1) \in \mathfrak{m}'_{\lambda_3}$ also $M_{\lambda_1}(1, 0, 0, 0) \in \mathfrak{m}'_{\lambda_1}$. Because $\mathfrak{m}'_{\lambda_1}$ is a complex subspace of $\mathfrak{m}_{\lambda_1}$, we then in fact have $M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \subset \mathfrak{m}'_{\lambda_1}$. This fact induces further relations between the root spaces of $\mathfrak{m}'$ besides (11), which we now explore.

For any $v := M_{\lambda_3}(d_1, d_2, d_3)$ we have

$$u := R(\lambda_2^\sharp, v)M_{\lambda_1}(1, 0, 0, 0) = -\frac{\sqrt{2}}{16}M_{\lambda_2}(i d_3, 0, i\overline{d_2}, -i d_1)$$

and

$$R(u, M_{\lambda_1}(1, 0, 0, 0))\lambda_1^\sharp = \frac{1}{128}v + \frac{1}{128}M_{\lambda_4}(-d_1, d_2, \overline{d_3}).$$

An analogous calculation applies starting with $v = M_{\lambda_4}(d_1, d_2, d_3)$, and in this way we see via the fact that $\mathfrak{m}'$ is a Lie triple system:

$$M_{\lambda_3}(d_1, d_2, d_3) \in \mathfrak{m}'_{\lambda_3} \Longrightarrow M_{\lambda_2}(i d_3, 0, i\overline{d_2}, -i d_1) \in \mathfrak{m}'_{\lambda_2}, \tag{12}$$

$$M_{\lambda_3}(d_1, d_2, d_3) \in \mathfrak{m}'_{\lambda_3} \iff M_{\lambda_4}(-d_1, d_2, \overline{d_3}) \in \mathfrak{m}'_{\lambda_4}. \tag{13}$$

Moreover for any $v := M_{\lambda_2}(c_1, c_2, c_3, c_4)$ we have

$$R(M_{\lambda_1}(1,0,0,0), v)\lambda_3^\sharp = \frac{\sqrt{2}}{8} M_{\lambda_3}(-c_4 i, -\overline{c_3} i, c_1 i)$$

and therefore, again by the fact that $\mathfrak{m}'$ is a Lie triple system

$$M_{\lambda_2}(c_1, c_2, c_3, c_4) \in \mathfrak{m}'_{\lambda_2} \Longrightarrow M_{\lambda_3}(-c_4 i, -\overline{c_3} i, c_1 i) \in \mathfrak{m}'_{\lambda_3}. \tag{14}$$

We can use these relations to draw the following consequences from the fact $M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \subset \mathfrak{m}'_{\lambda_1}$: First, from (11) we obtain $M_{\lambda_2}(\mathbb{C}, 0, 0, 0) \subset \mathfrak{m}'_{\lambda_2}$. By (14) therefrom $M_{\lambda_3}(0, 0, \mathbb{C}) \subset \mathfrak{m}'_{\lambda_3}$ follows, and therefrom we finally obtain by (13): $M_{\lambda_4}(0, 0, \mathbb{C}) \subset \mathfrak{m}'_{\lambda_4}$. Remember for the sequel also that we have $\mathfrak{m}_{2\lambda_k} = M_{2\lambda_k}(\mathbb{R})$ for $k \in \{1, 2\}$.

If $n = 2$ holds, then we in fact have $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, 0, 0, 0)$ and $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, 0, 0, 0)$; because of (12) we then have $\mathfrak{m}'_{\lambda_3} = M_{\lambda_3}(0, 0, \mathbb{C})$, and therefore $\mathfrak{m}'_{\lambda_4} = M_{\lambda_4}(0, 0, \mathbb{C})$ by (13). Thus we see by the root space decomposition (10) that

$$\begin{aligned} \mathfrak{m}' = \mathfrak{a} &\oplus M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, 0, 0, 0) \oplus M_{\lambda_3}(0, 0, \mathbb{C}) \oplus M_{\lambda_4}(0, 0, \mathbb{C}) \\ &\oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R}) \end{aligned}$$

holds, and thus $\mathfrak{m}'$ is of type $(\mathrm{G}_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 3)))$.

If $n = 4$ holds, then the Dynkin diagram with multiplicities corresponding to $\mathfrak{m}'$ is $\bullet^l \Rightarrow \odot^{4[1]}$ with some $1 \leq l \leq 6$; from the classification of irreducible Riemannian symmetric spaces (see, for example, [14], p. 119, 146) we see that $l = 2$ and $l = 4$ are the only possibilities. If $l = 2$ holds, we have $\mathfrak{m}'_{\lambda_3} = M_{\lambda_3}(0, 0, \mathbb{C})$ and $\mathfrak{m}'_{\lambda_4} = M_{\lambda_4}(0, 0, \mathbb{C})$. Because of $n = 4$ we see from (14) that $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0)$ and therefore by (11) also $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0)$ has to hold. Thus we see that

$$\begin{aligned} \mathfrak{m}' = \mathfrak{a} &\oplus M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_3}(0, 0, \mathbb{C}) \oplus M_{\lambda_4}(0, 0, \mathbb{C}) \\ &\oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R}) \end{aligned}$$

holds, and hence $\mathfrak{m}'$ is of type $(\mathrm{G}_2\mathbb{C}^6)$. On the other hand, if $l = 4$ holds, we let $v \in \mathfrak{m}'_{\lambda_1}$ be a unit vector which is orthogonal to $M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \subset \mathfrak{m}'_{\lambda_1}$; we have $v = M_{\lambda_1}(0, c_2, c_3, c_4)$ with some $c_2, c_3, c_4 \in \mathbb{C}$, and

$$\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \oplus \mathbb{R}v \oplus \mathbb{R}Jv \tag{15}$$

holds. Because of $v \in \mathfrak{m}'_{\lambda_1}$, we have $M_{\lambda_2}(0, c_2 i, -c_3 i, -c_4 i) \in \mathfrak{m}'_{\lambda_2}$ by (11), therefore $M_{\lambda_3}(-c_4, \overline{c_3}, 0) \in \mathfrak{m}'_{\lambda_3}$ by (14), thus $M_{\lambda_2}(0, 0, -ic_3, -ic_4) \in \mathfrak{m}'_{\lambda_2}$ by (12), and hence finally $M_{\lambda_1}(0, 0, c_3, c_4) \in \mathfrak{m}'_{\lambda_1}$ by (11). From (15) and the explicit description of J in Section 3.1 we see that

$$(0, 0, c_3, c_4) \in \mathbb{R}(0, c_2, c_3, c_4) \oplus \mathbb{R}(0, -ic_2, ic_3, -ic_4)$$

holds; this implies that we have either $c_2 = 0$ or $c_3 = c_4 = 0$. In fact, $c_3 = c_4 = 0$ is impossible, because then we would have $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0)$, therefore by (11) also $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0)$, and therefore by (12) $\mathfrak{m}'_{\lambda_3} \subset M_{\lambda_3}(0, 0, \mathbb{C})$, in contradiction to $l = 4$. Therefore we have $c_2 = 0$ and thus $v = M_{\lambda_1}(0, 0, c_3, c_4)$. We have $\mathrm{ad}(K_{\alpha_4}(2))M_{\lambda_1}(0, 0, c_3, c_4) = M_{\lambda_1}(0, 0, -\overline{c_4}, \overline{c_3})$, therefore by the application of a rotation $\mathrm{Ad}(\exp(H))$ with suitable $H \in \mathbb{R}\alpha_4^\sharp \oplus K_{\alpha_4}(\mathbb{C}) \cong \mathfrak{su}(2)$ to $\mathfrak{m}'$, we can arrange $c_4 = 0$, and thus $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, 0, \mathbb{C}, 0)$. Then we have $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, 0, \mathbb{C}, 0)$ by (11), $\mathfrak{m}'_{\lambda_3} = M_{\lambda_3}(0, \mathbb{C}, \mathbb{C})$ by (14) and the fact that $l = 4$, and $\mathfrak{m}'_{\lambda_4} = M_{\lambda_4}(0, \mathbb{C}, \mathbb{C})$ by (13). Therefore

$$\begin{aligned}\mathfrak{m}' = \mathfrak{a} &\oplus M_{\lambda_1}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_2}(\mathbb{C}, 0, \mathbb{C}, 0) \oplus M_{\lambda_3}(0, \mathbb{C}, \mathbb{C}) \oplus M_{\lambda_4}(0, \mathbb{C}, \mathbb{C}) \\ &\oplus M_{2\lambda_1}(\mathbb{R}) \oplus M_{2\lambda_2}(\mathbb{R})\end{aligned}$$

is of type (DIII).

The case $n = 6$ cannot occur, because the Dynkin diagram with multiplicities corresponding to $\mathfrak{m}'$ would then be $\bullet^l \Leftrightarrow \odot^{6[1]}$ with some $1 \leq l \leq 6$; (14) shows $l \geq 4$. But the classification of irreducible Riemannian symmetric spaces (see [14], p. 119, 146) shows that no symmetric space with such a diagram exists.

Finally, if $n = 8$ holds, we have $\mathfrak{m}'_{\lambda_k} = \mathfrak{m}_{\lambda_k}$ for $k \in \{1, 2\}$, from (14) we obtain $\mathfrak{m}'_{\lambda_3} = \mathfrak{m}_{\lambda_3}$, from (13) we then obtain $\mathfrak{m}'_{\lambda_4} = \mathfrak{m}_{\lambda_4}$, and we also have $\mathfrak{m}'_{2\lambda_k} = \mathfrak{m}_{2\lambda_k}$ for $k \in \{1, 2\}$. Thus we have $\mathfrak{m}' = \mathfrak{m}$.

Let us now consider the case where $2\lambda_1, 2\lambda_2 \notin \Delta'$. Then $\mathfrak{m}'_{\lambda_1}$ and $\mathfrak{m}'_{\lambda_2}$ are totally real subspaces of $\mathfrak{m}_{\lambda_1}$ resp. $\mathfrak{m}_{\lambda_2}$ by Proposition 3.7. We either have $\lambda_1, \lambda_2 \in \Delta'$ or $\lambda_1, \lambda_2 \notin \Delta'$ because of the invariance of Δ' under the Weyl transformation induced by $\lambda_3 \in \Delta'$. If $\lambda_1, \lambda_2 \notin \Delta'$, i.e. $\Delta' = \{\pm\lambda_3, \pm\lambda_4\}$ holds, then $\mathfrak{m}'$ is again contained in a Lie triple system $\hat{\mathfrak{m}}'$ of type (Q), and therefore, by the classification of Lie triple systems in $\hat{\mathfrak{m}}'$ given in [7], $\mathfrak{m}'$ is of type (Q, τ), where τ is one of the types listed in Theorem 4.1 of [7] for $m = 8$.

So we now suppose $\lambda_1, \lambda_2 \in \Delta'$. Once again using Proposition 3.1, we may suppose without loss of generality that $\mathfrak{m}'_{\lambda_1} \subset M_{\lambda_1}(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R})$ holds because $\mathfrak{m}'_{\lambda_1}$ is totally real, and also that $M_{\lambda_1}(1, 0, 0, 0) \in \mathfrak{m}'_{\lambda_1}$ holds. The proof of Equations (12)–(14) was based only on the fact that $M_{\lambda_1}(1, 0, 0, 0) \in \mathfrak{m}'_{\lambda_1}$ holds, and therefore these equations will again be valid in the present situation. Therefore we have $\mathfrak{m}'_{\lambda_2} \subset M_{\lambda_2}(i\mathbb{R}, i\mathbb{R}, i\mathbb{R}, i\mathbb{R})$ by (11), then $\mathfrak{m}'_{\lambda_3} \subset M_{\lambda_3}(\mathbb{R}, \mathbb{R}, \mathbb{R})$ by (14), and then $\mathfrak{m}'_{\lambda_4} \subset M_{\lambda_4}(\mathbb{R}, \mathbb{R}, \mathbb{R})$ by (13).

Therefore $\mathfrak{m}'$ is contained in the Lie triple system

$$\hat{\mathfrak{m}}' := \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}) \oplus M_{\lambda_2}(i\mathbb{R}, i\mathbb{R}, i\mathbb{R}, i\mathbb{R}) \oplus M_{\lambda_3}(\mathbb{R}, \mathbb{R}, \mathbb{R}) \oplus M_{\lambda_4}(\mathbb{R}, \mathbb{R}, \mathbb{R}),$$

of type $(\mathrm{G}_2\mathbb{H}^4)$. $\hat{\mathfrak{m}}'$ has the Dynkin diagram $\bullet^3 \Rightarrow \bullet^4$, and therefore the corresponding totally geodesic submanifold is locally isometric to $G_2(\mathbb{H}^4)$. $\mathfrak{m}'$ is also a Lie triple system of $\hat{\mathfrak{m}}'$, therefore we have either $\mathfrak{m}' = \hat{\mathfrak{m}}'$, or $\mathfrak{m}'$ is of one of the types described in the classification of Lie triple systems of $G_2(\mathbb{H}^{n+2})$ given in Theorem 5.3 of [8] for $n = 2$. It follows that $\mathfrak{m}'$ is either of type $(G_2\mathbb{H}^4)$ (if $\mathfrak{m}' = \hat{\mathfrak{m}}'$ holds), or of type $(G_2\mathbb{H}^4, \tau)$, where τ is one of the types of Lie triple systems of $\hat{\mathfrak{m}}'$ as described in Theorem 5.3 of [8] for $n = 2$.

CASE (b). Here we suppose that either, but not both, of λ_3 and λ_4 are in Δ'. Without loss of generality we may suppose $\lambda_3 \in \Delta'$, $\lambda_4 \notin \Delta'$. Because Δ' is invariant under its Weyl transformation group, we then have $\Delta' = \{\pm\lambda_3\}$ and therefore $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_3}$ is of type $(Q, (\mathrm{G2}, l, 1))$ with $l := 1 + n'_{\lambda_3}$, $2 \leq l \leq 7$.

CASE (c). So now we have $\lambda_3, \lambda_4 \notin \Delta'$. Let us first consider the case where at least one of the roots λ_1 and λ_2 is not in Δ'. Without loss of generality we suppose $\lambda_2 \notin \Delta'$, so that we have $\Delta' \subset \{\pm\lambda_1, \pm 2\lambda_1, \pm 2\lambda_2\}$. For $k \in \{1, 2\}$ we put $\mathbb{K}_k := \mathbb{C}$ if $2\lambda_k \in \Delta'$, $\mathbb{K}_k := \mathbb{R}$ if $2\lambda_k \notin \Delta'$. Proposition 3.7 then shows that $\mathfrak{m}'_{\lambda_1}$ is a linear subspace of $\mathfrak{m}_{\lambda_1}$ of type $(\mathbb{K}_1, \dim_{\mathbb{K}_1}(\mathfrak{m}'_{\lambda_1}))$. It follows that $\mathfrak{m}'$ is of type $(\mathbb{P}\times\mathbb{P}^1, (\mathbb{K}_1, 1 + \dim_{\mathbb{K}_1}(\mathfrak{m}'_{\lambda_1})), \mathbb{K}_2)$.

Now consider the case $\lambda_1, \lambda_2 \in \Delta'$. As before, we may use Proposition 3.1 to suppose without loss of generality that $M_{\lambda_1}(1, 0, 0, 0) \in \mathfrak{m}'_{\lambda_1}$ holds. Let $v \in \mathfrak{m}'_{\lambda_2}$ be given, say $v = M_{\lambda_2}(c_1, c_2, c_3, c_4)$ with $c_1, \ldots, c_4 \in \mathbb{C}$, then we have

$$\mathfrak{m}' \ni R(M_{\lambda_1}(1, 0, 0, 0), v)\lambda_3^\sharp = -\frac{\sqrt{2}}{8} M_{\lambda_3}(ic_4, i\overline{c_3}, -ic_1).$$

Because of $\lambda_3 \notin \Delta'$ it follows that we have $c_1 = c_3 = c_4 = 0$ and thus we have $\mathfrak{m}'_{\lambda_2} \subset M_{\lambda_2}(0, \mathbb{C}, 0, 0)$. Without loss of generality we may suppose $M_{\lambda_2}(0, 1, 0, 0) \in \mathfrak{m}'_{\lambda_2}$. Now let $v \in \mathfrak{m}'_{\lambda_1}$ be given, say $v = M_{\lambda_1}(c_1, c_2, c_3, c_4)$ with $c_1, \ldots, c_4 \in \mathbb{C}$, then we have

$$\mathfrak{m}' \ni R(M_{\lambda_2}(0, 1, 0, 0), v)\lambda_3^\sharp = \frac{\sqrt{2}}{8} M_{\lambda_3}(ic_3, i\overline{c_4}, ic_2).$$

Because of $\lambda_3 \notin \Delta'$ we obtain $c_2 = c_3 = c_4 = 0$, and thus $\mathfrak{m}'_{\lambda_1} \subset M_{\lambda_1}(\mathbb{C}, 0, 0, 0)$. Because $\mathfrak{m}_{\lambda_k}$ is either complex or totally real according as whether $2\lambda_k$ is or is not a member of Δ' by Proposition 3.7, we see that $\mathfrak{m}'$ is of the type $(\mathrm{G}_2\mathbb{C}^6, (\mathbb{P}\times\mathbb{P}, (\mathbb{K}_1, 2), (\mathbb{K}_2, 2)))$, where for $k \in \{1, 2\}$ we put $\mathbb{K}_k := \mathbb{C}$ if $2\lambda_k \in \Delta'$, $\mathbb{K}_k := \mathbb{R}$ if $2\lambda_k \notin \Delta'$.

This completes the classification of the Lie triple systems of EIII of rank 2.

We now turn our attention to the case where $\mathfrak{m}'$ is a Lie triple system of rank 1. Via the application of the isotropy action of EIII, we may suppose without loss of generality that $\mathfrak{m}'$ contains a unit vector H from the closure $\bar{\mathfrak{c}}$ of the positive Weyl chamber $\mathfrak{c}$ of $\mathfrak{m}$ (with respect to $\mathfrak{a}$ and our choice of positive roots). By Equations (8) and (9) we then have with $\varphi_0 := \varphi(H) \in [0, \pi/4]$

$$H = \cos(\varphi_0)\lambda_2^\sharp + \sin(\varphi_0)\lambda_1^\sharp. \tag{16}$$

Because of $\mathrm{rk}(\mathfrak{m}') = 1$, $\mathfrak{a}' := \mathbb{R}H$ is a Cartan subalgebra of $\mathfrak{m}'$, and we have $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$. It follows from Proposition 2.1 (a) that the root systems Δ' and Δ of $\mathfrak{m}'$ resp. $\mathfrak{m}$ with respect to $\mathfrak{a}'$ resp. to $\mathfrak{a}$ are related by

$$\Delta' \subset \{\lambda(H)\alpha_0 \mid \lambda \in \Delta, \lambda(H) \neq 0\} \tag{17}$$

with the linear form $\alpha_0\colon \mathfrak{a}' \to \mathbb{R}$, $tH \mapsto t$; moreover for $\mathfrak{m}'$ we have the root space decomposition

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha\in\Delta'_+} \mathfrak{m}'_\alpha \tag{18}$$

where for any root $\alpha \in \Delta'$, the corresponding root space $\mathfrak{m}'_\alpha$ is given by

$$\mathfrak{m}'_\alpha = \Bigg(\bigoplus_{\substack{\lambda\in\Delta \\ \lambda(H)=\alpha(H)}} \mathfrak{m}_\lambda \Bigg) \cap \mathfrak{m}'. \tag{19}$$

If $\Delta' = \varnothing$ holds, then we have $\mathfrak{m}' = \mathbb{R}H$, and therefore $\mathfrak{m}'$ is then of type (Geo, $\varphi = \varphi_0$). Otherwise by the same consideration as in my classification of the Lie triple systems in $G_2(\mathbb{H}^n)$ ([8], the beginning of Section 5.2), we see that

$$\varphi_0 \in \left\{0, \arctan\left(\frac{1}{3}\right), \arctan\left(\frac{1}{2}\right), \frac{\pi}{4}\right\}$$

holds; moreover in the cases $\varphi_0 = \arctan(1/3)$ and $\varphi_0 = \arctan(1/2)$, Δ' cannot have elementary roots in the sense of Definition 2.2.

In the sequel we consider the four possible values for φ_0 individually.

The case $\boldsymbol{\varphi_0 = 0}$. In this case we have $H = \lambda_2^\sharp$ by Equation (16) and therefore

$$\lambda_1(H) = 2\lambda_1(H) = 0, \quad \lambda_2(H) = \lambda_3(H) = \lambda_4(H) = 1, \quad 2\lambda_2(H) = 2.$$

Thus we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha\}$ with $\alpha := \lambda_2|\mathfrak{a}' = \lambda_3|\mathfrak{a}' = \lambda_4|\mathfrak{a}'$ by Equation (17), $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ by Equation (18), and $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$ and $\mathfrak{m}'_{2\alpha} \subset \mathfrak{m}_{2\lambda_2}$ by Equation (19).

We first note that if in fact $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_2}$ holds, then by the same argument as in the proof of Proposition 3.7, $\mathfrak{m}'_\alpha$ is either a complex or a totally real linear subspace of $\mathfrak{m}_{\lambda_2}$, depending on whether 2α is or is not a member of Δ'. Therefore $\mathfrak{m}'$ then is of type $(\mathbb{P}, \varphi = 0, (\mathbb{K}, l))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $l := \dim_{\mathbb{K}}(\mathfrak{m}'_\alpha) + 1$.

Also, if $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$ holds, then $\mathfrak{m}'$ is contained in a Lie triple system $\hat{\mathfrak{m}}'$ of type (Q), and therefore $\mathfrak{m}'$ is of type (Q, τ), where τ is a type given in [7], Theorem 4.1 for $m = 4$.

Thus we now suppose $\mathfrak{m}'_\alpha \not\subset \mathfrak{m}_{\lambda_2}$ and $\mathfrak{m}'_\alpha \not\subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$, in particular we have $\alpha \in \Delta'$. We will show that in this situation, $\mathfrak{m}'$ is conjugate under the isotropy action to another Lie triple system whose corresponding root space decomposition satisfies $\mathfrak{m}'_\alpha \subset \mathfrak{m}'_{\lambda_2}$ or $\mathfrak{m}'_\alpha \subset \mathfrak{m}'_{\lambda_3} \oplus \mathfrak{m}'_{\lambda_4}$. It then follows by the above discussion that $\mathfrak{m}'$ is of one of the types of the theorem.

It follows from our hypotheses $\mathfrak{m}'_\alpha \not\subset \mathfrak{m}_{\lambda_2}$ and $\mathfrak{m}'_\alpha \not\subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$ that there exists a unit vector $v_0 \in \mathfrak{m}'$, say

$$v_0 = M_{\lambda_2}(c_1, c_2, c_3, c_4) + M_{\lambda_3}(d_1, d_2, d_3) + M_{\lambda_4}(e_1, e_2, e_3),$$

with $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$ and $(d_1, d_2, d_3, e_1, e_2, e_3) \neq (0, \dots, 0)$. By virtue of Proposition 3.1 we may suppose without loss of generality that $(c_1, c_2, c_3, c_4) = (t, 0, 0, 0)$ holds with some $t \in \mathbb{R} \setminus \{0\}$, and furthermore that $(d_1, d_2, d_3) = (s, 0, 0)$ holds with $s \in \mathbb{R}$. Then we have

$$v_0 = M_{\lambda_2}(t, 0, 0, 0) + M_{\lambda_3}(s, 0, 0) + M_{\lambda_4}(e_1, e_2, e_3). \tag{20}$$

Because $R(H, v_0)v_0$ is a member of $\mathfrak{m}'$, the $\mathfrak{m}_{\lambda_1}$-component of this vector, which equals

$$t \cdot \frac{\sqrt{2}}{8} M_{\lambda_1}(ie_3, 0, -i\overline{e_2}, i(e_1 - s)),$$

must vanish (because of $\lambda_1(H) = 0$), and therefore we have $e_1 = s$, $e_2 = e_3 = 0$, and therefore

$$v_0 = M_{\lambda_2}(t, 0, 0, 0) + M_{\lambda_3}(s, 0, 0) + M_{\lambda_4}(s, 0, 0).$$

We have

$$\mathrm{ad}(K_{\alpha_{11}}(\sqrt{8}))M_{\lambda_2}(1, 0, 0, 0) = M_{\lambda_3}(1, 0, 0) + M_{\lambda_4}(1, 0, 0),$$

and

$$\mathrm{ad}(K_{\alpha_{11}}(\sqrt{8}))(M_{\lambda_3}(1, 0, 0) + M_{\lambda_4}(1, 0, 0)) = -M_{\lambda_2}(1, 0, 0, 0),$$

therefore the 1-parameter subgroup $\{\exp(K_{\alpha_{11}}(t))\}_{t\in\mathbb{R}}$ of the isotropy group acts as a rotation group on the plane $\mathbb{R}M_{\lambda_2}(1, 0, 0, 0) \oplus \mathbb{R}(M_{\lambda_3}(1, 0, 0) + M_{\lambda_4}(1, 0, 0))$; it follows that a suitable member of this 1-parameter group maps (via the isotropy action) v_0 onto $M_{\lambda_2}(1, 0, 0, 0)$. By replacing $\mathfrak{m}'$ with its image under the action of that element, we may therefore suppose that $M_{\lambda_2}(1, 0, 0, 0) \in \mathfrak{m}'$ holds.

If this replacement causes either $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_2}$ or $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$ to hold, then we are done. Otherwise, there exists another vector $v_1 \in \mathfrak{m}'_\alpha$, say

$$v_1 = M_{\lambda_2}(c_1, c_2, c_3, c_4) + M_{\lambda_3}(d_1, d_2, d_3) + M_{\lambda_4}(e_1, e_2, e_3),$$

with $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$ and $(d_1, d_2, d_3, e_1, e_2, e_3) \neq (0, \dots, 0)$, and which is orthogonal to $M_{\lambda_2}(1, 0, 0, 0) \in \mathfrak{m}'_\alpha$, whence we have $\mathrm{Re}(c_1) = 0$. By Proposition 3.1 we may suppose without loss of generality $(d_1, d_2, d_3) = (1, 0, 0)$ (whilst maintaining the condition $M_{\lambda_2}(1, 0, 0, 0) \in \mathfrak{m}'_\alpha$). Then we have

$$R(H, M_{\lambda_2}(1, 0, 0, 0))v_1 = \frac{\sqrt{2}}{16} M_{\lambda_1}(ie_3, 0, -i\overline{e_2}, i(e_1 - 1)) - \frac{1}{8} M_{2\lambda_2}(\mathrm{Im}(c_1)).$$

Because this vector is a member of $\mathfrak{m}'$, its $\mathfrak{m}_{\lambda_1}$-component must vanish. Thus we have $e_1 = 1$ and $e_2 = e_3 = 0$. Moreover: If $2\alpha \notin \Delta'$, also the $\mathfrak{m}_{2\lambda_2}$-component vanishes, and thus we have $\mathrm{Im}(c_1) = 0$, hence $c_1 = 0$. On the other hand, if $2\alpha \in \Delta'$, we also have $[K_{2\lambda_2}(1), v_0] = -(1/2)M_{\lambda_2}(i, 0, 0, 0) \in \mathfrak{m}'$, and therefore we can replace v_1 by $v_1 - \mathrm{Im}(c_1)M_{\lambda_2}(i, 0, 0, 0)$. Hence we can suppose $c_1 = 0$ in any case. Thus v_1 is of the form

$$v_1 = M_{\lambda_2}(0, c_2, c_3, c_4) + M_{\lambda_3}(1, 0, 0) + M_{\lambda_4}(1, 0, 0).$$

We now calculate $R(H, v_1)v_1 \in \mathfrak{m}'$:

$$R(H, v_1)v_1 = \left(\frac{\|c\|^2}{4} + \frac{1}{2}\right) \cdot H - \frac{\sqrt{2}}{4} M_{\lambda_1}(i\overline{c_4}, 0, i\overline{c_2}, 0).$$

The $\mathfrak{m}_{\lambda_1}$-component of this vector again vanishes, and thus we obtain $c_2 = c_4 = 0$. Thus we have

$$v_1 = M_{\lambda_2}(0, 0, c_3, 0) + M_{\lambda_3}(1, 0, 0) + M_{\lambda_4}(1, 0, 0).$$

We now consider the Lie subalgebra $\mathfrak{b} := \mathbb{R}\alpha_6^\sharp \oplus K_{\alpha_6}(\mathbb{C})$ of $\mathfrak{k}$, which is isomorphic to $\mathfrak{su}(2)$. For $c \in \mathbb{C}$ we have

$$\mathrm{ad}(K_{\alpha_6}(2))M_{\lambda_2}(0, 0, c, 0) = \frac{1}{\sqrt{2}}(M_{\lambda_3}(c, 0, 0) + M_{\lambda_4}(c, 0, 0)),$$

and

$$\mathrm{ad}(K_{\alpha_6}(2))\frac{1}{\sqrt{2}}(M_{\lambda_3}(c, 0, 0) + M_{\lambda_4}(c, 0, 0)) = -M_{\lambda_2}(0, 0, c, 0),$$

therefore the connected Lie subgroup B of K with Lie algebra $\mathfrak{b}$ acts on the complex 2-plane $M_{\lambda_2}(0, 0, \mathbb{C}, 0) \oplus \{M_{\lambda_3}(c, 0, 0) + M_{\lambda_4}(c, 0, 0) \mid c \in \mathbb{C}\}$ as $\mathrm{SU}(2)$, and further

$$\mathrm{ad}(K_{\alpha_6}(2))M_{\lambda_2}(1, 0, 0, 0) = 0,$$

therefore the action of B leaves $M_{\lambda_2}(1, 0, 0, 0)$ invariant. Hence, by replacing $\mathfrak{m}'$ with $\mathrm{Ad}(g)\mathfrak{m}'$ for an appropriate $g \in B$, we can transform v_1 into $M_{\lambda_2}(0, 0, 1, 0)$, while leaving $M_{\lambda_2}(1, 0, 0, 0)$ invariant. By replacing $\mathfrak{m}'$ with $\mathrm{Ad}(g)\mathfrak{m}'$, we can thus ensure besides $M_{\lambda_2}(1, 0, 0, 0) \in \mathfrak{m}'_\alpha$ also $M_{\lambda_2}(0, 0, 1, 0) \in \mathfrak{m}'_\alpha$.

If this replacement causes $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_2}$ or $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$ to hold, then we are done. Otherwise, there exists yet another vector $v_2 \in \mathfrak{m}'_\alpha$, say

$$v_2 = M_{\lambda_2}(c_1, c_2, c_3, c_4) + M_{\lambda_3}(d_1, d_2, d_3) + M_{\lambda_4}(e_1, e_2, e_3),$$

with $(d_1, d_2, d_3, e_1, e_2, e_3) \neq (0, \dots, 0)$, which is orthogonal to $M_{\lambda_2}(1, 0, 0, 0)$, $M_{\lambda_2}(0, 0, 1, 0) \in$

$\mathfrak{m}'_\alpha$, whence we have $\mathrm{Re}(c_1) = \mathrm{Re}(c_3) = 0$. By an analogous argument as previously, we in fact obtain $c_1 = c_3 = 0$. Then we calculate

$$\begin{aligned} &R(H, v_2)M_{\lambda_2}(1, 0, 0, 0) \\ &= \frac{\sqrt{2}}{16} M_{\lambda_1}((e_3 - \overline{d_3})i, 0, -(\overline{d_2} + \overline{e_2})i, (e_1 - d_1)i) + \frac{1}{8} M_{2\lambda_2}(\mathrm{Im}(c_1)) \end{aligned}$$

and

$$\begin{aligned} &R(H, v_2)M_{\lambda_2}(0, 0, 1, 0) \\ &= \frac{\sqrt{2}}{16} M_{\lambda_1}((e_2 - d_2)i, (e_1 - d_1)i, (d_3 + \overline{e_3})i, 0) + \frac{1}{8} M_{2\lambda_2}(\mathrm{Im}(c_3)). \end{aligned}$$

Because these vectors are elements of $\mathfrak{m}'$, their $\mathfrak{m}_{\lambda_1}$-components vanish. From this fact, we derive the equations $e_1 = d_1$ and $d_2 = d_3 = e_2 = e_3 = 0$. Using the fact that these equations hold, we now calculate

$$R(H, v_2)v_2 = \left(\frac{\|c\|^2}{4} + \frac{|d_1|^2}{2} \right) H + \frac{\sqrt{2}}{4} M_{\lambda_1}(i\overline{c_4}d_1, 0, i\overline{c_2}d_1, 0).$$

Also this vector is an element of $\mathfrak{m}'$, and thus its $\mathfrak{m}_{\lambda_1}$-component once again vanishes, whence it follows (because $d_1 \neq 0$) that we have $c_2 = c_4 = 0$, hence $v_2 = M_{\lambda_3}(d_1, 0, 0) + M_{\lambda_4}(d_1, 0, 0)$.

Now let $Z := K_{\alpha_1}(2)$. Then we have $\mathrm{ad}(Z)v_2 = 0$,

$$\begin{aligned} \mathrm{ad}(Z)M_{\lambda_2}(1, 0, 0, 0) &= \frac{1}{\sqrt{2}}(M_{\lambda_3}(0, 0, 1) + M_{\lambda_4}(0, 0, 1)) =: v'_0, \\ \mathrm{ad}(Z)v'_0 &= -M_{\lambda_2}(1, 0, 0, 0), \end{aligned}$$

and

$$\begin{aligned} \mathrm{ad}(Z)M_{\lambda_2}(0, 0, 1, 0) &= \frac{1}{\sqrt{2}}(M_{\lambda_3}(0, 1, 0) + M_{\lambda_4}(0, 1, 0)) =: v'_1, \\ \mathrm{ad}(Z)v'_1 &= -M_{\lambda_2}(0, 0, 1, 0). \end{aligned}$$

These equations show that the adjoint action of the one-parameter subgroup B of K tangential to Z leaves the element v_2 of $\mathfrak{m}'$ invariant, whereas it acts as a rotation on the 2-planes spanned by $M_{\lambda_2}(1, 0, 0, 0)$ and $M_{\lambda_3}(0, 0, 1) + M_{\lambda_4}(0, 0, 1)$, resp. by $M_{\lambda_2}(0, 0, 1, 0)$ and $M_{\lambda_3}(0, 1, 0) + M_{\lambda_4}(0, 1, 0)$. It follows that there exists $g \in B$ so that we have $\mathrm{Ad}(g)M_{\lambda_2}(1, 0, 0, 0) = v'_0$, $\mathrm{Ad}(g)M_{\lambda_2}(0, 0, 1, 0) = v'_1$ and $\mathrm{Ad}(g)v_2 = v_2$ holds. We replace $\mathfrak{m}'$ by the Lie triple system $\mathrm{Ad}(g)\mathfrak{m}'$. Then we have $v'_0, v'_1, v_2 \in \mathfrak{m}'_\alpha$. For $v \in \mathfrak{m}'_\alpha$, we can evaluate $R(H, v)v'_0$, $R(H, v)v'_1$ and $R(H, v)v_2$; by the application of arguments analogous to those used above it turns out that any such v which is orthogonal to the $\mathbb{C}$-span of v'_0, v'_1, v_2 is necessarily zero. Therefore we now have $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4}$.

This completes the treatment of the case $\varphi_0 = 0$.

The case $\varphi_0 = \arctan(1/3)$. We have by Equation (16):

$$H = \frac{3}{\sqrt{10}}\lambda_2^\sharp + \frac{1}{\sqrt{10}}\lambda_1^\sharp$$

and therefore

$$\lambda_1(H) = \frac{1}{\sqrt{10}}, \quad \lambda_2(H) = \frac{3}{\sqrt{10}}, \quad \lambda_3(H) = \frac{2}{\sqrt{10}}, \quad \lambda_4(H) = \frac{4}{\sqrt{10}},$$
$$2\lambda_1(H) = \frac{2}{\sqrt{10}}, \quad 2\lambda_2(H) = \frac{6}{\sqrt{10}}.$$

Because there are no elementary roots (Definition 2.2) in the present case, it follows by Equation (17) that we have $\Delta' \subset \{\pm\alpha\}$ with $\alpha := \lambda_3|\mathfrak{a}' = (2\lambda_1)|\mathfrak{a}'$, and by Equations (18), (19) we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ with $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{2\lambda_1}$.

It follows that $\mathfrak{m}'$ is contained in the Lie triple system $\hat{\mathfrak{m}}' := \mathfrak{a} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4} \oplus \mathfrak{m}_{2\lambda_1} \oplus \mathfrak{m}_{2\lambda_2}$ of type (Q). $\hat{\mathfrak{m}}'$ corresponds to a complex quadric of complex dimension 8, and therefore the Lie triple systems contained in $\hat{\mathfrak{m}}'$ have been classified in [7]. $\mathfrak{m}'$ is a Lie triple system of rank 1, and its isotropy angle arctan(1/3) corresponds to the isotropy angle $\pi/4 - \arctan(1/3) = \arctan(1/2)$ in $\hat{\mathfrak{m}}'$, as has been explained in Remark 3.4. It therefore follows from the classification in [7], Theorem 4.1 that $\mathfrak{m}'$ is, as Lie triple system of $\hat{\mathfrak{m}}'$, of type (A). Thus $\mathfrak{m}'$ is as Lie triple system of $\mathfrak{m}$ of type $(Q, \text{(A)})$.

The case $\varphi_0 = \arctan(1/2)$. In this case we have by Equation (16):

$$H = \frac{2}{\sqrt{5}}\lambda_2^\sharp + \frac{1}{\sqrt{5}}\lambda_1^\sharp$$

and therefore

$$\lambda_1(H) = \frac{1}{\sqrt{5}}, \quad \lambda_2(H) = \frac{2}{\sqrt{5}}, \quad \lambda_3(H) = \frac{1}{\sqrt{5}}, \quad \lambda_4(H) = \frac{3}{\sqrt{5}},$$
$$2\lambda_1(H) = \frac{2}{\sqrt{5}}, \quad 2\lambda_2(H) = \frac{4}{\sqrt{5}}.$$

Because there are no elementary roots (Definition 2.2) in the present case, it follows by Equation (17) that we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_3|\mathfrak{a}'$, $2\alpha = \lambda_2|\mathfrak{a}' = (2\lambda_1)|\mathfrak{a}'$, and by Equations (18), (19) we have

$$\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha} \tag{21}$$

with $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_3}$ and $\mathfrak{m}'_{2\alpha} \subset \mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{2\lambda_1}$.

We have $\alpha^\sharp = (3/5)\lambda_1^\sharp + (2/5)\lambda_3^\sharp$ and $(2\alpha)^\sharp = (1/5)(2\lambda_1)^\sharp + (4/5)\lambda_2^\sharp$. By Proposition 2.4 it follows that there exist linear subspaces $\mathfrak{m}'_{\lambda_3} \subset \mathfrak{m}_{\lambda_3}$, $\mathfrak{m}'_{2\lambda_1} \subset \mathfrak{m}_{2\lambda_1}$ and isometric linear maps $\Phi_\alpha : \mathfrak{m}'_{\lambda_3} \to \mathfrak{m}_{\lambda_1}$, $\Phi_{2\alpha} : \mathfrak{m}'_{2\lambda_1} \to \mathfrak{m}_{\lambda_2}$ so that

$$\mathfrak{m}'_\alpha = \left\{ x + \sqrt{\frac{3}{2}}\Phi_\alpha(x) \;\middle|\; x \in \mathfrak{m}'_{\lambda_3} \right\} \quad \text{and} \quad \mathfrak{m}'_{2\alpha} = \{x + 2\Phi_{2\alpha}(x) \mid x \in \mathfrak{m}'_{2\lambda_1}\} \tag{22}$$

holds; in particular we have for the multiplicities of the roots of $\mathfrak{m}'$: $n'_\alpha \leq 6$ and $n'_{2\alpha} \leq 1$. We now consider the cases $2\alpha \in \Delta'$ and $2\alpha \notin \Delta'$ separately.

First suppose $2\alpha \in \Delta'$. Then we have $n'_{2\alpha} = 1$ and $\mathfrak{m}'_{2\lambda_1} = \mathfrak{m}_{2\lambda_1} = M_{2\lambda_1}(\mathbb{R})$. $\Phi_{2\alpha}(M_{2\lambda_1}(1))$ is a unit vector in $\mathfrak{m}_{\lambda_2}$, and via Proposition 3.1, we may suppose without loss of generality that $\Phi_{2\alpha}(M_{2\lambda_1}(1)) = M_{\lambda_2}(1,0,0,0)$ holds. By Equation (22) $\mathfrak{m}'_{2\alpha}$ is then spanned by the vector

$$v_{2\alpha} := M_{2\lambda_1}(1) + 2M_{\lambda_2}(1,0,0,0). \tag{23}$$

We now let $v \in \mathfrak{m}'_\alpha$ be given, say $v = M_{\lambda_1}(a_1,a_2,a_3,a_4)+M_{\lambda_3}(b_1,b_2,b_3)$ with $a_k, b_l \in \mathbb{C}$. Because $\mathfrak{m}'$ is a Lie triple system, we have $v_R := R(H, v_{2\alpha})v \in \mathfrak{m}'$. The root space decomposition (21) together with Equations (22) shows that therefore the $\mathfrak{m}_{\lambda_4}$-component of v_R, which is equal to

$$\frac{\sqrt{5}}{20} M_{\lambda_4}(i(-b_1+\sqrt{2}a_4),\, i(b_2+\sqrt{2}\overline{a_3}),\, i(\overline{b_3}+\sqrt{2}a_1))$$

must vanish, and thus we have

$$b_1 = \sqrt{2}a_4, \quad b_2 = -\sqrt{2}\overline{a_3}, \quad b_3 = -\sqrt{2}\overline{a_1}. \tag{24}$$

By Equation (22) we have $\Phi_\alpha(M_{\lambda_3}(b_1,b_2,b_3)) = \sqrt{2/3}M_{\lambda_1}(a_1,a_2,a_3,a_4)$; because Φ_α is isometric, it follows that

$$\frac{2}{3}\sum_k |a_k|^2 = \sum_k |b_k|^2 \overset{(24)}{=} 2(|a_4|^2+|a_3|^2+|a_1|^2)$$

and hence

$$|a_2|^2 = 2(|a_1|^2+|a_3|^2+|a_4|^2)$$

holds. It follows that the projection map

$$\mathfrak{m}'_\alpha \to \mathbb{C}, \quad v = M_{\lambda_1}(a_1,a_2,a_3,a_4)+M_{\lambda_3}(b_1,b_2,b_3) \mapsto a_2$$

is injective, and hence we have $n'_\alpha \leq 2$. We now give $v_R = R(H, v_{2\alpha})v$ explicitly for the situation where v satisfies Equations (24):

$$v_R = \frac{\sqrt{5}}{20}(M_{\lambda_1}(ia_1, ia_2, ia_3, -ia_4) + \sqrt{2}M_{\lambda_3}(-ia_4, i\overline{a_3}, i\overline{a_1})).$$

Because $v_R \in \mathfrak{m}'_\alpha$ is therefore orthogonal to v, we see that $n'_\alpha \in \{0,2\}$ holds. If $n'_\alpha = 0$, then we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_{2\alpha}$, and thus we see that $\mathfrak{m}'$ is of type $(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{C}, 1)))$. If $n'_\alpha = 2$ then by Proposition 3.1 we may suppose without loss of generality that $v = M_{\lambda_1}(1, \sqrt{2}, 0, 0) + M_{\lambda_3}(0, 0, -\sqrt{2})$ holds; then we have $\mathfrak{m}' =$

$\mathbb{R}H \oplus \mathbb{R}v \oplus \mathbb{R}v_R \oplus \mathbb{R}v_{2\alpha}$ with $4v_R = M_{\lambda_1}(i, i\sqrt{2}, 0, 0) + M_{\lambda_3}(0, 0, \sqrt{2}i)$. Therefore $\mathfrak{m}'$ is then of type $(G_2\mathbb{C}^6, (\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{C}, 2)))$.

Let us now consider the case $2\alpha \notin \Delta'$, so that $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ holds. If $\alpha \notin \Delta'$, then $\mathfrak{m}' = \mathbb{R}H$ is of type $(\mathrm{Geo}, \varphi = \arctan(1/2))$, otherwise because of Proposition 3.1, we may suppose without loss of generality that

$$v_\alpha := M_{\lambda_1}(1, \sqrt{2}, 0, 0) + M_{\lambda_3}(0, 0, -\sqrt{2}) \in \mathfrak{m}'_\alpha.$$

If $n'_\alpha = 1$ holds, then we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_\alpha$, and therefore $\mathfrak{m}'$ is then of type $(G_2\mathbb{H}^4, (\mathbb{S}, \varphi = \arctan(1/3), 2))$ (note that the isotropy angle $\varphi = \arctan(1/2)$ of $\mathfrak{m}'$ corresponds to the isotropy angle $\pi/4 - \arctan(1/2) = \arctan(1/3)$ within the type $(G_2\mathbb{H}^4)$ by Remark 3.4). Otherwise, we let another vector $v \in \mathfrak{m}'_\alpha$ which is orthogonal to v_α be given, say

$$v = M_{\lambda_1}(a_1, a_2, a_3, a_4) + M_{\lambda_3}(b_1, b_2, b_3)$$

with $a_k, b_l \in \mathbb{C}$, and consider $v_R := R(H, v_\alpha)v \in \mathfrak{m}'$. Both the $\mathfrak{m}_{\lambda_2}$-component and the $\mathfrak{m}_{2\lambda_1}$-component must vanish (because of $2\alpha \notin \Delta'$). The $\mathfrak{m}_{\lambda_2}$-component of v_R is proportional to

$$M_{\lambda_2}(i(\sqrt{2}b_3 + 2a_1), i(2\overline{b_3} + 2a_2), i(\sqrt{2}\,\overline{b_2} - 2a_3 - 2b_1), i(-2\overline{b_2} - 2a_4 - \sqrt{2}b_1))$$

and so we have

$$b_3 = -\sqrt{2}a_1, \quad b_3 = -\overline{a_2}, \quad -2b_1 + \sqrt{2}\,\overline{b_2} = 2a_3$$

and

$$-\sqrt{2}b_1 - 2\overline{b_2} = 2a_4,$$

hence

$$b_1 = -\frac{2}{3}a_3 - \frac{\sqrt{2}}{3}a_4, \quad b_2 = \frac{\sqrt{2}}{3}\overline{a_3} - \frac{2}{3}\overline{a_4}, \quad b_3 = -\overline{a_2} \quad \text{and} \quad a_2 = \sqrt{2}\overline{a_1}. \tag{25}$$

Moreover the $\mathfrak{m}_{2\lambda_1}$-component of v_R is proportional to $M_{2\lambda_1}(\mathrm{Im}(a_1 - \sqrt{2}a_2))$ and so we have $\mathrm{Im}(a_1 - \sqrt{2}a_2) = 0$, hence $\mathrm{Im}(a_1) = \sqrt{2}\,\mathrm{Im}(a_2) \overset{(25)}{=} -2\,\mathrm{Im}(a_1)$, and thus

$$\mathrm{Im}(a_1) = \mathrm{Im}(a_2) = 0. \tag{26}$$

Further, the condition that v is orthogonal to v_α gives

$$0 = \langle v, v_\alpha \rangle = \mathrm{Re}(a_1) + \sqrt{2}\,\mathrm{Re}(a_2) - \sqrt{2}\,\mathrm{Re}(b_3) \overset{(25)}{=} \mathrm{Re}(a_1 - \sqrt{2}a_2)$$

and therefore $\mathrm{Re}(a_1) = \sqrt{2}\,\mathrm{Re}(a_2) \overset{(25)}{=} 2\,\mathrm{Re}(a_1)$, hence

$$\mathrm{Re}(a_1) = \mathrm{Re}(a_2) = 0. \tag{27}$$

From Equations (26) and (27) we obtain $a_1 = a_2 = 0$. By the remaining equations from (25) we now see that

$$v = M_{\lambda_1}(0, 0, c, d) + M_{\lambda_3}\left(-\frac{2}{3}c - \frac{\sqrt{2}}{3}d,\ \frac{\sqrt{2}}{3}\overline{c} - \frac{2}{3}\overline{d},\ 0\right)$$

holds with some constants $c, d \in \mathbb{C}$.

We now consider the Lie subalgebra $\mathfrak{b} := \mathbb{R}\alpha_4^\sharp \oplus K_{\alpha_4}(\mathbb{C})$ of $\mathfrak{k}_0$, which is isomorphic to $\mathfrak{su}(2)$. For $z \in \mathbb{C}$, we have $\mathrm{ad}(K_{\alpha_4}(z))H = \mathrm{ad}(K_{\alpha_4}(z))v_\alpha = 0$, whereas $\mathrm{ad}(K_{\alpha_4}(z))$ acts on the complex plane

$$\mathfrak{w} := \left\{ M_{\lambda_1}(0, 0, c, d) + M_{\lambda_3}\left(-\frac{2}{3}c - \frac{\sqrt{2}}{3}d,\ \frac{\sqrt{2}}{3}\overline{c} - \frac{2}{3}\overline{d},\ 0\right) \,\middle|\, c, d \in \mathbb{C} \right\}$$

as a skew-adjoint, invertible endomorphism for $z \neq 0$. It follows that the adjoint action of the connected Lie group $B \subset K$ with Lie algebra $\mathfrak{b}$ on $\mathfrak{m}$ leaves H and v_α invariant, whereas it acts on $\mathfrak{w}$ as $SU(2)$ does. Therefore there exists $g \in B$ so that $\mathrm{Ad}(g)$ leaves H and v_α invariant, and satisfies $\mathrm{Ad}(g)v = M_{\lambda_1}(0, 0, 3, 0) + M_{\lambda_3}(-2, \sqrt{2}, 0)$. By replacing $\mathfrak{m}'$ with the Lie triple system $\mathrm{Ad}(g)\mathfrak{m}'$ from the same congruence class, we can thus arrange that

$$v = M_{\lambda_1}(0, 0, 3, 0) + M_{\lambda_3}(-2, \sqrt{2}, 0)$$

holds. Hence we see that in the case $n'_\alpha = 2$, $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_\alpha \oplus \mathbb{R}v$ is of type $(G_2\mathbb{H}^4, (\mathbb{S}, \varphi = \arctan(1/3), 2))$.

Finally we show that the case $n'_\alpha \geq 3$ cannot happen: Let $v' \in \mathfrak{m}'_\alpha$ be orthogonal to both v_α and v. Then, as above, the $\mathfrak{m}_{\lambda_2}$-component and the $\mathfrak{m}_{2\lambda_1}$-component of both $R(H, v_\alpha)v'$ and $R(H, v)v'$ have to vanish, and these conditions yield $v' = 0$.

The case $\varphi_0 = \pi/4$. In this case we have by Equation (16): $H = (1/\sqrt{2})\lambda_2^\sharp + (1/\sqrt{2})\lambda_1^\sharp$ and therefore

$$\lambda_1(H) = \frac{1}{\sqrt{2}}, \quad \lambda_2(H) = \frac{1}{\sqrt{2}}, \quad \lambda_3(H) = 0, \quad \lambda_4(H) = \frac{2}{\sqrt{2}},$$
$$2\lambda_1(H) = \frac{2}{\sqrt{2}}, \quad 2\lambda_2(H) = \frac{2}{\sqrt{2}}.$$

It follows by Equation (17) that we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_2|\mathfrak{a}'$, $2\alpha = \lambda_4|\mathfrak{a}' = (2\lambda_1)|\mathfrak{a}' = (2\lambda_2)|\mathfrak{a}'$, and by Equations (18), (19) we have

$$\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha} \tag{28}$$

with $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_2}$ and $\mathfrak{m}'_{2\alpha} \subset \mathfrak{m}_{2\lambda_1} \oplus \mathfrak{m}_{2\lambda_2} \oplus \mathfrak{m}_{\lambda_4}$.

Further information on the structure of elements of $\mathfrak{m}'_\alpha$ resp. of $\mathfrak{m}'_{2\alpha}$ can be obtained: First let $v \in \mathfrak{m}'_\alpha$ be given, say $v = M_{\lambda_1}(a_1, a_2, a_3, a_4) + M_{\lambda_2}(b_1, b_2, b_3, b_4)$ with $a_k, b_k \in \mathbb{C}$. By Proposition 2.4 and the fact that $\alpha^\sharp = (1/2)(\lambda_1^\sharp + \lambda_2^\sharp)$ holds, we see that we have $\|(a_1, \dots, a_4)\| = \|(b_1, \dots, b_4)\|$, in particular $n'_\alpha \le 8$.

Similarly, for any $v \in \mathfrak{m}'_{2\alpha}$, say $v = M_{\lambda_4}(c_1, c_2, c_3) + M_{2\lambda_1}(t) + M_{2\lambda_2}(s)$ with $c_k \in \mathbb{C}$ and $t, s \in \mathbb{R}$, we consider the vector $v_R := R(H, v)v \in \mathfrak{m}'$. The $\mathfrak{a}$-component of v_R must be proportional to H, and this condition yields $|t| = |s|$, hence $t = \pm s$. Moreover, because of $\lambda_3(H) = 0$, the $\mathfrak{m}_{\lambda_3}$-component of v_R, which is proportional to

$$M_{\lambda_3}(ic_1(s-t), ic_2(t-s), i\overline{c_3}(s-t)),$$

has to vanish, and thus we have either $c_1 = c_2 = c_3 = 0$ or $t = s$. If we have $t = -s$, and hence $c_1 = c_2 = c_3 = 0$, we put $Y := K_{2\lambda_1}(\sqrt{8}) - K_{2\lambda_2}(\sqrt{8})$, then we have $\mathrm{ad}(Y)H = M_{2\lambda_1}(1) - M_{2\lambda_2}(1)$ and $\mathrm{ad}(Y)(M_{2\lambda_1}(1) + M_{2\lambda_2}(1)) = 4\sqrt{2}\lambda_3^\sharp$. These equations show that a Lie triple system $\mathfrak{m}'$ where the case $t = -s$ occurs is congruent under the adjoint action of a member of the 1-parameter subgroup of K induced by Y to a Lie triple system corresponding to the case $t = s$. By replacing $\mathfrak{m}'$ with the latter Lie triple system, we may suppose without loss of generality that in any case

$$\mathfrak{m}'_{2\alpha} \subset \mathfrak{m}_{\lambda_4} \oplus \mathbb{R}(M_{2\lambda_1}(1) + M_{2\lambda_2}(1)) =: \hat{\mathfrak{m}}'_{2\alpha}$$

holds.

In the case $\alpha \notin \Delta'$ it now follows immediately that $\mathfrak{m}'$ is of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{S}^{1+n'_{2\alpha}})$.

So let us now turn our attention to the case $\alpha \in \Delta'$. $\mathfrak{m}'$ corresponds to a Riemannian symmetric space of rank 1; the classification of these spaces gives that we have $n'_{2\alpha} \in \{0, 1, 3, 7\}$ (corresponding to the projective spaces over the reals, the complex numbers, the quaternions, and the octonions, respectively), and that $n'_{2\alpha} + 1$ divides n'_α.

We continue our investigation of the structure of $\mathfrak{m}'_\alpha$: Let $v \in \mathfrak{m}'_\alpha$ be given, say $v = M_{\lambda_1}(a_1, \dots, a_4) + M_{\lambda_2}(b_1, \dots, b_4)$ with $a_k, b_k \in \mathbb{C}$. Then the $\mathfrak{m}_{\lambda_3}$-component of $R(H, v)v \in \mathfrak{m}'$ equals

$$M_{\lambda_3}\Big(-\frac{i}{8}(a_4 b_1 + a_1 b_4 + b_2 a_3 + a_2 b_3), -\frac{i}{8}(\overline{a_4} b_2 - \overline{a_3} b_1 + a_1 \overline{b_3} - a_2 \overline{b_4}), \frac{i}{8}(b_1 \overline{a_1} - \overline{a_4} b_4 + \overline{b_3} a_3 - \overline{b_2} a_2)\Big).$$

Because of $\lambda_3(H) = 0$, this has to vanish. In this way it follows that

$$\mathfrak{m}'_\alpha \subset \{M_{\lambda_1}(a_1, a_2, a_3, a_4) + M_{\lambda_2}(a_2, a_1, -a_4, -a_3) \mid a_1, \dots, a_4 \in \mathbb{C}\} =: \hat{\mathfrak{m}}'_\alpha$$

holds.

Therefore in any case $\mathfrak{m}'$ is contained in the Lie triple system $\hat{\mathfrak{m}}' := \mathbb{R}H \oplus \hat{\mathfrak{m}}'_\alpha \oplus \hat{\mathfrak{m}}'_{2\alpha}$ of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$. The totally geodesic submanifold corresponding to $\hat{\mathfrak{m}}'$ is a Cayley plane $\mathbb{O}\mathrm{P}^2$, and $\mathfrak{m}'$ also is a Lie triple system of $\hat{\mathfrak{m}}'$. Therefore it follows from the classification of the Lie triple systems of $\mathbb{O}\mathrm{P}^2$ (see [15], Section 3), that $\mathfrak{m}'$ is of one of the types $(\mathbb{P}, \varphi = \pi/4, \mathbb{K}\mathrm{P}^2)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

This completes the proof of the classification of Lie triple systems in EIII.

3.3. Totally geodesic submanifolds in $E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$. We now study the geometry of the totally geodesic submanifolds of EIII associated to the Lie triple systems found in Theorem 3.3. Of course, the local isometry type of the submanifolds can easily be obtained by determining the restricted root system (with multiplicities) of the corresponding Lie triple systems as they are given in Theorem 3.3. But to obtain the global isometry type, and also to understand the position of the submanifolds in EIII better, we describe the totally geodesic submanifolds of EIII explicitly. We will also want information on how the transvection groups of the respective submanifolds are embedded in E_6, the transvection group of EIII.

In this way, we obtain the results of the following table. Herein, we ascribe the type of a Lie triple system also to the corresponding totally geodesic submanifold (or to a corresponding totally geodesic embedding). For $l \in \mathbb{N}$ and $r > 0$ we denote by $\mathbb{S}^l_r$ the l-dimensional sphere of radius r, and for $\varkappa > 0$ we denote by $\mathbb{R}\mathrm{P}^l_\varkappa$ the l-dimensional real projective space of sectional curvature $\varkappa$, and by $\mathbb{C}\mathrm{P}^l_\varkappa$ the l-dimensional complex projective space of constant holomorphic curvature $4\varkappa$. Note that with these notations, $\mathbb{R}\mathrm{P}^l_\varkappa$ is a real form of $\mathbb{C}\mathrm{P}^l_\varkappa$. Moreover, with $\mathbb{H}\mathrm{P}^l_\varkappa$ resp. $\mathbb{O}\mathrm{P}^2_\varkappa$ we denote the l-dimensional quaternionic projective space resp. the Cayley projective plane, with their invariant Riemannian metrics scaled in such a way that the *minimal* sectional curvature equals $\varkappa$. Also for the irreducible Riemannian symmetric spaces of rank 2, their invariant Riemannian metrics are a priori only defined up to a positive constant; in the table below we describe the appropriate metrics of these spaces by giving the length a of the shortest restricted root of the space in the index $_{\mathrm{srr}=a}$. We continue to use also the additional names of types introduced in Remark 3.5.

type of Lie triple system	isometry type	properties[5]
$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
$(\mathbb{P}, \varphi = 0, (\mathbb{K}, l))$	$\mathbb{K}\mathrm{P}^l_{\varkappa=1}$	$(\mathbb{K}, l) = (\mathbb{C}, 1)$: Helgason sphere
$(\mathbb{P}, \varphi = \pi/4, \mathbb{S}^k)$	$\mathbb{S}^k_{r=1/\sqrt{2}}$	
$(\mathbb{P}, \varphi = \pi/4, \mathbb{K}\mathrm{P}^2)$	$\mathbb{K}\mathrm{P}^2_{\varkappa=1/2}$	$\mathbb{K} = \mathbb{O}$: reflective, real form, maximal
$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}_1, l), \mathbb{K}_2)$	$\mathbb{K}_1\mathrm{P}^l_{\varkappa=1} \times \mathbb{K}_2\mathrm{P}^1_{\varkappa=1}$	$(\mathbb{K}_1, l, \mathbb{K}_2) = (\mathbb{C}, 5, \mathbb{C})$: meridian for (DIII), maximal
(Q)	$Q^8_{\mathrm{srr}=\sqrt{2}}$	polar, meridian for itself, maximal
(Q, τ)	see [7], Section 5	
$(G_2\mathbb{C}^6)$	$G_2(\mathbb{C}^6)_{\mathrm{srr}=1}$	reflective, maximal
$(G_2\mathbb{C}^6, \tau)$	see [8], Section 7	
$(G_2\mathbb{H}^4)$	$(G_2(\mathbb{H}^4)/\mathbb{Z}_2)_{\mathrm{srr}=1}$	reflective, real form, maximal
$(G_2\mathbb{H}^4, \tau)$	see [8], Section 6	
(DIII)	$\mathrm{SO}(10)/\mathrm{U}(5)_{\mathrm{srr}=1}$	polar, maximal

[5]The polars and meridians are also reflective, without this fact being noted explicitly in the table.

For the application of the information from [7] and [8] it should be noted that these two papers use different conventions regarding the metrics used on the spaces under investigation: In the investigation of the complex quadric in [7], the metric is normalized such that the shortest restricted roots of Q^m have length $\sqrt{2}$, whereas in the investigation of $G_2(\mathbb{C}^n)$ and $G_2(\mathbb{H}^n)$ in [8], the metrics on these spaces are normalized such that those linear forms which are the shortest roots in $G_2(\mathbb{K}^n)$ for $n \geq 5$ have length 1 (notice that they are not actually roots of $G_2(\mathbb{K}^4)$, because their multiplicities then degenerate to zero). Also for the investigation of EIII in the present paper, we normalize the metric such that the shortest roots of this space have length 1.

By looking at the root systems of the totally geodesic submanifolds of type (Q), $(G_2\mathbb{C}^6)$ and $(G_2\mathbb{H}^4)$ of EIII (see Remark 3.4), it follows that the data given in the cited papers on the metric properties of totally geodesic submanifolds can be carried over without any change to the present situation for the totally geodesic submanifolds Q^8 and $G_2(\mathbb{C}^6)$ of EIII. However, for $G_2(\mathbb{H}^4)/\mathbb{Z}_2$ it is necessary to scale the data given in [8], as this manifold is considered with $\mathrm{srr} = \sqrt{2}$ in [8], whereas it has $\mathrm{srr} = 1$ here.

For the proof of the data in the table, and to obtain the desired information on the position of the totally geodesic submanifolds of EIII, it is sufficient to consider the maximal totally geodesic submanifolds. In the case of EIII every maximal totally geodesic submanifold is reflective (see [13]), and therefore a connected component of the fixed point set of an involutive isometry of EIII. We will describe these submanifolds in this way in the first instance.

To prove that the fixed point sets of the involutive isometries of EIII we investigate below are indeed of the isometry type claimed above, we will then construct totally geodesic, equivariant embeddings of the appropriate manifolds onto these fixed point sets for many of the types of maximal totally geodesic submanifolds of EIII. We will also describe the subgroups of the transvection group E_6 of EIII which correspond to the transvection groups of these totally geodesic submanifolds.

For these investigations, we need a model of EIII in which we can carry out calculations explicitly. For this purpose, we use the explicit presentations of EIII and of the exceptional Lie group E_6 given by Yokota in [16] and by Atsuyama in [3].

To describe these presentations, we denote by $\mathbb{R}$, $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$, $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ and $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e$ the four normed real division algebras: the field of real numbers, the field of complex numbers, the skew-field of quaternions, and the division algebra of octonions. For $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $x \in \mathbb{K}$, we have the conjugate $\overline{x}$ of x. We will also consider the complexification $\mathbb{K}^{\mathbb{C}} := \mathbb{K} \otimes_{\mathbb{R}} \mathbb{R}^{\mathbb{C}}$ of $\mathbb{K}$ with respect to another "copy" $\mathbb{R}^{\mathbb{C}} = \mathbb{R} \oplus \mathbb{R}I$ of the field of complex numbers; we linearly extend the conjugation map $x \mapsto \overline{x}$ of $\mathbb{K}$ to $\mathbb{K}^{\mathbb{C}}$. Notice that the algebras $\mathbb{C}^{\mathbb{C}}$, $\mathbb{H}^{\mathbb{C}}$ and $\mathbb{O}^{\mathbb{C}}$ have zero divisors.

Let $M(n \times m, \mathbb{K})$ be the linear space of $(n \times m)$-matrices over $\mathbb{K}$, abbreviate $M(n, \mathbb{K}) := M(n \times n, \mathbb{K})$, and let $\mathfrak{J}(n, \mathbb{K}) := \{X \in M(n, \mathbb{K}) \mid X^* = X\}$ be the subspace of Hermitian matrices; via the multiplication map

$$\mathfrak{J}(n, \mathbb{K}) \times \mathfrak{J}(n, \mathbb{K}) \to \mathfrak{J}(n, \mathbb{K}), \quad (X, Y) \mapsto X \circ Y := \frac{1}{2}(XY + YX),$$

$\mathfrak{J}(n,\mathbb{K})$ becomes a real Jordan algebra for $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ or $\mathbb{K} = \mathbb{O}$, $n = 3$; it becomes a complex Jordan algebra for $\mathbb{K} \in \{\mathbb{R}^{\mathbb{C}},\mathbb{C}^{\mathbb{C}},\mathbb{H}^{\mathbb{C}}\}$ or $\mathbb{K} = \mathbb{O}^{\mathbb{C}}$, $n = 3$. $\mathfrak{J}(3,\mathbb{O})$ resp. $\mathfrak{J} := \mathfrak{J}(3, \mathbb{O}^{\mathbb{C}})$ is the real resp. complex exceptional Jordan algebra.

We now consider the complex projective space over $\mathfrak{J}$, which we denote by $[\mathfrak{J}] \cong \mathbb{C}\mathrm{P}^{26}$. For $X \in \mathfrak{J} \setminus \{0\}$, we denote by $[X] := (\mathbb{R}^{\mathbb{C}})X$ the projective line through X; for a subset $M \subset \mathfrak{J}$, we put $[M] := \{[X] \mid X \in M \setminus \{0\}\}$. Following Atsuyama ([3]), we consider the submanifold

$$\widetilde{\mathrm{EIII}} := \left\{ X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \in \mathfrak{J} \;\middle|\; \begin{array}{l} \xi_1, \xi_2, \xi_3 \in \mathbb{R}^{\mathbb{C}},\ x_1, x_2, x_3 \in \mathbb{O}^{\mathbb{C}}, \\ \xi_2\xi_3 = |x_1|^2,\ \xi_3\xi_1 = |x_2|^2,\ \xi_1\xi_2 = |x_3|^2, \\ x_2x_3 = \xi_1\overline{x_1},\ x_3x_1 = \xi_2\overline{x_2},\ x_1x_2 = \xi_3\overline{x_3} \end{array} \right\}$$

of $\mathfrak{J}$. Then Atsuyama has shown ([3], Lemma 3.1) that $[\widetilde{\mathrm{EIII}}] \subset [\mathfrak{J}]$ is a model of the exceptional symmetric space EIII. In the sequel, we denote by EIII this model.

We will also use the fact that the exceptional Lie group E_6, which is the transvection group of EIII, can be realized as a subgroup of the group $\mathrm{Aut}(\mathfrak{J})$ of complex-linear automorphisms of $(\mathfrak{J}, \circ)$. More specifically, consider the inner product $\langle\cdot,\cdot\rangle$ and the operation $A\Delta B$ defined on $\mathfrak{J}$ in [3], §1. Then Atsuyama showed in [3], Lemma 1.5 (2) that

$$E_6 = \{ f \in \mathrm{Aut}(\mathfrak{J}) \mid \forall X, Y \in \mathfrak{J}\colon\ f(X\Delta Y) = (fX)\Delta(fY),\ \langle fX, fY\rangle = \langle X, Y\rangle\}$$

is a model of the exceptional Lie group E_6. This model acts transitively on the model of EIII described above.

We now define several involutive isometries on EIII (see also [16] Section 3, where the involutive automorphisms on the exceptional Lie group E_6 are classified):

- The conjugation map $\lambda_0\colon \mathbb{O}^{\mathbb{C}} \to \mathbb{O}^{\mathbb{C}}$ induced by the real form $\mathbb{O}$ of $\mathbb{O}^{\mathbb{C}}$ (i.e. the orthogonal involution $\lambda_0\colon \mathbb{O}^{\mathbb{C}} \to \mathbb{O}^{\mathbb{C}}$ characterized by $\mathrm{Fix}(\lambda_0) = \mathbb{O}$) induces via the map

$$\widetilde{\mathrm{EIII}} \to \widetilde{\mathrm{EIII}},\quad \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_0(\xi_1) & \lambda_0(x_3) & \overline{\lambda_0(x_2)} \\ \overline{\lambda_0(x_3)} & \lambda_0(\xi_2) & \lambda_0(x_1) \\ \lambda_0(x_2) & \overline{\lambda_0(x_1)} & \lambda_0(\xi_3) \end{pmatrix}$$

an isometric involution $\lambda\colon \mathrm{EIII} \to \mathrm{EIII}$.

- The orthogonal involution $\gamma_0\colon \mathbb{O}^{\mathbb{C}} \to \mathbb{O}^{\mathbb{C}}$ characterized by $\mathrm{Fix}(\gamma_0) = \mathbb{H}^{\mathbb{C}}$ induces via the map

$$\widetilde{\mathrm{EIII}} \to \widetilde{\mathrm{EIII}},\quad \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & \gamma_0(x_3) & \overline{\gamma_0(x_2)} \\ \overline{\gamma_0(x_3)} & \xi_2 & \gamma_0(x_1) \\ \gamma_0(x_2) & \overline{\gamma_0(x_1)} & \xi_3 \end{pmatrix}$$

another isometric involution $\gamma\colon \mathrm{EIII} \to \mathrm{EIII}$.

- The linear map

$$\widetilde{\mathrm{EIII}} \to \widetilde{\mathrm{EIII}}, \quad \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

induces yet another isometric involution $\sigma\colon \mathrm{EIII} \to \mathrm{EIII}$. σ is the geodesic symmetry of the symmetric space EIII at the point $p_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{EIII}$.

With help of these involutions we can describe the reflective submanifolds of EIII explicitly.

The type ($\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2$). The fixed point set of λ equals $[\mathrm{EIII} \cap \mathfrak{J}(3, \mathbb{O})] \cong \mathbb{O}\mathrm{P}^2$, a totally geodesic submanifold of EIII of type $(\mathbb{P}, \varphi = \pi/4, \mathbb{O}\mathrm{P}^2)$. Notice that this is a real form of the Hermitian symmetric space EIII.

The types ($G_2\mathbb{C}^6$) and ($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{C}$, 5), $\mathbb{C}$). The fixed point set of the involutive isometry $\gamma\colon \mathrm{EIII} \to \mathrm{EIII}$ has two connected components:

$$F_1^\gamma := [\widetilde{\mathrm{EIII}} \cap \mathfrak{J}(3, \mathbb{H}^{\mathbb{C}})],$$

and

$$F_2^\gamma := \left\{ \begin{bmatrix} 0 & a_3 e & -a_2 e \\ -a_3 e & 0 & a_1 e \\ a_2 e & -a_1 e & 0 \end{bmatrix} \middle|\, a_k \in \mathbb{H}^{\mathbb{C}},\ a_1\overline{a_2} = a_2\overline{a_3} = a_3\overline{a_1} = 0 \right\}.$$

It turns out that the totally geodesic submanifolds F_1^γ and F_2^γ of EIII are of type ($G_2\mathbb{C}^6$) and ($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{C}$, 5), $\mathbb{C}$), respectively. To show that they are isomorphic to $G_2(\mathbb{C}^6)$ resp. to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5$, we will now explicitly construct isometries $f_1\colon G_2(\mathbb{C}^6) \to F_1^\gamma$ and $f_2\colon \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5 \to F_2^\gamma$ which are compatible with the group actions on the symmetric spaces involved.

For this purpose, we note that E_6 contains a subgroup which is isomorphic to $(\mathrm{Sp}(1) \times \mathrm{SU}(6))/\mathbb{Z}_2$, and which is the fixed point group of the Lie group automorphism $E_6 \to E_6$, $g \mapsto \gamma \cdot g \cdot \gamma^{-1}$. This subgroup has been described explicitly by Yokota ([16], Section 3.5) in the following way:

To associate to a given $(b, B) \in \mathrm{Sp}(1) \times \mathrm{SU}(6)$ a member of $E_6 \subset \mathrm{Aut}(\mathfrak{J})$, we need to describe an action of (b, B) on $\mathfrak{J}$. For this purpose we note that $\mathfrak{J}$ is $(\mathbb{R}^{\mathbb{C}})$-linear isomorphic to $\mathfrak{J}(3, \mathbb{H}^{\mathbb{C}}) \oplus (\mathbb{H}^{\mathbb{C}})^3$ by the map

$$\varphi_1\colon \mathfrak{J}(3, \mathbb{H}^{\mathbb{C}}) \oplus (\mathbb{H}^{\mathbb{C}})^3 \to \mathfrak{J}, \quad (X, x) \mapsto X + \begin{pmatrix} 0 & x_3 e & x_2 e \\ -x_3 e & 0 & x_1 e \\ x_2 e & -x_1 e & 0 \end{pmatrix}.$$

Furthermore, $M(3, \mathbb{H}^{\mathbb{C}}) \supset \mathfrak{J}(3, \mathbb{H}^{\mathbb{C}})$ is $(\mathbb{R}^{\mathbb{C}})$-linear isomorphic to

$$M(6, \mathbb{C}^{\mathbb{C}})_J := \{X \in M(6, \mathbb{C}^{\mathbb{C}}) \mid JX = \overline{X}J\},$$

and $(\mathbb{H}^{\mathbb{C}})^3$ is $(\mathbb{R}^{\mathbb{C}})$-linear isomorphic to

$$M(2\times 6,\mathbb{C}^{\mathbb{C}})_J := \{X \in M(2\times 6,\mathbb{C}^{\mathbb{C}}) \mid J'X = \overline{X}J\},$$

where we put $J' := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M(2,\mathbb{R})$ and $J := \operatorname{diag}(J', J', J') \in M(6,\mathbb{R})$. These isomorphisms are exhibited by the maps

$$\varphi_2\colon M(3,\mathbb{H}^{\mathbb{C}}) \to M(6,\mathbb{C}^{\mathbb{C}})_J \quad \text{resp.} \quad \varphi_2'\colon (\mathbb{H}^{\mathbb{C}})^3 \to M(2\times 6,\mathbb{C}^{\mathbb{C}})_J$$

which transform any given matrix $X \in M(3,\mathbb{H}^{\mathbb{C}})$ resp. any given row vector $x \in (\mathbb{H}^{\mathbb{C}})^3$ into a matrix $\varphi_2(X) \in M(6,\mathbb{C}^{\mathbb{C}})$ resp. $\varphi_2'(x) \in M(2\times 6,\mathbb{C}^{\mathbb{C}})$ by mapping every entry $a + bj \in \mathbb{H}^{\mathbb{C}}$ $(a, b \in \mathbb{C}^{\mathbb{C}})$ of X into a (2×2)-block component $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ of $\varphi_2(X)$ resp. $\varphi_2'(x)$. We put $\mathfrak{J}(6,\mathbb{C}^{\mathbb{C}})_J := \varphi_2(\mathfrak{J}(3,\mathbb{H}^{\mathbb{C}})) \subset M(6,\mathbb{C}^{\mathbb{C}})_J$. In this way we obtain an isomorphism between $\mathfrak{J}$ and $\mathbb{V} := \mathfrak{J}(6,\mathbb{C}^{\mathbb{C}})_J \oplus M(2\times 6,\mathbb{C}^{\mathbb{C}})_J$:

$$\varphi := (\varphi_2 \oplus \varphi_2') \circ \varphi_1^{-1}\colon \mathfrak{J} \to \mathbb{V},$$

which we will use to describe the action of $\mathrm{Sp}(1)\times\mathrm{SU}(6)$ on $\mathfrak{J}$.

To do so, we consider for $\mathbb{K} \in \{\mathbb{C}, \mathbb{C}^{\mathbb{C}}\}$ besides $\mathrm{SU}(6,\mathbb{K}) = \{A \in M(6,\mathbb{K}) \mid A^*A = \mathrm{id},\ \det(A) = 1\}$ also $\mathrm{SU}^*(6,\mathbb{K}) := \{A \in M(6,\mathbb{K}) \mid JA = \overline{A}J,\ \det(A) = 1\}$. Then we have the isomorphism of Lie groups

$$\Phi\colon \mathrm{SU}(6,\mathbb{C}^{\mathbb{C}}) \to \mathrm{SU}^*(6,\mathbb{C}^{\mathbb{C}}), \quad A \mapsto \varepsilon A - \overline{\varepsilon} J \overline{A} J,$$

where we put $\varepsilon := (1/2)(1 + iI) \in \mathbb{C}^{\mathbb{C}}$.

We now consider the action $F_0\colon (\mathrm{Sp}(1)\times\mathrm{SU}^*(6,\mathbb{C}^{\mathbb{C}}))\times\mathbb{V} \to \mathbb{V}$ given by

$$F_0(b, B)(X + x) = BXB^* + (\varphi_2' b (\varphi_2')^{-1}) x B^{-1}$$

for all $(b, B) \in \mathrm{Sp}(1)\times\mathrm{SU}^*(6,\mathbb{C}^{\mathbb{C}})$ and $X + x \in \mathbb{V}$. F_0 induces an action $F\colon (\mathrm{Sp}(1)\times\mathrm{SU}(6))\times\mathfrak{J} \to \mathfrak{J}$ which is characterized by the fact that the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{Sp}(1)\times\mathrm{SU}^*(6,\mathbb{C}^{\mathbb{C}}))\times\mathbb{V} \arrow[r, "F_0"] & \mathbb{V} \\
(\mathrm{Sp}(1)\times\mathrm{SU}(6))\times\mathfrak{J} \arrow[r, "F"'] \arrow[u, "(\mathrm{id}_{\mathrm{Sp}(1)}\times\Phi)\times\varphi"] & \mathfrak{J}. \arrow[u, "\varphi"']
\end{tikzcd}$$

It has been shown by Yokota ([16], Theorem 3.5.11 and its proof) that $F(b, B) \in E_6$ holds for all $(b, B) \in \mathrm{Sp}(1)\times\mathrm{SU}(6)$. In this way we obtain a homomorphism of Lie groups $F\colon \mathrm{Sp}(1)\times\mathrm{SU}(6) \to E_6$ with $\ker(F) = \{\pm(\mathrm{id}, \mathrm{id})\}$.

We now denote for $U \in G_2(\mathbb{C}^6)$ by $P_U \in M(6,\mathbb{C})$ the orthogonal projection onto U. Then we have $Q_U := \varepsilon P_U - \overline{\varepsilon} J \overline{P_U} J \in \mathfrak{J}(6,\mathbb{C}^{\mathbb{C}})_J$, and therefore the map

$$f_1\colon G_2(\mathbb{C}^6) \to [\mathfrak{J}], \quad U \mapsto [\varphi^{-1}(Q_U + 0_{M(2\times 6,\mathbb{C}^{\mathbb{C}})})]$$

is well-defined. It turns out that f_1 is an isometric embedding and equivariant in the sense that for every $B \in \mathrm{SU}(6)$, $U \in G_2(\mathbb{C}^6)$ we have

$$F(\mathrm{id}, B)\, f_1(U) = f_1(BU).$$

As a consequence of this property and the fact that $f_1(\mathbb{C}e_1 \oplus \mathbb{C}e_2) = p_0 \in \mathrm{EIII}$ holds, f_1 maps into EIII, and hence it maps into $\mathrm{EIII} \cap [\mathfrak{J}(3, \mathbb{H}^{\mathbb{C}})] = F_1^{\gamma}$. Because both F_1^{γ} and $G_2(\mathbb{C}^6)$ are compact and connected, and are of the same (real) dimension 16, it follows that the isometric embedding f_1 in fact maps $G_2(\mathbb{C}^6)$ onto the totally geodesic submanifold F_1^{γ} of EIII.

To similarly construct a map $f_2 \colon \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5 \to F_2^{\gamma}$, we identify $\mathbb{C}^2$ with $\mathbb{H}$. In this way, we can regard $\mathbb{C}\mathrm{P}^1$ as the space $\{l\mathbb{C} \mid l \in \mathbb{S}(\mathbb{H})\}$. We also identify $\mathbb{C}^6$ with $\mathbb{H}^3$. Using these identifications, we can interpret for any $l \in \mathbb{C}^2 \cong \mathbb{H}$ and $v \in \mathbb{C}^6 \cong \mathbb{H}^3$ the expression $l\varepsilon v$ as a member of $(\mathbb{H}^{\mathbb{C}})^3$; via this expression we define the map

$$f_2 \colon \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5 \to [\mathfrak{J}], \quad (l\mathbb{C}, [v]) \mapsto [\varphi^{-1}(0_{\mathfrak{J}(6,\mathbb{C}^{\mathbb{C}})_J} + l\varepsilon v^*)],$$

which turns out to be a well-defined isometric embedding, which is equivariant in the following sense: For all $(b, B) \in \mathrm{Sp}(1) \times \mathrm{SU}(6)$, $(l\mathbb{C}, [v]) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5$, we have

$$F(b, B)\, f_2(l\mathbb{C}, [v]) = f_2(bl\mathbb{C}, [Bv]).$$

Because of this property, and the fact that $f_2(1\mathbb{C}, [e_1]) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon e \\ 0 & -\varepsilon e & 0 \end{bmatrix} \in F_2^{\gamma} \subset \mathrm{EIII}$, f_2 maps into EIII, and hence it maps into $\mathrm{EIII} \cap [\varphi(0_{\mathfrak{J}(6,\mathbb{C}^{\mathbb{C}})_J} \oplus (\mathbb{H}^{\mathbb{C}})^3)] = F_2^{\gamma}$. Because both F_2^{γ} and $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5$ are compact and connected, and they are of the same (real) dimension 12, it follows that the isometric embedding f_2 in fact maps $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^5$ onto the totally geodesic submanifold F_2^{γ} of EIII.

The type ($G_2\mathbb{H}^4$). Notice that the involutive isometries λ and γ commute with each other, and therefore $\lambda \circ \gamma$ is another involutive isometry of EIII. The fixed point set of the latter involution equals

$$F^{\lambda\gamma} := \left\{ p := \begin{bmatrix} r_1 & p_3 + q_3 eI & \overline{p_2} - q_2 eI \\ \overline{p_3} - q_3 eI & r_2 & p_1 + q_1 eI \\ p_2 + q_2 eI & \overline{p_1} - q_1 eI & r_3 \end{bmatrix} \;\middle|\; \begin{array}{l} r_k \in \mathbb{R},\ p_k, q_k \in \mathbb{H}, \\ p \in \mathrm{EIII} \end{array} \right\}.$$

It turns out that the totally geodesic submanifold $F^{\lambda\gamma}$ of EIII corresponds to the type $(G_2\mathbb{H}^4)$. We will show that $F^{\lambda\gamma}$ is isometric to $G_2(\mathbb{H}^4)/\mathbb{Z}_2$.

E_6 contains a subgroup isomorphic to $\mathrm{Sp}(4)/\mathbb{Z}_2$, which is the fixed point group of the Lie group automorphism $E_6 \to E_6$, $g \mapsto (\lambda\gamma)g(\lambda\gamma)^{-1}$. Also this subgroup has been described explicitly by Yokota ([16], Section 3.4). We will use his construction, which we now describe, to obtain an action of $\mathrm{Sp}(4)$ on $\mathfrak{J}$.

We continue to use the space $\mathbb{V}$ and the linear isomorphism $\varphi \colon \mathfrak{J} \to \mathbb{V}$ from the previous construction, put $\mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0 := \{X \in \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}}) \mid \mathrm{tr}(X) = 0\}$, and consider the isomorphism of linear spaces $\psi\colon \mathfrak{J} \to \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0$ given in the following way: For $A \in \mathfrak{J}$, say $\varphi(A) = X + x \in \mathbb{V}$, we put

$$\psi(A) = \begin{pmatrix} \frac{1}{2}\,\mathrm{tr}(X) & Ix \\ Ix^* & X - \frac{1}{2}\,\mathrm{tr}(X)\cdot \mathrm{id}_{(\mathbb{H}^{\mathbb{C}})^3} \end{pmatrix},$$

where the right-hand expression is to be read as a block matrix with respect to the decomposition $(\mathbb{H}^{\mathbb{C}})^4 = \mathbb{H}^{\mathbb{C}} \oplus (\mathbb{H}^{\mathbb{C}})^3$.

Notice that $\mathrm{Sp}(4)$ acts on $\mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0$ in the canonical way, i.e. by the action

$$F_0\colon \mathrm{Sp}(4) \times \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0 \to \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0, \quad (B, X) \mapsto BXB^*.$$

Via the linear isomorphism ψ, F_0 induces an action $F\colon \mathrm{Sp}(4) \times \mathfrak{J} \to \mathfrak{J}$, characterized by the fact that the diagram

$$\begin{tikzcd} \mathrm{Sp}(4) \times \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0 \arrow[r, "F_0"] & \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0 \\ \mathrm{Sp}(4) \times \mathfrak{J} \arrow[u, "\mathrm{id}_{\mathrm{Sp}(4)}\times\psi"] \arrow[r, "F"'] & \mathfrak{J} \arrow[u, "\psi"'] \end{tikzcd}$$

commutes. It has been shown by Yokota ([16], the proof of Theorem 3.4.2) that for any $B \in \mathrm{Sp}(4)$, $F(B) \in E_6$ holds, and $F(B)$ commutes with $\lambda\gamma \in E_6$. In this way, we obtain a homomorphism of Lie groups $F\colon \mathrm{Sp}(4) \to E_6$ with $\ker(F) = \{\pm \mathrm{id}\}$.

We now consider the map

$$f\colon G_2(\mathbb{H}^4) \to [\mathfrak{J}], \quad U \mapsto [\psi^{-1}(Z_U)],$$

where for any $U \in G_2(\mathbb{H}^4)$ we denote by $Z_U \in \mathfrak{J}(4, \mathbb{H}^{\mathbb{C}})_0$ the linear map characterized by $Z_U|U = (1/2)\mathrm{id}_U$, $Z_U|U^\perp = -(1/2)\mathrm{id}_{U^\perp}$. It is easy to see that f is a well-defined isometric two-fold covering map onto its image with fibers $\{U, U^\perp\}$ for $U \in G_2(\mathbb{H}^4)$, and that f is equivariant, i.e. that for any $U \in G_2(\mathbb{H}^4)$ and $B \in \mathrm{Sp}(4)$ we have

$$F(B)\,f(U) = f(BU).$$

Because of the latter property and the fact that $f(\mathbb{H}e_1 \oplus \mathbb{H}e_2) = p_0 \in \mathrm{EIII}$ holds, f maps into EIII. Moreover, we have $p_0 \in F^{\lambda\gamma}$ and for every $B \in \mathrm{Sp}(4)$, $F(B) \in E_6$ commutes with $\lambda\gamma$, and therefore f maps into the totally geodesic submanifold $F^{\lambda\gamma}$ of EIII. Because both $F^{\lambda\gamma}$ and $G_2(\mathbb{H}^4)$ are compact and connected, and they are of the same dimension 16, it follows that the isometric immersion f in fact maps $G_2(\mathbb{H}^4)$ onto $F^{\lambda\gamma}$. Because f is a two-fold covering map with fibers $\{U, U^\perp\}$, we conclude that $F^{\lambda\gamma}$ is isometric to $G_2(\mathbb{H}^4)/\mathbb{Z}_2$.

The types (Q) and (DIII). The connected components $\neq \{p_0\}$ of the fixed point set of the geodesic symmetry at p_0 are the polars of the symmetric space. (Also see [5], §2, especially Theorem 2.8, where the polars are denoted by M_+.) In the case of EIII, the polars have also been investigated by Atsuyama in [3], §3.

It is easily seen that the fixed point set of σ consists of two connected components besides $\{p_0\}$, namely

$$F_1^\sigma := \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x_1} & \xi_3 \end{bmatrix} \in [\mathfrak{J}] \;\middle|\; \xi_2\xi_3 = |x_1|^2 \right\},$$

and

$$F_2^\sigma := \left\{ \begin{bmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix} \in [\mathfrak{J}] \;\middle|\; x_2x_3 = 0,\ x_2\overline{x_2} = 0,\ x_3\overline{x_3} = 0 \right\}.$$

It turns out that the totally geodesic submanifolds F_1^σ and F_2^σ are of type (Q) and (DIII), respectively.

The complex-8-dimensional submanifold F_1^σ of the complex projective space $[\mathfrak{J}]$ is defined by a single non-degenerate quadratic equation, which is adapted to the Fubini–Study metric of $[\mathfrak{J}]$. Hence F_1^σ is isometric to the complex quadric Q^8.

Furthermore, it has been shown by Atsuyama that the reflective submanifold F_2^σ is isometric to SO(10)/U(5), see [3], the remark after Lemma 3.2 and [2], the Remark (2) after Proposition 5.4.

3.4. Totally geodesic submanifolds in Sp(2). Our next objective is the classification of the Lie triple systems in the Lie group Sp(2), regarded as a Riemannian symmetric space. We will use this result also in our classification of Lie triple systems of SO(10)/U(5) in Section 3.5 below.

We will base our classification on the fact that Sp(2) is a maximal totally geodesic submanifold of $G_2(\mathbb{H}^4)$ (of type (Sp_2) according to the classification in [8], Theorem 5.3). Because the Lie triple systems of $G_2(\mathbb{H}^4)$ have been classified in [8], we can therefore obtain a classification of the Lie triple systems by determining which of the Lie triple systems of $G_2(\mathbb{H}^4)$ are contained in a Lie triple system of type (Sp_2).

To do so, we will work in the setting of [8] in the present section. We consider the space $G_2(\mathbb{H}^4) = \mathrm{Sp}(4)/(\mathrm{Sp}(2) \times \mathrm{Sp}(2))$. We let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the canonical decomposition associated with this space, i.e. we have $\mathfrak{g} = \mathfrak{sp}(4)$, $\mathfrak{k} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \subset \mathfrak{g}$, and $\mathfrak{m}$ is isomorphic to the tangent space of $G_2(\mathbb{H}^4)$. We will use the notations of Section 5 of [8] in the sequel, especially we use the types of Lie triple systems defined in Theorem 5.3 of [8] for $G_2(\mathbb{H}^4)$, i.e. for $n = 2$. We let $\mathfrak{m}_1 \subset \mathfrak{m}$ be a Lie triple system of $\mathfrak{m}$ of type (Sp_2).

Theorem 3.8. *Exactly the following types of Lie triple systems of* $\mathfrak{m}$, *as defined in* Theorem 5.3 *of* [8], *have representatives which are contained in* $\mathfrak{m}_1$:

- $(\mathrm{Geo}, \varphi = t)$ *with* $t \in [0, \pi/4]$,
- $(\mathbb{S}, \varphi = \arctan(1/3), l)$ *with* $l \in \{2, 3\}$,
- $(\mathbb{P}, \varphi = \pi/4, \tau)$ *with* $\tau \in \{(\mathbb{R}, 1), (\mathbb{C}, 1), (\mathbb{S}^3), (\mathbb{H}, 1)\}$,
- $(\mathbb{P} \times \mathbb{P}, \tau_1, \tau_2)$ *with* $\tau_1, \tau_2 \in \{(\mathbb{R}, 1), (\mathbb{C}, 1), (\mathbb{S}^3)\}$,
- $(\mathbb{S}^1 \times \mathbb{S}^5, l)$ *with* $2 \leq l \leq 3$,
- (Q_3).

The maximal Lie triple systems of $\mathfrak{m}_1$ *are those which are of the types*: $(\mathbb{S}, \varphi = \arctan(1/3), 3)$, $(\mathbb{P}, \varphi = \pi/4, (\mathbb{H}, 1))$, $(\mathbb{P} \times \mathbb{P}, (\mathbb{S}^3), (\mathbb{S}^3))$ *and* (Q_3).

REMARK 3.9. The maximal totally geodesic submanifolds of $\mathrm{Sp}(2)$ of types $(\mathbb{S}, \varphi = \arctan(1/3), 3)$ and (Q_3) are missing from [5], Table VIII. Their isometry types are that of a 3-sphere of radius $(\pi/2)\sqrt{10}$ resp. of a complex quadric Q^3. The totally geodesic submanifolds of the former type are once again in a "skew" position in the sense that their geodesic diameter is strictly larger than the geodesic diameter π of $\mathrm{Sp}(2)$.

Proof of Theorem 3.8. It is easily seen that the prototypes for the types listed, as they are given in [8], Theorem 5.3, are contained in Lie triple systems of type (Sp_2). Therefore, we only need to show that no other types of Lie triple systems of $G_2(\mathbb{H}^4)$ have representatives which are contained in $\mathfrak{m}_1$.

For this purpose, we let $\mathfrak{m}'$ be a Lie triple system of $\mathfrak{m}$ which is contained in $\mathfrak{m}_1$. We are to show that $\mathfrak{m}'$ is of one of the types listed in Theorem 3.8.

If $\mathfrak{m}'$ is of rank 2, then for $\mathfrak{m}'$ to be contained in $\mathfrak{m}_1$, it is necessary that all the roots of $\mathfrak{m}'$ have at most the multiplicity of the corresponding root in $\mathfrak{m}_1$. Because the Dynkin diagram of $\mathfrak{m}_1$ is $\bullet^2 \Rightarrow \bullet^2$, we see by this argument that $\mathfrak{m}'$ cannot be of one of the types (G_2, τ), $(\mathbb{P} \times \mathbb{P}, \tau_1, \tau_2)$ where either of the $\mathbb{H}$P-types[6] τ_1 and τ_2 has dimension ≥ 2 or width 4, or $(\mathbb{S}^1 \times \mathbb{S}^5, l)$ where $l \geq 4$. This already shows that among the types of Lie triple systems of rank 2 of $G_2(\mathbb{H}^4)$, only those which are listed in Theorem 3.8 remain.

If $\mathfrak{m}'$ is of rank 1, we note that if $\mathfrak{m}'$ is of type $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 1))$ or of type $(\mathbb{P}, \varphi = \arctan(1/2), \tau)$ with $\tau \neq (\mathbb{R}, 1)$, it cannot be contained in $\mathfrak{m}_1$ because the roots $2\lambda_k$ are not present in $\mathfrak{m}_1$. Because the types $(\mathbb{P}, \varphi = 0, (\mathbb{R}, 1))$ and $(\mathbb{P}, \varphi = \arctan(1/2), (\mathbb{R}, 1))$ are identical to $(\mathrm{Geo}, \varphi = t)$ with $t = 0$ resp. with $t = \arctan(1/2)$, this argument again leaves only the types of rank 1 which have been listed in the theorem.

For the statements on the maximality: $(\mathbb{P} \times \mathbb{P}, (\mathbb{S}^3), (\mathbb{S}^3))$ and (Q_3) are Lie triple systems of rank 2, and therefore can be contained only in other Lie triple systems of this rank. Because they have the same dimension 6 and are clearly not isomorphic, neither of them can be contained in the other, and also for reason of dimension, nei-

[6] See [8], Definition 5.1.

ther can be contained in $(\mathbb{S}^1 \times \mathbb{S}^5, l)$ with $l \le 3$. Therefore these two types are maximal in $\mathfrak{m}_1$. From a consideration of the root systems it can also be seen that $(\mathbb{S}, \varphi = \arctan(1/3), 3)$ and $(\mathbb{P}, \varphi = \pi/4, (\mathbb{H}, 1))$ are maximal. On the other hand, $(\mathrm{Geo}, \varphi = t)$, $(\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 1))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(\mathbb{P} \times \mathbb{P}, \tau_1, \tau_2)$ with $\tau_1, \tau_2 \in \{(\mathbb{R}, 1), (\mathbb{C}, 1), (\mathbb{S}^3)\}$ and $(\mathbb{S}^1 \times \mathbb{S}^5, l)$ with $l \le 3$ are all contained in $(\mathbb{P} \times \mathbb{P}, (\mathbb{S}^3), (\mathbb{S}^3))$, whereas $(\mathbb{S}, \varphi = \arctan(1/3), 2)$ is contained in $(\mathbb{S}, \varphi = \arctan(1/3), 3)$. Therefore these types cannot be maximal. □

We can obtain the global isometry types of the totally geodesic submanifolds corresponding to the Lie triple systems of Sp(2) as listed in Theorem 3.8 from the totally geodesic embeddings into $G_2(\mathbb{H}^n)$ described in [8], Section 6. When applying the information from that paper, one needs to take into account, however, that in the Sp(2) as totally geodesic submanifold of $G_2(\mathbb{H}^n)$ (with the Riemannian metric considered in that paper) the shortest restricted root has length $\sqrt{2}$, whereas here we want to view Sp(2) with the metric so that the shortest restricted root has length 1. Therefore the curvatures of the projective spaces have to be multiplied with 1/2, and the radii of the spheres have to be multiplied with $\sqrt{2}$, to translate from the situation in [8] to the present situation. In this way, we obtain the following information on the totally geodesic submanifolds of $\mathrm{Sp}(2)_{\mathrm{srr}=1}$, where we again use the notations introduced in Section 3.3.

type of Lie triple system	isometry type	properties
$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
$(\mathbb{S}, \varphi = \arctan(1/3), l)$	$\mathbb{S}^l_{r=\sqrt{5}}$	$l = 3$: maximal
$(\mathbb{P}, \varphi = \pi/4, (\mathbb{K}, 1))$	$\mathbb{K}\mathrm{P}^1_{\varkappa=1/4}$	$\mathbb{K} = \mathbb{H}$: polar, maximal
$(\mathbb{P}, \varphi = \pi/4, (\mathbb{S}^3))$	$\mathbb{S}^3_{r=1}$	
$(\mathbb{P} \times \mathbb{P}, (\mathbb{K}_1, 1), (\mathbb{K}_2, 1))$	$\mathbb{K}_1\mathrm{P}^1_{\varkappa=1/2} \times \mathbb{K}_2\mathrm{P}^1_{\varkappa=1/2}$	
$(\mathbb{P} \times \mathbb{P}, (\mathbb{K}, 1), (\mathbb{S}^3))$	$\mathbb{K}\mathrm{P}^1_{\varkappa=1/2} \times \mathbb{S}^3_{r=1/\sqrt{2}}$	$\mathbb{K} = \mathbb{R}$: meridian to (Q_3)
$(\mathbb{P} \times \mathbb{P}, (\mathbb{S}^3), (\mathbb{S}^3))$	$\mathbb{S}^3_{r=1/\sqrt{2}} \times \mathbb{S}^3_{r=1/\sqrt{2}}$	meridian to $(\mathbb{P}, \varphi = \pi/4, (\mathbb{H}, 1))$, maximal
$(\mathbb{S}^1 \times \mathbb{S}^5, l)$	$(\mathbb{S}^1_{r=1} \times \mathbb{S}^l_{r=1})/\mathbb{Z}_2$	
(Q_3)	$Q^3_{\mathrm{srr}=1}$	polar, maximal

3.5. Totally geodesic submanifolds in SO(10)/U(5). We now want to classify the Lie triple systems of SO(10)/U(5). Note that this symmetric space occurs as a maximal totally geodesic submanifold of EIII. We will use the classification of Lie triple systems of EIII from Section 3.2 to obtain the classification for SO(10)/U(5) in an analogous way as we used the classification in $G_2(\mathbb{H}^4)$ to obtain the classification for Sp(2) in the previous section.

Thus we now return to the situation studied in Section 3.2. We consider the Riemannian symmetric space EIII, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g} = \mathfrak{e}_6$ associated with this space, i.e. we have $\mathfrak{k} = \mathbb{R} \oplus \mathfrak{o}(10)$ and $\mathfrak{m}$ is isomorphic to the tangent space of EIII. We will use the names for the types of Lie triple systems of $\mathfrak{m}$ as introduced in Theorem 3.3 and Remark 3.5.

Further, we let $\mathfrak{m}_1$ be a Lie triple system of $\mathfrak{m}$ of type (DIII), i.e. the totally geodesic submanifold of EIII corresponding to $\mathfrak{m}_1$ is isometric to SO(10)/U(5).

Theorem 3.10. *Exactly the following types of Lie triple systems of* EIII *have representatives which are contained in* $\mathfrak{m}_1$:

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/4]$.
- ($\mathbb{P}$, $\varphi = 0$, ($\mathbb{K}$, 4)) *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- ($\mathbb{P}$, $\varphi = \pi/4$, ($\mathbb{S}^k$)) *with* $k \in \{5, 6\}$.
- ($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{K}_1$, 3), $\mathbb{K}_2$) *with* $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$.
- (Q, (G1, 6)).
- *The types* (Q, τ), *where* τ *is one of the types of Lie triple systems in the complex quadric as defined in* [7], Theorem 4.1, *for* $m = 6$, *i.e.* τ *is one of the following*: (G1, k) *with* $k \le 5$, (G2, k_1, k_2) *with* $k_1 + k_2 \le 6$, (G3), (P1, k) *with* $k \le 6$, (P2), (A), (I1, k) *with* $k \le 3$, *and* (I2, k) *with* $k \le 3$.
- ($G_2\mathbb{C}^6$, (G_2, ($\mathbb{C}$, 3))).
- *The types* ($G_2\mathbb{C}^6$, τ), *where* τ *is one of the following*: ($\mathbb{P}$, $\varphi = \arctan(1/2)$, ($\mathbb{K}$, k)) *with* ($\mathbb{K}$, k) $\in \{(\mathbb{R}, 1), (\mathbb{C}, 1), (\mathbb{R}, 2)\}$, ($\mathrm{G}_2$, ($\mathbb{R}$, k)) *with* $k \le 3$, (G_2, ($\mathbb{C}$, k)) *with* $k \le 2$, *and* ($\mathbb{P} \times \mathbb{P}$, ($\mathbb{K}_1$, k_1), ($\mathbb{K}_2$, k_2)) *with* $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k_1 + k_2 \le 3$.
- ($G_2\mathbb{H}^4$, (Sp_2)).
- *The types* ($G_2\mathbb{H}^4$, τ), *where* τ *is one of the following*: ($\mathbb{S}$, $\varphi = \arctan(1/3)$, 3), ($\mathbb{P}$, $\varphi = \pi/4$, ($\mathbb{K}$, 1)) *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *and* ($\mathbb{S}^1 \times \mathbb{S}^5$, 3).

The maximal Lie triple systems of $\mathfrak{m}_1$ *are those of the types*: ($\mathbb{P}$, $\varphi = 0$, ($\mathbb{C}$, 4)), ($\mathbb{P} \times \mathbb{P}^1$, ($\mathbb{C}$, 3), $\mathbb{C}$), (Q, (G1, 6)), ($G_2\mathbb{C}^6$, (G_2, ($\mathbb{C}$, 3))) *and* ($G_2\mathbb{H}^4$, (Sp_2)).

REMARK 3.11. Chen and Nagano correctly list the local isometry type of all the maximal totally geodesic submanifolds of SO(10)/U(5) in their Table VIII of [5]. However the isometry types of the types (Q, (G1, 6)) resp. ($G_2\mathbb{H}^4$, (Sp_2))) are Q^6 resp. SO(5) (where Chen/Nagano claim $G_2(\mathbb{R}^8) \cong Q^6/\mathbb{Z}_2$ and $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ respectively). Moreover, it should be mentioned that also SO(10)/U(5) contains "skew" totally geodesic submanifolds, namely the totally geodesic submanifolds of the types (Q, (A)) and ($G_2\mathbb{H}^4$, ($\mathbb{S}$, $\varphi = \arctan(1/3)$, 3)), which are isometric to a 2-sphere resp. a 3-sphere of radius $\sqrt{5}$, so that their geodesic diameter $\sqrt{5}\pi$ is strictly larger than the geodesic diameter π of SO(10)/U(5). They are not maximal in SO(10)/U(5); their presence can not be inferred from Table VIII of [5] because of the missing entries for the spaces $G_2^+(\mathbb{R}^5)$ and Sp(2).

Proof of Theorem 3.10. For the maximal ones among the types listed, the corresponding totally geodesic embeddings into SO(10)/U(5) are described below, so we know that these types, and therefore also all the other types listed, have representatives contained in $\mathfrak{m}_1$. Therefore, we only need to show that no other types of Lie triple systems of EIII have representatives which are contained in $\mathfrak{m}_1$.

For this purpose, we let $\mathfrak{m}'$ be a Lie triple system of $\mathfrak{m}$ which is contained in $\mathfrak{m}_1$. We are to show that $\mathfrak{m}'$ is of one of the types listed in Theorem 3.10.

If $\mathfrak{m}'$ is of rank 2, all the roots of $\mathfrak{m}'$ have at most the multiplicity of the corresponding root in $\mathfrak{m}_1$. Because the Dynkin diagram of $\mathfrak{m}_1$ is $\bullet^4 \Leftrightarrow \odot^{4[1]}$, we see by this argument that $\mathfrak{m}'$ cannot be one of the types $(\mathbb{P} \times \mathbb{P}^1, (\mathbb{K}_1, l), \mathbb{K}_2)$ with $l \geq 4$, (Q), $(G_2\mathbb{C}^6)$ and $(G_2\mathbb{H}^4)$. Moreover, we note that the intersection of $\mathfrak{m}_1$ with a Lie triple system of type (Q) is of type $(Q, (\mathrm{G1}, 6))$ (corresponding to $Q^6 \subset Q^8 \subset \mathrm{EIII}$), with a Lie triple system of type $(G_2\mathbb{C}^6)$ is of type $(G_2\mathbb{C}^6, (G_2, (\mathbb{C}, 3)))$ (corresponding to $G_2(\mathbb{C}^5) \subset G_2(\mathbb{C}^6) \subset \mathrm{EIII}$), and with a Lie triple system of type $G_2(\mathbb{H}^4)$ is of type $(G_2\mathbb{H}^4, (\mathrm{Sp}_2))$ (corresponding to $\mathrm{SO}(5) \subset G_2(\mathbb{H}^4)/\mathbb{Z}_2 \subset \mathrm{EIII}$). Therefrom everything about the rank 2 Lie triple systems follows.

For the spaces of rank 1 a similar consideration of the multiplicities of the roots shows that $\mathfrak{m}'$ is of one of the types listed in the theorem. □

Because $\mathrm{SO}(10)/\mathrm{U}(5)$ is a totally geodesic submanifold of EIII, the isometry types of the totally geodesic submanifolds in $\mathrm{SO}(10)/\mathrm{U}(5)$ corresponding to the various types of Lie triple systems are the same as the isometry types of the totally geodesic submanifolds in EIII of those types, which were described in Section 3.3. In particular, the isometry types of the maximal totally geodesic submanifolds of $\mathrm{SO}(10)/\mathrm{U}(5)$, and some of their properties, are as follows:

type	isometry type	properties
$(\mathbb{P}, \varphi = 0, (\mathbb{C}, 4))$	$\mathbb{C}\mathrm{P}^4_{\varkappa=1}$	polar
$(\mathbb{P} \times \mathbb{P}^1, (\mathbb{C}, 3), \mathbb{C})$	$\mathbb{C}\mathrm{P}^3_{\varkappa=1} \times \mathbb{C}\mathrm{P}^1_{\varkappa=1}$	meridian for $(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 3)))$
$(Q, (\mathrm{G1}, 6))$	$Q^6_{\mathrm{srr}=\sqrt{2}}$	meridian for $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 4))$
$(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 3)))$	$G_2(\mathbb{C}^5)_{\mathrm{srr}=1}$	polar
$(G_2\mathbb{H}^4, (\mathrm{Sp}_2))$	$\mathrm{SO}(5)_{\mathrm{srr}=1}$	reflective

To elucidate the position of the maximal totally geodesic submanifolds, we describe totally geodesic embeddings for these types:

The types $(\mathbb{P}, \varphi = 0, (\mathbb{C}, 4))$ and $(G_2\mathbb{C}^6, (\mathrm{G}_2, (\mathbb{C}, 3)))$. The totally geodesic submanifolds of these types are the polars in $\mathrm{SO}(10)/\mathrm{U}(5)$, and can therefore be obtained as $\mathrm{U}(5)$-orbits through points of $\mathrm{SO}(10)/\mathrm{U}(5)$ which are antipodal to the origin point $p_0 := \mathrm{U}(5) \in \mathrm{SO}(10)/\mathrm{U}(5)$ in this space.

For an explicit construction, we consider both $\mathrm{U}(5)$ and $\mathrm{SO}(10)$ acting on $\mathbb{C}^5$; in the latter case the action is only $\mathbb{R}$-linear on $\mathbb{C}^5 \cong \mathbb{R}^{10}$. We fix a real form V of $\mathbb{C}^5$ (i.e. a 5-dimensional real linear subspace $V \subset \mathbb{C}^5$ so that $i \cdot V$ is orthogonal to V with respect to the real inner product on $\mathbb{C}^5$). Then we can describe $\mathrm{U}(5)$ as a subgroup of $\mathrm{SO}(10)$ by

$$\mathrm{U}(5) = \left\{ g \in \mathrm{SO}(10) \;\middle|\; g = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},\; A, B \in M(5 \times 5, \mathbb{R}) \right\},$$

where the matrix expression is to be read as a block matrix with respect to the splitting $\mathbb{C}^5 = V \oplus i \cdot V$. In the same way, we can describe the involutive automorphism describing the symmetric structure of SO(10)/U(5):

$$\sigma : \mathrm{SO}(10) \to \mathrm{SO}(10), \quad \begin{pmatrix} A & C \\ B & D \end{pmatrix} \mapsto \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

Via the linearization of σ, we obtain the space $\mathfrak{m}$ in the splitting $\mathfrak{o}(10) = \mathfrak{u}(5) \oplus \mathfrak{m}$ induced by the symmetric structure of SO(10)/U(5):

$$\mathfrak{m} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \,\middle|\, A, B \in \mathfrak{o}(5) \right\}.$$

We now fix a $2k$-dimensional real linear subspace $W \subset V$ (with $k \in \{1, 2\}$) and a "partial complex structure with respect to W", i.e. a skew-adjoint transformation $J \colon V \to V$ with $J^3 = -J$ and $J(V) = W$. Then we have $X := \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \in \mathfrak{m}$, and therefore $\gamma \colon \mathbb{R} \to \mathrm{SO}(10)/\mathrm{U}(5)$, $t \mapsto \exp(tX) \cdot p_0$ is a geodesic of SO(10)/U(5). For $t \in \mathbb{R}$ and $w \in W$ we have

$$\exp(tX)w = \cos(t)w + \sin(t)Jw$$

and

$$\exp(tX)iw = \cos(t)iw - \sin(t)iJw$$

as well as $\exp(tX)w' = w'$ for any $w' \in (W \oplus iW)^\perp$. We have $\gamma(t) = p_0$ if and only if $\exp(tX) \in \mathrm{U}(5)$; from the above description it follows that this is the case if and only if $\sin(t) = 0$ holds, i.e. if we have $t \in \pi\mathbb{Z}$. Hence the geodesic γ is periodic with period π, and therefore $p_1 := \gamma(\pi/2)$ is an antipodal point of p_0 in SO(10)/U(5). By general results (see [5], Lemma 2.1), it is known that the polar $M := \mathrm{U}(5) \cdot p_1$ is a totally geodesic submanifold of SO(10)/U(5).

To determine the isometry type of the totally geodesic submanifold M, we calculate the isotropy group of the action of U(5) at p_1: We have $p_1 = S \cdot \mathrm{U}(5)$ with $S := \exp((\pi/2)X) \in \mathrm{SO}(10)$; from the explicit description of X we obtain the explicit description

$$S|W = J|W, \quad S|iW = -J|iW, \quad S|(W \oplus iW)^\perp = \mathrm{id}_{(W\oplus iW)^\perp} \tag{29}$$

of S. Therefore we have for $g \in \mathrm{U}(5)$:

$$\begin{aligned} g \cdot p_1 = p_1 &\iff g \cdot S \cdot \mathrm{U}(5) = S \cdot \mathrm{U}(5) \iff S^{-1} g S \in \mathrm{U}(5) \\ &\iff g(W \oplus iW) = W \oplus iW, \end{aligned}$$

where the last equivalence follows from Equations (29). Therefore the isotropy group of the action of U(5) at p_1 is isomorphic to $\mathrm{U}(W \oplus iW) \times \mathrm{U}((W \oplus iW)^\perp) \cong \mathrm{U}(2k) \times$

$\mathrm{U}(5-2k)$. It follows that the totally geodesic submanifold M of $\mathrm{SO}(10)/\mathrm{U}(5)$ is isometric to $\mathrm{U}(5)/(\mathrm{U}(2k)\times\mathrm{U}(5-2k))$.

In the case $k=1$, M is thus isometric to $\mathrm{U}(5)/(\mathrm{U}(2)\times\mathrm{U}(3)) \cong G_2(\mathbb{C}^5)$; this totally geodesic submanifold turns out to be of type $(G_2\mathbb{C}^6, (\mathbb{C}, 3))$.

In the case $k=2$, M is isometric to $\mathrm{U}(5)/(\mathrm{U}(4)\times\mathrm{U}(1)) \cong \mathbb{C}\mathrm{P}^4$; this totally geodesic submanifold is of type $(\mathbb{P}, \varphi=0, (\mathbb{C}, 4))$.

The types $(\mathbb{P}\times\mathbb{P}^1, (\mathbb{C}, 3), \mathbb{C})$ and $(Q, (\mathrm{G1}, 6))$. These types are the meridians corresponding to the polars of type $(G_2\mathbb{C}^6, (\mathbb{C}, 3))$ and $(\mathbb{P}, \varphi=0, (\mathbb{C}, 4))$, respectively. This means that each of them is a totally geodesic submanifold which intersects the corresponding polar orthogonally and transversally in one point.

However, in the present situation there is an easier way to describe the totally geodesic submanifolds of these types. Note that there are canonical embeddings $\mathrm{SO}(4)\times\mathrm{SO}(6)\subset\mathrm{SO}(10)$ and $\mathrm{SO}(8)\subset\mathrm{SO}(10)$ which are compatible with the symmetric structure of $\mathrm{SO}(10)/\mathrm{U}(5)$. In this way, we get totally geodesic embeddings of $(\mathrm{SO}(4)/\mathrm{U}(2))\times(\mathrm{SO}(6)/\mathrm{U}(3))\cong\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^3$ and of $\mathrm{SO}(8)/\mathrm{U}(4)\cong Q^6$ into $\mathrm{SO}(10)/\mathrm{U}(5)$; they are of type $(\mathbb{P}\times\mathbb{P}^1, (\mathbb{C}, 3), \mathbb{C})$ and $(Q, (\mathrm{G1}, 6))$, respectively.

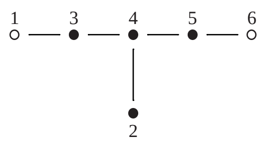
The type $(G_2\mathbb{H}^4, (\mathrm{Sp}_2))$. Consider the map

$$\Phi\colon \mathrm{SO}(5)\to\mathrm{SO}(10),\quad B\mapsto\begin{pmatrix} B & 0\\ 0 & B^{-1}\end{pmatrix}.$$

For $B\in\mathrm{SO}(5)$ we have $\Phi(B)\in\mathrm{U}(5)\iff B=\mathrm{id}$, and therefore Φ induces an embedding $\underline{\Phi}\colon\mathrm{SO}(5)\to\mathrm{SO}(10)/\mathrm{U}(5)$. Its linearization maps $X\in\mathfrak{o}(5)$ onto $\begin{pmatrix} X & 0\\ 0 & -X\end{pmatrix}\in\mathfrak{m}$, and therefore $\underline{\Phi}$ is totally geodesic. It turns out to be of type $(G_2\mathbb{H}^4, (\mathrm{Sp}_2))$.

4. The symmetric spaces E_6/F_4, SU(6)/Sp(3), SU(3) and SU(3)/SO(3)

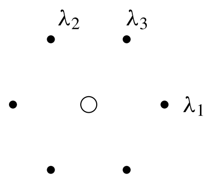
4.1. The geometry of E_6/F_4. In this section we will study the Riemannian symmetric space $\mathrm{EIV} := E_6/F_4$, which has the Satake diagram



EIV does not have an invariant Hermitian structure.

We consider the Lie algebra $\mathfrak{g}:=\mathfrak{e}_6$ of the transvection group E_6 of EIV and the splitting $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ induced by the symmetric structure of EIV. Herein $\mathfrak{k}=\mathfrak{f}_4$ is the Lie algebra of the isotropy group of EIV, and $\mathfrak{m}$ is isomorphic to the tangent space of EIV in the origin. The E_6-invariant Riemannian metric on EIV induces an $\mathrm{Ad}(F_4)$-invariant Riemannian metric on $\mathfrak{m}$. As was explained in Section 2, this metric is only unique up to a factor; we choose the factor in such a way that the restricted roots of EIV (see below) have the length 1.

The root space decomposition. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ which is maximally non-compact, i.e. it is such that the flat subspace $\mathfrak{a} := \mathfrak{t} \cap \mathfrak{m}$ of $\mathfrak{m}$ is of the maximal dimension 2, and hence a Cartan subalgebra of $\mathfrak{m}$. Then we consider the root system $\Delta^{\mathfrak{g}} \subset \mathfrak{t}^*$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, as well as the restricted root system $\Delta \subset \mathfrak{a}^*$ of the symmetric space EIV with respect to $\mathfrak{a}$. EIV has the restricted Dynkin diagram $\bullet^{8}\!\!-\!\!\!-\!\!\bullet^{8}$, in other words: its restricted root system Δ is of type A_2, i.e. we have $\Delta = \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3\}$, where (λ_1, λ_2) is a system of simple roots of Δ, these two roots are at an angle of $(2/3)\pi$ and have the same length, and we have $\lambda_3 = \lambda_1 + \lambda_2$. All roots in Δ have the multiplicity 8, and Δ has the following graphical representation:



To be able to apply the results from [9] and the corresponding computer package for the calculation of the curvature tensor of EIV, we again need to describe the relationship between the restricted roots of the symmetric space EIV and the (non-restricted) roots of the Lie algebra $\mathfrak{e}_6$. For this purpose, we again denote the positive roots of $\mathfrak{e}_6$ by $\alpha_1, \ldots, \alpha_{36}$ in the way described in Section 3.1. To find out which restricted root of EIV corresponds to each root of $\mathfrak{e}_6$, we tabulate the orbits of the action of σ on the root system $\Delta^{\mathfrak{g}}$, and the restricted root of EIV corresponding to each orbit (compare Section 4 of [9]):

orbit	$\{\alpha_1, -\alpha_{30}\}$	$\{\alpha_7, -\alpha_{27}\}$	$\{\alpha_{12}, -\alpha_{22}\}$	$\{\alpha_{17}, -\alpha_{18}\}$
corresp. restr. root	λ_1	λ_1	λ_1	λ_1

orbit	$\{\alpha_6, -\alpha_{31}\}$	$\{\alpha_{11}, -\alpha_{28}\}$	$\{\alpha_{16}, -\alpha_{24}\}$	$\{\alpha_{20}, -\alpha_{21}\}$
corresp. restr. root	λ_2	λ_2	λ_2	λ_2

orbit	$\{\alpha_{23}, -\alpha_{36}\}$	$\{\alpha_{26}, -\alpha_{35}\}$	$\{\alpha_{29}, -\alpha_{34}\}$	$\{\alpha_{32}, -\alpha_{33}\}$
corresp. restr. root	λ_3	λ_3	λ_3	λ_3

Moreover, we have $\sigma(\alpha_k) = \alpha_k$ for $k \in \{2, 3, 4, 5, 8, 9, 10, 13, 14, 15, 19, 25\}$.

Again using the notations $K_\alpha(c)$ and $M_\alpha(c)$ introduced in Section 3.1, we now put for $c_1, \ldots, c_4 \in \mathbb{C}$ and $t \in \mathbb{R}$, and where A denotes one of the letters K and M:

$$\begin{aligned}
A_{\lambda_1}(c_1, c_2, c_3, c_4) &:= A_{\alpha_1}(c_1) + A_{\alpha_7}(c_2) + A_{\alpha_{12}}(c_3) + A_{\alpha_{17}}(c_4),\\
A_{\lambda_2}(c_1, c_2, c_3, c_4) &:= A_{\alpha_6}(c_1) + A_{\alpha_{11}}(c_2) + A_{\alpha_{16}}(c_3) + A_{\alpha_{20}}(c_4),\\
A_{\lambda_3}(c_1, c_2, c_3, c_4) &:= A_{\alpha_{23}}(c_1) + A_{\alpha_{26}}(c_2) + A_{\alpha_{29}}(c_3) + A_{\alpha_{32}}(c_4).
\end{aligned}$$

Then we have $\mathfrak{m}_{\lambda_k} = M_{\lambda_k}(\mathbb{C}, \mathbb{C}, \mathbb{C}, \mathbb{C})$ for $k \in \{1, 2, 3\}$.

The action of the isotropy group. We now look at the isotropy action of EIV. Regarding it, we again use the notations introduced at the end of Section 2, in particular we have the continuous function $\varphi\colon \mathfrak{m} \setminus \{0\} \to [0, \pi/3]$ parametrizing the orbits of the isotropy action. For the elements of the closure $\bar{\mathfrak{c}}$ of the positive Weyl chamber $\mathfrak{c} := \{v \in \mathfrak{a} \mid \lambda_1(v) \geq 0, \lambda_2(v) \geq 0\}$ we again explicitly describe the relation to their isotropy angle: $((\lambda_1^\sharp + \lambda_3^\sharp)/\sqrt{3}, \lambda_2^\sharp)$ is an orthonormal basis of $\mathfrak{a}$ so that with $v_t := \cos(t)((\lambda_1^\sharp + \lambda_3^\sharp)/\sqrt{3}) + \sin(t)\lambda_2^\sharp$ we have

$$\bar{\mathfrak{c}} = \left\{ s \cdot v_t \;\middle|\; t \in \left[0, \frac{\pi}{3}\right], s \in \mathbb{R}_{\geq 0} \right\}, \tag{30}$$

and because the Weyl chamber $\mathfrak{c}$ is bordered by the two vectors $v_0 = (\lambda_1^\sharp + \lambda_3^\sharp)/\sqrt{3}$ with $\varphi(v_0) = 0$ and $v_{\pi/3} = (\lambda_2^\sharp + \lambda_3^\sharp)/\sqrt{3}$ with $\varphi(v_{\pi/3}) = \pi/3$, we have

$$\varphi(s \cdot v_t) = t \quad \text{for all} \quad t \in \left[0, \frac{\pi}{3}\right], s \in \mathbb{R}_+. \tag{31}$$

The isotropy action of $K = F_4$ on $\mathfrak{m}$ corresponds to the irreducible 26-dimensional representation of F_4 (see [1], Lemma 14.4 (i), p. 95). It can be described as an action of F_4 on the 26-dimensional space $\mathfrak{J}(3, \mathbb{O})_0 := \{X \in M(3 \times 3, \mathbb{O}) \mid X^* = X, \operatorname{tr}(X) = 0\}$ of trace-free, Hermitian (3×3)-matrices over the division algebra of octonions $\mathbb{O}$, for the details see [1], Chapter 16. Under the identification of $\mathfrak{m}$ with $\mathfrak{J}(3, \mathbb{O})_0$ induced thereby, the Cartan subalgebra $\mathfrak{a}$ corresponds to the space of trace-free diagonal matrices, and the three root spaces $\mathfrak{m}_{\lambda_k}$ $(k = 1, 2, 3)$ correspond to the subspaces $\mathfrak{J}_k := \{X = (x_{ij}) \in \mathfrak{J}(3, \mathbb{O})_0 \mid x_{11} = x_{22} = x_{33} = x_{kl} = x_{km} = 0\}$ of $\mathfrak{J}(3, \mathbb{O})_0$, where l and m are the two members of $\{1, 2, 3\} \setminus \{k\}$.

The subgroup K_0 of F_4 with Lie algebra $\mathfrak{k}^{\mathfrak{a}} := \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}$ consists of those $g \in F_4$ which leave all the subspaces $\mathfrak{J}_k$ invariant, and is therefore isomorphic to Spin(8) (see [1], Theorem 16.7 (iii)). K_0 acts on the three spaces $\mathfrak{J}_k$ as the three irreducible 8-dimensional representations of Spin(8): the vector representation, and the two spin representations; these representations are "intertwined" by the triality automorphism of Spin(8).

Proposition 4.1. *We regard $\mathbb{R}^8$ as the real linear space underlying $\mathbb{C}^4$ and for $k \in \{1, 2, 3\}$ we consider the linear isometry*

$$\varphi\colon \mathbb{R}^8 \to \mathfrak{m}_{\lambda_k}, \quad (c_1, c_2, c_3, c_4) \mapsto M_{\lambda_k}(c_1, c_2, c_3, c_4).$$

Then there exists an isomorphism of Lie groups $\Phi\colon \mathrm{Spin}(8) \to K_0$ so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spin}(8) \times \mathbb{R}^8 \arrow[r, "\Phi\times\varphi"] \arrow[d] & K_0 \times \mathfrak{m}_{\lambda_1} \arrow[d, "\mathrm{Ad}"] \\ \mathbb{R}^8 \arrow[r, "\varphi"'] & \mathfrak{m}_{\lambda_1}, \end{tikzcd}$$

where the left vertical arrow represents the canonical action of $\mathrm{Spin}(8)$ *on* $\mathbb{R}^8$.

If we fix $v_1 \in \mathfrak{m}_{\lambda_1} \setminus \{0\}$, *then the Lie subgroup* $\mathrm{Spin}' := \{B \in \mathrm{Spin}(8) \mid B(\varphi^{-1}v_1) = \varphi^{-1}v_1\}$ *of* $\mathrm{Spin}(8)$ *is isomorphic to* $\mathrm{Spin}(7)$, *and the subgroup* $K_0' := \Phi(\mathrm{Spin}')$, *which is isomorphic to* $\mathrm{Spin}(7)$, *acts transitively on* $\mathfrak{m}_{\lambda_2}$.

If we now also fix $v_2 \in \mathfrak{m}_{\lambda_2} \setminus \{0\}$, *then the Lie subgroup* $\mathrm{Spin}'' := \{B \in \mathrm{Spin}' \mid B(\varphi^{-1}v_2) = \varphi^{-1}v_2\}$ *of* Spin' *is isomorphic to the exceptional Lie group* G_2, *and hence* $\Phi(\mathrm{Spin}'')$ *is also isomorphic to* G_2.

These statements are also true for an arbitrary permutation of the indices 1, 2, 3 *of the root spaces* $\mathfrak{m}_{\lambda_k}$.

Proof. Most statements follow from the preceding discussion of the isotropy action. For the transitivity statements, see [1], Lemma 14.13, p. 100. □

4.2. Lie triple systems in E_6/F_4.

Theorem 4.2. *The linear subspaces* $\mathfrak{m}' \subset \mathfrak{m}$ *given in the following are Lie triple systems, and every Lie triple system* $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{m}$ *is congruent under the isotropy action to one of them.*

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/3]$.
  $\mathfrak{m}' = \mathbb{R}(\cos(t)((\lambda_1^\sharp + \lambda_3^\sharp)/\sqrt{3}) + \sin(t)\lambda_2^\sharp)$ (*compare* Equation (30)).
- ($\mathbb{S}$, $\varphi = \pi/6$, l) *with* $2 \leq l \leq 9$.
  $\mathfrak{m}'$ *is an* l*-dimensional linear subspace of* $\mathbb{R}\lambda_1^\sharp \oplus \mathfrak{m}_{\lambda_1}$.
- ($\mathbb{P}$, $\varphi = \pi/6$, $(\mathbb{K}, l)$) *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ *and* $l \in \{2, 3\}$, *or with* $(\mathbb{K}, l) = (\mathbb{O}, 2)$.
  We define the following vectors:

$$v_0 := M_{\lambda_1}(1, 0, 0, 0) + M_{\lambda_2}(1, 0, 0, 0), \quad v_1 := M_{\lambda_1}(i, 0, 0, 0) + M_{\lambda_2}(-i, 0, 0, 0),$$

$$v_0^C := M_{\lambda_1}(0, 0, 0, i) + M_{\lambda_2}(0, 0, 0, -i), \quad v_1^C := M_{\lambda_1}(0, 0, 0, 1) + M_{\lambda_2}(0, 0, 0, 1),$$

$$v_0^H := M_{\lambda_1}(0, 0, i, 0) + M_{\lambda_2}(0, 0, -i, 0), \quad v_1^H := M_{\lambda_1}(0, 0, 1, 0) + M_{\lambda_2}(0, 0, 1, 0),$$

$$v_0^{CH} := M_{\lambda_1}(0, 1, 0, 0) + M_{\lambda_2}(0, 1, 0, 0), \quad v_1^{CH} := M_{\lambda_1}(0, -i, 0, 0) + M_{\lambda_2}(0, i, 0, 0),$$

$$v_0^O := M_{\lambda_1}(i, 0, 0, 0) + M_{\lambda_2}(i, 0, 0, 0),$$

$$v_0^{CO} := M_{\lambda_1}(0, 0, 0, 1) + M_{\lambda_2}(0, 0, 0, -1),$$

$$v_0^{HO} := M_{\lambda_1}(0, 0, 1, 0) + M_{\lambda_2}(0, 0, -1, 0),$$

$$v_0^{CHO} := M_{\lambda_1}(0, -i, 0, 0) + M_{\lambda_2}(0, -i, 0, 0),$$

$$H := \lambda_3^\sharp, \quad w_4 := M_{\lambda_3}(0, 0, 0, 1),$$

$$w_1 := M_{\lambda_3}(1, 0, 0, 0), \quad w_5 := M_{\lambda_3}(-i, 0, 0, 0),$$

$$w_2 := M_{\lambda_3}(0, 1, 0, 0), \quad w_6 := M_{\lambda_3}(0, i, 0, 0),$$

$$w_3 := M_{\lambda_3}(0, 0, i, 0), \quad w_7 := M_{\lambda_3}(0, 0, 1, 0).$$

Then $\mathfrak{m}'$ *is spanned by the following vectors, in dependence of* $(\mathbb{K}, l)$:

For $(\mathbb{K}, l) = (\mathbb{R}, 2)$: H, v_0.
For $(\mathbb{K}, l) = (\mathbb{R}, 3)$: H, v_0, v_1.
For $(\mathbb{K}, l) = (\mathbb{C}, 2)$: H, v_0, v_0^C, w_1.
For $(\mathbb{K}, l) = (\mathbb{C}, 3)$: $H, v_0, v_0^C, v_1, v_1^C, w_1$.
For $(\mathbb{K}, l) = (\mathbb{H}, 2)$: $H, v_0, v_0^C, v_0^H, v_0^{CH}, w_1, w_2, w_3$.
For $(\mathbb{K}, l) = (\mathbb{H}, 3)$: $H, v_0, v_0^C, v_0^H, v_0^{CH}, v_1, v_1^C, v_1^H, v_1^{CH}, w_1, w_2, w_3$.
For $(\mathbb{K}, l) = (\mathbb{O}, 2)$: $H, v_0, v_0^C, v_0^H, v_0^{CH}, v_0^O, v_0^{CO}, v_0^{HO}, v_0^{CHO}, w_1, w_2, w_3, w_4, w_5, w_6, w_7$.

- (AI).
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{R}, 0, 0, 0) \oplus M_{\lambda_2}(\mathbb{R}, 0, 0, 0) \oplus M_{\lambda_3}(0, 0, 0, i\mathbb{R})$.
- (A_2).
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, 0, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, 0, 0, 0) \oplus M_{\lambda_3}(0, 0, 0, \mathbb{C})$.
- (AII).
  $\mathfrak{m}' = \mathfrak{a} \oplus M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0) \oplus M_{\lambda_3}(0, 0, \mathbb{C}, \mathbb{C})$.
- $(\mathbb{S} \times \mathbb{S}^1, l)$ *with* $l \leq 9$.
  $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1}$ *with an* $(l-1)$*-dimensional linear subspace* $\mathfrak{m}'_{\lambda_1} \subset \mathfrak{m}_{\lambda_1}$.

We call the full name (Geo, $\varphi = t$) *etc. given in the above table the type of the Lie triple systems which are congruent under the adjoint action to the space given in that entry. Then every Lie triple system of* $\mathfrak{m}$ *is of exactly one type.*

The Lie triple systems $\mathfrak{m}'$ *of the various types have the properties given in the following table. The column "isometry type" again gives the isometry type of the totally geodesic submanifolds corresponding to the Lie triple systems of the respective type in abbreviated form, for the details see* Section 4.3.

type of $\mathfrak{m}'$	$\dim(\mathfrak{m}')$	$\mathrm{rk}(\mathfrak{m}')$	$\mathfrak{m}'$ *maximal*	*isometry type*
(Geo, $\varphi = t$)	1	1	*no*	$\mathbb{R}$ *or* $\mathbb{S}^1$
$(\mathbb{S}, \varphi = \pi/6, l)$	l	1	*no*	$\mathbb{S}^l$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l))$	$\dim_{\mathbb{R}} \mathbb{K} \cdot l$	1	*for* $(\mathbb{K}, l) \in \{(\mathbb{H}, 3), (\mathbb{O}, 2)\}$	$\mathbb{K}\mathrm{P}^l$
(AI)	5	2	*no*	$(\mathrm{SU}(3)/\mathrm{SO}(3))/\mathbb{Z}_3$
(A_2)	8	2	*no*	$\mathrm{SU}(3)/\mathbb{Z}_3$
(AII)	14	2	*yes*	$(\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3$
$(\mathbb{S} \times \mathbb{S}^1, l)$	$l+1$	2	*for* $l = 9$	$(\mathbb{S}^l \times \mathbb{S}^1)/\mathbb{Z}_4$

REMARK 4.3. For the symmetric space EIV, Chen and Nagano correctly list the *local* isometry types of the maximal totally geodesic submanifolds. However, the global isometry types of the totally geodesic submanifolds of type (AII) resp. $(\mathbb{S} \times \mathbb{S}^1, 9)$ is $(\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3$ resp. $(\mathbb{S}^1 \times \mathbb{S}^9)/\mathbb{Z}_4$ (and not $\mathrm{SU}(6)/\mathrm{Sp}(3)$ resp. $\mathbb{S}^1 \times \mathbb{S}^9$, as Chen and Nagano claim).

Proof of Theorem 4.2. We first mention that it is easily checked using the Maple implementation that the spaces defined in the theorem, and therefore also the linear subspaces $\mathfrak{m}' \subset \mathfrak{m}$ which are congruent to one of them, are Lie triple systems. It is also easily seen that the information in the table concerning the dimension and the rank of the Lie triple systems is correct. The information on the isometry type of the corresponding totally geodesic submanifolds will be discussed in Section 4.3. Note that no two types of Lie triple systems correspond to the same isometry type of totally geodesic submanifold, therefore no Lie triple system can be of more than one type.

We next show that the information on the maximality of the Lie triple systems given in the table is correct. For this purpose, we presume that the list of Lie triple systems given in the theorem is in fact complete; this will be proved in the remainder of the present section.

Proof that the Lie triple systems which are claimed to be maximal in the table indeed are: This is clear for the type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{O}, 2))$, because it has the maximal dimension among all the Lie triple systems of EIV. It is also clear for the type (AII) because it has rank 2 and maximal dimension among all the Lie triple systems of EIV of that rank. For the type $(\mathbb{S} \times \mathbb{S}^1, 9)$: For reason of dimension and rank, a Lie triple system $\mathfrak{m}'$ of this type could only be contained in a Lie triple system of type (AII); however $\mathfrak{m}'$ has a root of multiplicity 8, whereas all the roots of Lie triple systems of type (AII) have multiplicity 4, so such an inclusion is impossible. For the type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{H}, 3))$: For reason of dimension, a Lie triple system $\mathfrak{m}'$ of this type could again only be contained in a Lie triple system of type (AII); however this is impossible because $\mathfrak{m}'$ requires the multiplicity 8 for the "collapsing" roots λ_1 and λ_2.

That no Lie triple systems are maximal besides those mentioned above follows from the following table:

Every Lie triple system of type...	is contained in a Lie triple system of type...
$(\mathrm{Geo}, \varphi = t)$	$(\mathbb{S} \times \mathbb{S}^1, 1)$
$(\mathbb{S}, \varphi = \pi/6, l)$	$(\mathbb{S} \times \mathbb{S}^1, l)$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, 2))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$	$(\mathbb{P}, \varphi = \pi/6, (\mathbb{O}, 2))$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, 3))$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$	$(\mathbb{P}, \varphi = \pi/6, (\mathbb{H}, 3))$
(AI)	(A_2)
(A_2)	(AII)
$(\mathbb{S} \times \mathbb{S}^1, l)$ with $l \leq 8$	$(\mathbb{S} \times \mathbb{S}^1, 9)$

We now turn to the proof that the list of Lie triple systems of EIV given in Theorem 4.2 is indeed complete. For this purpose, we let an arbitrary Lie triple system $\mathfrak{m}'$ of $\mathfrak{m}$, $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{m}$, be given. Because the symmetric space EIV is of rank 2, the rank of $\mathfrak{m}'$ is either 1 or 2. We will handle these two cases separately in the sequel.

We first suppose that $\mathfrak{m}'$ is a Lie triple system of rank 2. Let us fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}'$; because of $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{m})$, $\mathfrak{a}$ is then also a Cartan subalgebra of $\mathfrak{m}$. In relation to this situation, we use the notations introduced in Sections 2 and 4.1. In particular, we consider the positive root system $\Delta_+ := \{\lambda_1, \lambda_2, \lambda_3\}$ of the root system $\Delta := \Delta(\mathfrak{m}, \mathfrak{a})$ of $\mathfrak{m}$, and also the root system $\Delta' := \Delta(\mathfrak{m}', \mathfrak{a})$ of $\mathfrak{m}'$. By Proposition 2.1 (b), Δ' is a root subsystem of Δ, and therefore $\Delta'_+ := \Delta' \cap \Delta_+$ is a positive system of roots for Δ'. Moreover, in the root space decompositions of $\mathfrak{m}$ and $\mathfrak{m}'$

(32) $$\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta_+} \mathfrak{m}_\lambda \quad \text{and} \quad \mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta'_+} \mathfrak{m}'_\lambda$$

the root space $\mathfrak{m}'_\lambda$ of $\mathfrak{m}'$ with respect to $\lambda \in \Delta'_+$ is related to the corresponding root space $\mathfrak{m}_\lambda$ of $\mathfrak{m}$ by $\mathfrak{m}'_\lambda = \mathfrak{m}_\lambda \cap \mathfrak{m}'$.

Because the subset Δ' of Δ is invariant under its own Weyl transformation group, we have (up to Weyl transformation) only the following possibilities for Δ'_+, which we will treat individually in the sequel:

$$\Delta'_+ = \Delta_+, \quad \Delta'_+ = \{\lambda_1\} \quad \text{and} \quad \Delta'_+ = \varnothing.$$

CASE $\Delta'_+ = \Delta_+$. In this case, the restricted Dynkin diagram with multiplicities of $\mathfrak{m}'$ is $\bullet^{n'_{\lambda_1}} — \bullet^{n'_{\lambda_2}}$, and the classification of the Riemannian symmetric spaces (see, for example, [14], p. 119, 146) shows that $n' := n'_{\lambda_1} = n'_{\lambda_2} = n'_{\lambda_3} \in \{1, 2, 4, 8\}$ holds.

If $n' = 1$ holds, we may suppose without loss of generality by Proposition 4.1 that $\mathfrak{m}'_{\lambda_k}$ is spanned by $v_k := M_{\lambda_k}(1, 0, 0, 0)$ for $k \in \{1, 2\}$. Then we have $\mathfrak{m}' \ni R(\lambda_1^\sharp, v_1)v_2 = (\sqrt{2}/8)M_{\lambda_3}(0, 0, 0, i)$, and therefore $\mathfrak{m}'_{\lambda_3}$ is spanned by $v_3 := M_{\lambda_3}(0, 0, 0, i)$. Thus $\mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{k=1}^3 \mathfrak{m}'_{\lambda_k}$ is of type (AI).

If $n' = 2$ holds, we may suppose without loss of generality $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, 0, 0, 0)$ and $v_2 \in \mathfrak{m}'_{\lambda_2}$. We then obtain $v_3 \in \mathfrak{m}'_{\lambda_3}$ as before, also from the equality $\mathfrak{m}' \ni R(\lambda_1^\sharp, M_{\lambda_1}(i, 0, 0, 0))v_2 = -(\sqrt{2}/8)M_{\lambda_3}(0, 0, 0, 1)$ the fact $M_{\lambda_3}(0, 0, 0, 1) \in \mathfrak{m}'_{\lambda_3}$ and then from the equality $\mathfrak{m}' \ni R(\lambda_1^\sharp, v_1)M_{\lambda_3}(0, 0, 0, 1) = -(\sqrt{2}/8)M_{\lambda_2}(i, 0, 0, 0)$ the fact $M_{\lambda_2}(i, 0, 0, 0) \in \mathfrak{m}'_{\lambda_2}$. Thus we have besides $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, 0, 0, 0)$ also $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, 0, 0, 0)$ and $\mathfrak{m}'_{\lambda_3} = M_{\lambda_3}(0, 0, 0, \mathbb{C})$, and therefore $\mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{k=1}^3 \mathfrak{m}'_{\lambda_k}$ is of type (A_2).

If $n' = 4$ holds, we may suppose without loss of generality $\mathfrak{m}'_{\lambda_1} = M_{\lambda_1}(\mathbb{C}, \mathbb{C}, 0, 0)$ and $v_2 \in \mathfrak{m}'_{\lambda_2}$. Then as above we obtain $M_{\lambda_2}(\mathbb{C}, 0, 0, 0) \subset \mathfrak{m}'_{\lambda_2}$ and $M'_{\lambda_3}(0, 0, 0, \mathbb{C}) \subset \mathfrak{m}'_{\lambda_3}$. Moreover for $c \in \mathbb{C}$ we have

$$\mathfrak{m}' \ni R(\lambda_1^\sharp, M_{\lambda_1}(0, c, 0, 0))v_2 = \frac{\sqrt{2}}{8} M_{\lambda_3}(0, 0, \bar{c}i, 0),$$

hence $\mathfrak{m}'_{\lambda_3} = M_{\lambda_3}(0, 0, \mathbb{C}, \mathbb{C})$, and

$$\mathfrak{m}' \ni R(\lambda_1^\sharp, v_1)M_{\lambda_3}(0, 0, c, 0) = \frac{\sqrt{2}}{8} M_{\lambda_2}(0, \bar{c}i, 0, 0),$$

hence $\mathfrak{m}'_{\lambda_2} = M_{\lambda_2}(\mathbb{C}, \mathbb{C}, 0, 0)$. This shows that $\mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{k=1}^3 \mathfrak{m}'_{\lambda_k}$ is of type (AII).

Finally, if $n' = 8$ holds, we have $\mathfrak{m}'_{\lambda_k} = \mathfrak{m}_{\lambda_k}$ for $k \in \{1, 2, 3\}$ and therefore $\mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{k=1}^3 \mathfrak{m}'_{\lambda_k} = \mathfrak{m}$.

CASE $\Delta'_+ = \{\lambda_1\}$. In this case we have $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1}$ with a linear subspace $\mathfrak{m}'_{\lambda_1} \subset \mathfrak{m}_{\lambda_1}$, and therefore $\mathfrak{m}'$ is of type $(\mathbb{S} \times \mathbb{S}^1, l)$ with $l := 1 + n'_{\lambda_1} \leq 9$.

CASE $\Delta'_+ = \varnothing$. In this case we have $\mathfrak{m}' = \mathfrak{a}$, and therefore $\mathfrak{m}'$ is of type $(\mathbb{S} \times \mathbb{S}^1, 1)$.

We now turn our attention to the case where $\mathfrak{m}'$ is a Lie triple system of rank 1. Via the application of the isotropy action of EIV, we may suppose without loss of generality that $\mathfrak{m}'$ contains a unit vector H from the closure $\bar{\mathfrak{c}}$ of the positive Weyl chamber $\mathfrak{c}$ of $\mathfrak{m}$ (with respect to $\mathfrak{a}$ and our choice of positive roots). Then we have by Equations (30) and (31) with $\varphi_0 := \varphi(H) \in [0, \pi/3]$

$$H = \cos(\varphi_0)\frac{\lambda_1^\sharp + \lambda_3^\sharp}{\sqrt{3}} + \sin(\varphi_0)\lambda_2^\sharp. \tag{33}$$

Because of $\operatorname{rk}(\mathfrak{m}') = 1$, $\mathfrak{a}' := \mathbb{R}H$ is a Cartan subalgebra of $\mathfrak{m}'$, and we have $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$. It follows from Proposition 2.1 (a) that the root systems Δ' and Δ of $\mathfrak{m}'$ resp. $\mathfrak{m}$ with respect to $\mathfrak{a}'$ resp. to $\mathfrak{a}$ are related by

$$\Delta' \subset \{\lambda(H)\alpha_0 \mid \lambda \in \Delta,\ \lambda(H) \neq 0\} \tag{34}$$

with the linear form $\alpha_0 \colon \mathfrak{a}' \to \mathbb{R}$, $tH \mapsto t$; moreover for $\mathfrak{m}'$ we have the root space decomposition

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha \tag{35}$$

where for any root $\alpha \in \Delta'$, the corresponding root space $\mathfrak{m}'_\alpha$ is given by

$$\mathfrak{m}'_\alpha = \left(\bigoplus_{\substack{\lambda \in \Delta \\ \lambda(H) = \alpha(H)}} \mathfrak{m}_\lambda \right) \cap \mathfrak{m}'. \tag{36}$$

If $\Delta' = \varnothing$ holds, then we have $\mathfrak{m}' = \mathbb{R}H$, and therefore $\mathfrak{m}'$ is then of type $(\mathrm{Geo}, \varphi = \varphi_0)$. Otherwise it follows from Proposition 2.3 that one of the following two conditions holds: Either H is proportional to a root vector $\lambda^\sharp$ with $\lambda \in \Delta$, or there exist two $\lambda, \mu \in \Delta$ $(\lambda \neq \mu)$ so that H is orthogonal to $\lambda^\sharp - \mu^\sharp$. Evaluating all possible values for λ and μ, we see that $\varphi_0 \in \{0, \pi/6, \pi/3\}$ holds.

In the sequel we consider the three possible values for φ_0 individually.

CASE $\varphi_0 = 0$. In this case we have $H = (1/\sqrt{3})(\lambda_1^\sharp + \lambda_3^\sharp) = (1/\sqrt{3})(2\lambda_1^\sharp + \lambda_2^\sharp)$ by Equation (33) and therefore

$$\lambda_1(H) = \frac{1}{2}\sqrt{3}, \quad \lambda_2(H) = 0, \quad \lambda_3(H) = \frac{1}{2}\sqrt{3}.$$

Thus we have $\Delta' = \{\pm\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_3|\mathfrak{a}'$ by Equation (34), $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ by Equation (35), and $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_3}$ by Equation (36).

Assume that $\mathfrak{m}'_\alpha \neq \{0\}$ holds. We have $\alpha^\sharp = (1/2)\sqrt{3}H = (1/2)\lambda_1^\sharp + (1/2)\lambda_3^\sharp$ and therefore by Proposition 2.4, for any $v \in \mathfrak{m}'_\alpha$, say $v = M_{\lambda_1}(a_1, \dots, a_4) + M_{\lambda_3}(b_1, \dots, b_4)$ with $a_1, \dots, a_4, b_1, \dots, b_4 \in \mathbb{C}$, we have $\|a\| = \|b\|$. Therefore we can suppose without loss of generality via Proposition 4.1 that $v_0 := M_{\lambda_1}(1, 0, 0, 0) + M_{\lambda_3}(1, 0, 0, 0) \in \mathfrak{m}'_\alpha$ holds. Then we have

$$\mathfrak{m}' \ni R(H, v_0)v_0 = \frac{3}{4}H + \frac{\sqrt{6}}{8}M_{\lambda_2}(0, 0, 0, i).$$

However, this is a contradiction to the fact that because of $\lambda_2(H) = 0$, no element of $\mathfrak{m}'$ can have a non-zero $\mathfrak{m}_{\lambda_2}$-component. So we in fact have $\mathfrak{m}'_\alpha = \{0\}$, hence $\mathfrak{m}' = \mathbb{R}H$. This shows that (for $\dim(\mathfrak{m}') \geq 2$) the case $\varphi_0 = 0$ cannot in fact occur.

CASE $\varphi_0 = \pi/6$. In this case we have $H = (\sqrt{3}/2)\cdot(1/\sqrt{3})(\lambda_1^\sharp + \lambda_3^\sharp) + (1/2)\lambda_2^\sharp = \lambda_3^\sharp$ by Equation (33) and therefore

$$\lambda_1(H) = \frac{1}{2}, \quad \lambda_2(H) = \frac{1}{2}, \quad \lambda_3(H) = 1.$$

Thus we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_2|\mathfrak{a}'$ by Equation (34), $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ by Equation (35), and $\mathfrak{m}'_\alpha \subset \mathfrak{m}_{\lambda_1} \oplus \mathfrak{m}_{\lambda_2}$ and $\mathfrak{m}'_{2\alpha} \subset \mathfrak{m}_{\lambda_3}$ by Equation (36).

If $\alpha \notin \Delta'$ holds, we thus have $\mathfrak{m}' = \mathbb{R}\lambda_3^\sharp \oplus \mathfrak{m}'_{2\alpha} \subset \mathbb{R}\lambda_3^\sharp \oplus \mathfrak{m}_{\lambda_3}$, and therefore $\mathfrak{m}'$ then is of type $(\mathbb{S}, \varphi = \pi/6, l)$ with $l := 1 + n'_{2\alpha}$.

So we now suppose $\alpha \in \Delta'$. By the classification of the Riemannian symmetric spaces of rank 1 we then have $n'_{2\alpha} \in \{0, 1, 3, 7\}$, and the totally geodesic submanifold corresponding to $\mathfrak{m}'$ is isometric either to $\mathbb{R}\mathrm{P}^k$, to $\mathbb{C}\mathrm{P}^k$, to $\mathbb{H}\mathrm{P}^k$ or to the Cayley projective plane $\mathbb{O}\mathrm{P}^{k=2}$, depending on whether $n'_{2\alpha}$ equals 0, 1, 3 or 7, respectively; here we have $k = n'_\alpha/(n'_{2\alpha} + 1)$.

It should also be noted that we have $\alpha^\sharp = (1/2)H = (1/2)\lambda_1^\sharp + (1/2)\lambda_2^\sharp$, and therefore we have for any $c_1, \dots, c_4, d_1, \dots, d_4 \in \mathbb{C}$ by Proposition 2.4

$$M_{\lambda_1}(c_1, \dots, c_4) + M_{\lambda_2}(d_1, \dots, d_4) \in \mathfrak{m}'_\alpha \Longrightarrow \|c\| = \|d\|. \tag{37}$$

In the sequel, we consider the four possible values for $n'_{2\alpha}$ individually. In our calculations we will use the vectors $v_0, v_0^C, \dots$ as they are defined in the entry for the types $(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l))$ in Theorem 4.2.

Let us first suppose $n'_{2\alpha} = 0$, i.e. $\Delta' = \{\pm\alpha\}$. By Proposition 4.1 and because of (37) we may suppose without loss of generality that $v_0 \in \mathfrak{m}'_\alpha$ holds. If $n'_\alpha = 1$ holds, we then have $\mathfrak{m}'_\alpha = \mathbb{R}v_0$ and therefore $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 2))$. Otherwise we choose $v \in \mathfrak{m}'_\alpha$ to be orthogonal to v_0, say $v = M_{\lambda_1}(c_1, \dots, c_4) + M_{\lambda_2}(d_1, \dots, d_4)$. Then we have

$$\begin{aligned}\mathfrak{m}' \ni R(H, v_0)v = {}& \frac{1}{4}\operatorname{Re}(c_1)\lambda_1^\sharp + \frac{1}{4}\operatorname{Re}(d_1)\lambda_2^\sharp \\ & + \frac{\sqrt{2}}{16}M_{\lambda_3}(i(\overline{c_4} - \overline{d_4}), i(-\overline{c_3} + \overline{d_3}), -i(\overline{c_2} + \overline{d_2}), i(\overline{d_1} - c_1)).\end{aligned}$$

Because the $\mathfrak{a}$-component of this vector is proportional to H, we have $\mathrm{Re}(c_1) = \mathrm{Re}(d_1)$; this equation together with our requirement that v be orthogonal to v_0 shows $\mathrm{Re}(c_1) = \mathrm{Re}(d_1) = 0$ and hence $c_1, d_1 \in i\mathbb{R}$. Moreover, because of $2\alpha \notin \Delta'$, the $\mathfrak{m}_{\lambda_3}$-component of the above vector vanishes, and thus we have

$$c_1 = -d_1 \in i\mathbb{R}, \quad c_2 = -d_2, \quad c_3 = d_3 \quad \text{and} \quad c_4 = d_4,$$

hence

$$v = M_{\lambda_1}(it, c_2, c_3, c_4) + M_{\lambda_2}(-it, -c_2, c_3, c_4) \tag{38}$$

with $t \in \mathbb{R}$. By application of another isotropy transformation, we can now arrange that v is proportional to v_1. Thus we have $v_0, v_1 \in \mathfrak{m}'_\alpha$, and therefore in the case $n'_\alpha = 2$, $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3))$. We now show that the case $n'_\alpha \geq 3$ does not occur. For this purpose, we again let $v \in \mathfrak{m}'_\alpha$ be given, but now suppose that v is orthogonal to both v_0 and v_1. Then v again has the form of Equation (38), however the requirement that v be orthogonal to v_1 implies $t = 0$. Moreover, we have

$$\mathfrak{m}' \ni R(H, v)v_1 = \frac{\sqrt{2}}{16} M_{\lambda_3}(\overline{c_4}i, -\overline{c_3}i, -\overline{c_2}i, 0).$$

Because of $2\alpha \notin \Delta'$, the $\mathfrak{m}_{\lambda_3}$-component of this vector vanishes, and thus we have $c_2 = c_3 = c_4 = 0$, hence $v = 0$. This shows that $n'_\alpha \geq 3$ is impossible.

Next we suppose $n'_{2\alpha} = 1$. Then the Lie triple system $\mathfrak{m}'$ corresponds to a complex projective space $\mathbb{C}\mathrm{P}^l$, which is a Hermitian symmetric space. Let $\mathfrak{m}'' \subset \mathfrak{m}'$ be the tangent space of a real form of this space, then $\mathfrak{m}''$ will also be a Lie triple system of $\mathfrak{m}$, it will be of rank 1 and correspond to the isotropy angle $\varphi = \pi/6$, and it will have only the root α, not 2α. As a consequence of the preceding classification of the Lie triple systems with these properties, $\mathfrak{m}''$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l))$ with $l \in \{2, 3\}$. Without loss of generality, we may therefore suppose that $\mathfrak{m}''$ is the prototype Lie triple system of the type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l))$ as given in Theorem 4.2. Thus we have $v_0 \in \mathfrak{m}'_\alpha$ and in the case $l = 3$ also $v_1 \in \mathfrak{m}'_\alpha$. Further we may suppose without loss of generality $\mathfrak{m}'_{2\alpha} = \mathbb{R}w_1$. Then we have

$$R(w_1, v_k)H = \frac{\sqrt{2}}{16} v_k^C$$

for $k \in \{0, 1\}$, and therefore $v_k \in \mathfrak{m}'_\alpha$ implies also $v_k^C \in \mathfrak{m}'_\alpha$. This shows that $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, l))$.

Now we suppose $n'_{2\alpha} = 3$. Then $\mathfrak{m}'$ corresponds to a quaternionic projective space $\mathbb{H}\mathrm{P}^l$, and therefore an analogous argument as in the treatment of the case $n'_{2\alpha} = 1$ shows that $\mathfrak{m}'$ contains as a complex form a Lie triple system $\mathfrak{m}''$ of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, l))$ with $l \in \{2, 3\}$. Without loss of generality, we may suppose that $\mathfrak{m}''$ is the prototype Lie triple system of that type as given in Theorem 4.2, and therefore

$w_1 \in \mathfrak{m}'_{2\alpha}$ and $v_k, v_k^C \in \mathfrak{m}'_\alpha$ holds, where $k = 0$ for $l = 2$ and $k = 0, 1$ for $l = 3$. Further we may suppose without loss of generality that also $w_2 \in \mathfrak{m}'_{2\alpha}$ holds. We have for $k \in \{0, 1\}$

$$R(w_2, v_k)H = -\frac{\sqrt{2}}{16} v_k^H \quad \text{and} \quad R(w_2, v_k^C)H = -\frac{\sqrt{2}}{16} v_k^{CH}.$$

Therefore $v_k, v_k^C \in \mathfrak{m}'_\alpha$ implies also $v_k^H, v_k^{CH} \in \mathfrak{m}'_\alpha$ Moreover, we have

$$R(v_0, v_0^{CH})H = \frac{\sqrt{2}}{4} w_3,$$

and therefore $w_3 \in \mathfrak{m}'_{2\alpha}$. Thus $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{H}, l))$.

Finally we suppose $n'_{2\alpha} = 7$. Then $n'_\alpha = 8$ is the only possibility by the classification of the Riemannian symmetric spaces of rank 1, and $\mathfrak{m}'$ corresponds to the Cayley projective plane $\mathbb{O}\mathrm{P}^2$. $\mathbb{O}\mathrm{P}^2$ contains a $\mathbb{H}\mathrm{P}^2$ as totally geodesic submanifold, and thus by an analogous argument as before, we see that $\mathfrak{m}'$ contains a Lie triple system $\mathfrak{m}''$ of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{H}, 2))$; without loss of generality we may suppose that $\mathfrak{m}''$ is the prototype Lie triple system of that type. Thus we have $v_0, v_0^C, v_0^H, v_0^{CH} \in \mathfrak{m}'_\alpha$ and $w_1, w_2, w_3 \in \mathfrak{m}'_{2\alpha}$. Without loss of generality we may further suppose $w_4 \in \mathfrak{m}'_{2\alpha}$. We have

$$R(w_4, v_0)H = \frac{\sqrt{2}}{16} v_0^O, \quad R(w_4, v_0^C)H = \frac{\sqrt{2}}{16} v_0^{CO},$$
$$R(w_4, v_0^H)H = \frac{\sqrt{2}}{16} v_0^{HO} \quad \text{and} \quad R(w_4, v_0^{CH})H = \frac{\sqrt{2}}{16} v_0^{CHO},$$

and therefore $\mathfrak{m}'_\alpha$ is spanned by $v_0, v_0^C, v_0^H, v_0^{CH}, v_0^O, v_0^{CO}, v_0^{HO}, v_0^{CHO}$. Moreover, we have

$$R(v_0, v_0^{CO})H = \frac{\sqrt{2}}{4} w_5, \quad R(v_0, v_0^{HO})H = \frac{\sqrt{2}}{4} w_6$$

and

$$R(v_0, v_0^{CHO})H = \frac{\sqrt{2}}{4} w_7$$

and therefore $\mathfrak{m}'_{2\alpha}$ is spanned by $w_1, \ldots, w_7$. Therefore $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{O}, 2))$.

CASE $\varphi_0 = \pi/3$. By an analogous argument as in the case $\varphi_0 = 0$, one shows that this case cannot occur.

This completes the classification of the Lie triple systems in the Riemannian symmetric space EIV. □

4.3. Totally geodesic submanifolds in E_6/F_4. We are interested in determining the global isometry types of the totally geodesic submanifolds of EIV corresponding to the various types of Lie triple systems as they were classified in Theorem 4.2. In the case of EIV all the maximal totally geodesic submanifolds are reflective, so we can derive this information from the classification of reflective submanifolds due to Leung, see [13].

Using the information from that paper, we obtain the results of the following table. In it, we again use the notations introduced at the beginning of Section 3.3.

type of Lie triple system	isometry type	properties[7]
(Geo, $\varphi = t$)	$\mathbb{R}$ or $\mathbb{S}^1$	
($\mathbb{S}$, $\varphi = \pi/6$, l)	$\mathbb{S}^l_{r=1}$	$l = 9$: Helgason sphere
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, l))	$\mathbb{K}\mathrm{P}^l_{\varkappa=1/4}$	$(\mathbb{K}, l) = (\mathbb{O}, 2)$: polar, maximal $(\mathbb{K}, l) = (\mathbb{H}, 3)$: reflective, maximal
(AI)	$((\mathrm{SU}(3)/\mathrm{SO}(3))/\mathbb{Z}_3)_{\mathrm{srr}=1}$	
(A_2)	$(\mathrm{SU}(3)/\mathbb{Z}_3)_{\mathrm{srr}=1}$	
(AII)	$((\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3)_{\mathrm{srr}=1}$	reflective, maximal
($\mathbb{S}\times\mathbb{S}^1$, l)	$\left(\mathbb{S}^l_{r=1}\times\mathbb{S}^1_{r=\sqrt{3}}\right)/\mathbb{Z}_4$	$l = 9$: meridian for ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{O}$, 2)), maximal

4.4. Totally geodesic submanifolds in SU(6)/Sp(3). Similarly as we derived the classification of the Lie triple systems resp. the totally geodesic submanifolds in SO(10)/U(5) from that classification in EIII in Section 3.5, we now derive the classification for SU(6)/Sp(3) from the classification in EIV, using the fact that SU(6)/Sp(3) is the local isometry type of a maximal totally geodesic submanifold of EIV.

Thus we remain in the situation studied in Section 4.2. We consider the Riemannian symmetric space EIV, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g} = \mathfrak{e}_6$ associated with this space, i.e. we have $\mathfrak{k} = \mathfrak{f}_4$ and $\mathfrak{m}$ is isomorphic to the tangent space of EIV. We will use the names for the types of Lie triple systems of $\mathfrak{m}$ as introduced in Theorem 4.2.

Further, we let $\mathfrak{m}_1$ be a Lie triple system of $\mathfrak{m}$ of type (AII), i.e. $\mathfrak{m}_1$ corresponds to a totally geodesic submanifold which is locally isometric to SU(6)/Sp(3).

Theorem 4.4. *Exactly the following types of Lie triple systems of* EIV *have representatives which are contained in* $\mathfrak{m}_1$:

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/3]$,
- ($\mathbb{S}$, $\varphi = \pi/6$, l) *with* $l \leq 5$,
- ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, 2)) *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$,
- ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, 3)) *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,
- (AI),
- (A_2),
- ($\mathbb{S} \times \mathbb{S}^1$, l) *with* $l \leq 5$.

The maximal Lie triple systems of $\mathfrak{m}_1$ *are those of the types*: ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{H}$, 2)), ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{C}$, 3)), (A_2) *and* ($\mathbb{S} \times \mathbb{S}^1$, 5).

[7]The polars and meridians are also reflective, without this fact being noted explicitly in the table.

Proof. Similar to the proofs of Theorems 3.8 and 3.10. $\square$

REMARK 4.5. Chen/Nagano incorrectly state in [5] that the Lie triple systems of type (AI) (corresponding to $SU(3)/SO(3)$) were maximal in $SU(6)/Sp(3)$, rather these Lie triple systems are contained in Lie triple systems of type (A_2) (corresponding to $SU(3)$).

Also for $SU(6)/Sp(3)$, the maximal totally geodesic submanifolds are all reflective. Using the information from [13], we obtain the following information on the global isometry type of the totally geodesic submanifolds of $SU(6)/Sp(3)$ corresponding to the various types of Lie triple systems:

type of Lie triple system	isometry type	properties[8]
$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
$(\mathbb{S}, \varphi = \pi/6, l)$	$\mathbb{S}^l_{r=1}$	$l = 5$: Helgason sphere
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l))$	$\mathbb{K}\mathrm{P}^l_{\varkappa=1/4}$	$(\mathbb{K}, l) = (\mathbb{H}, 2)$: polar, maximal $(\mathbb{K}, l) = (\mathbb{C}, 3)$: reflective, maximal
(AI)	$(SU(3)/SO(3))_{\mathrm{srr}=1}$	
(A_2)	$SU(3)_{\mathrm{srr}=1}$	reflective
$(\mathbb{S} \times \mathbb{S}^1, l)$	$\left(\mathbb{S}^l_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}}\right)/\mathbb{Z}_2$	$l = 5$: meridian for $(\mathbb{P}, \varphi = \pi/6, (\mathbb{H}, 2))$, maximal

4.5. Totally geodesic submanifolds in SU(3). Using the same strategy as before, we next classify the totally geodesic submanifolds of $SU(3)$, regarded as a Riemannian symmetric space. We again let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the splitting corresponding to EIV, and let $\mathfrak{m}_1$ now be a Lie triple system of $\mathfrak{m}$ of type (A_2); then the totally geodesic submanifold of EIV corresponding to $\mathfrak{m}_1$ is locally isometric to $SU(3)$.

Theorem 4.6. *Exactly the following types of Lie triple systems of* EIV *have representatives which are contained in* $\mathfrak{m}_1$:
- $(\mathrm{Geo}, \varphi = t)$ *with* $t \in [0, \pi/3]$,
- $(\mathbb{S}, \varphi = \pi/6, l)$ *with* $l \leq 3$,
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, 2))$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3))$,
- (AI),
- $(\mathbb{S} \times \mathbb{S}^1, l)$ *with* $l \leq 3$.

The maximal Lie triple systems of $\mathfrak{m}_1$ *are those of the types*: $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2))$, $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3))$, (AI) *and* $(\mathbb{S} \times \mathbb{S}^1, 3)$.

Proof. Similar to the proofs of Theorems 3.8 and 3.10. $\square$

[8] The polars and meridians are also reflective, without this fact being noted explicitly in the table.

REMARK 4.7. Chen/Nagano incorrectly state in [5] that SU(3) contains totally geodesic submanifolds isometric to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SU}(3)/(\mathrm{SU}(2) \times \mathrm{SU}(2))$. This is impossible, because $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has the same rank as SU(3), but whereas the former group has two orthogonal roots, the latter has not.

Once again, also for the Riemannian symmetric space SU(3), all the maximal totally geodesic submanifolds are reflective. Using the classification of the reflective submanifolds by Leung (in [11], Theorem 3.3 for the group manifolds, see also [12]), we obtain the following information on the global isometry type of the totally geodesic submanifolds of SU(3) corresponding to the various types of Lie triple systems:

type of Lie triple system	isometry type	properties[9]
$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
$(\mathbb{S}, \varphi = \pi/6, l)$	$\mathbb{S}^l_{r=1}$	$l = 3$: Helgason sphere
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l))$	$\mathbb{K}\mathrm{P}^l_{\varkappa=1/4}$	$(\mathbb{K}, l) = (\mathbb{C}, 2)$: polar, maximal $(\mathbb{K}, l) = (\mathbb{R}, 3)$: reflective, maximal
(AI)	$(\mathrm{SU}(3)/\mathrm{SO}(3))_{\mathrm{srr}=1}$	reflective, maximal
$(\mathbb{S} \times \mathbb{S}^1, l)$	$\left(\mathbb{S}^l_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}}\right)/\mathbb{Z}_2$	$l = 3$: meridian for $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2))$, maximal

4.6. Totally geodesic submanifolds in SU(3)/SO(3). The totally geodesic submanifolds of SU(3)/SO(3) have already been classified in [9], Section 6. Because the totally geodesic submanifolds of EIV of type (AI) are locally isometric to SU(3)/SO(3), the Lie triple systems of SU(3)/SO(3) also occur as Lie triple systems of EIV. In the following table, we list the correspondence between the types of Lie triple systems of SU(3)/SO(3) as defined in [9], Proposition 6.1, and types of Lie triple systems of EIV as defined in Theorem 4.2 of the present paper. We also give the isometry type of the corresponding totally geodesic submanifolds, as it has been determined in [9], Section 6; for the application of this information it should be noted that there the metric of SU(3)/SO(3) has been normalized in such a way that the roots have length $\sqrt{2}$, whereas we now want to normalize the metric in such a way that the roots have length 1.

type ([9], Proposition 6.1)	type (Theorem 4.2)	isometry type	properties
(G)	$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
(T)	$(\mathbb{S} \times \mathbb{S}^1, 1)$	$\left(\mathbb{S}^1_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}}\right)/\mathbb{Z}_2$	
(S)	$(\mathbb{S}, \varphi = \pi/6, 2)$	$\mathbb{S}^2_{r=1}$	Helgason sphere
(M)	$(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 2))$	$\mathbb{R}\mathrm{P}^2_{\varkappa=1/4}$	polar, maximal
(P)	$(\mathbb{S} \times \mathbb{S}^1, 2)$	$\left(\mathbb{S}^2_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}}\right)/\mathbb{Z}_2$	meridian, maximal

[9] The polars and meridians are also reflective, without this fact being noted explicitly in the table.

5. The symmetric spaces G_2 and $G_2/\mathrm{SO}(4)$

5.1. The geometry of the Lie group G_2, regarded as a symmetric space. In this section we will study the exceptional compact Lie group G_2, regarded as a Riemannian symmetric space. In particular we need to obtain its curvature tensor. The usual way to do so would be to regard G_2 as the quotient space $(G_2 \times G_2)/\Delta(G_2)$, where $\Delta(G_2) := \{(g, g) \mid g \in G_2\}$ is the diagonal of the product $G_2 \times G_2$, and then to apply the method of [9] to that space to compute its curvature tensor.

However, we can reduce the effort involved in the calculations by noting that in that model of the symmetric space G_2, the space $\mathfrak{m}$ which corresponds to the tangent space at the origin, is given by $\mathfrak{m} = \{(X, -X) \mid X \in \mathfrak{g}_2\} \subset \mathfrak{g}_2 \oplus \mathfrak{g}_2$, and that for elements $(X, -X), (Y, -Y), (Z, -Z) \in \mathfrak{m}$, the curvature tensor is given by

$$-[[(X, -X), (Y, -Y)], (Z, -Z)] = -([[X, Y], Z], -[[X, Y], Z]).$$

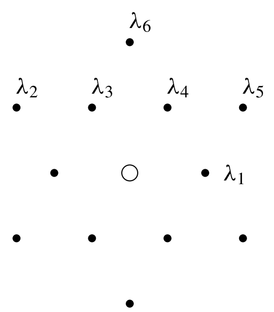
Under the canonical isomorphism $\mathfrak{m} \to \mathfrak{g}_2$, $(X, -X) \mapsto X$, the curvature tensor of these elements of $\mathfrak{m}$ therefore corresponds to $-[[X, Y], Z] \in \mathfrak{g}_2$, hence the Lie triple systems in $\mathfrak{m} \subset \mathfrak{g}_2 \oplus \mathfrak{g}_2$ correspond to the Lie triple systems in $\mathfrak{g}_2$ (i.e. to the linear subspaces of $\mathfrak{g}_2$ which are invariant under the Lie triple bracket $[[\,\cdot\,,\,\cdot\,],\,\cdot\,]$ of $\mathfrak{g}_2$). Moreover, the isotropy action of $\Delta(G_2)$ on $\mathfrak{m}$ corresponds to the adjoint action of G_2 on $\mathfrak{g}_2$. For this reason, we can carry out the classification of Lie triple systems by calculation in $\mathfrak{g}_2$ itself (instead of in $\mathfrak{m} \subset \mathfrak{g}_2 \oplus \mathfrak{g}_2$). In doing so, we will only need the description of the root system and the Lie bracket of $\mathfrak{g}_2$, which we obtain by application of the results of Sections 2, 3 of [9].

In the sequel, we will consider also an $\mathrm{Ad}(G_2)$-invariant inner product on $\mathfrak{g}_2$. Such an inner product is unique up to a positive constant, which we choose so that the shortest roots of $\mathfrak{g}_2$ (see below) have the length 1.

We now fix a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}_2$ and a choice of positive roots in the root system Δ of $\mathfrak{g}_2$ with respect to $\mathfrak{a}$. The Dynkin diagram of $\mathfrak{g}_2$ is $\bullet \Lleftarrow \bullet$, and therefore the simple roots of $\mathfrak{g}_2$, which we denote by λ_1 and λ_2, have an angle of $5\pi/6$ to each other, where λ_2 is the longer root by a factor of $\sqrt{3}$. The other positive roots of G_2 are

$$\lambda_3 := \lambda_1 + \lambda_2, \quad \lambda_4 := 2\lambda_1 + \lambda_2, \quad \lambda_5 := 3\lambda_1 + \lambda_2 \quad \text{and} \quad \lambda_6 := 3\lambda_1 + 2\lambda_2.$$

In this way we obtain the following root diagram for G_2:



In the sequel, we will use the notation $V_{\lambda_k}(c)$ defined as in Equation (7) for $k \in \{1, \dots, 6\}$ and $c \in \mathbb{C}$ to denote an element of the root space of $\mathfrak{g}_2$ corresponding to the root λ_k. Then the root space corresponding to λ_k equals $V_{\lambda_k}(\mathbb{C})$.

We will also use the isotropy angle function φ defined at the end of Section 2 for $\mathfrak{g}_2$; remember that in the present situation, the isotropy action of the symmetric space G_2 is given simply by the adjoint action of G_2 on $\mathfrak{g}_2$. We have $\varphi_{\max} = \pi/6$ and thus we obtain an isotropy angle function $\varphi\colon \mathfrak{g}_2 \setminus \{0\} \to [0, \pi/6]$. For the elements of the closure $\bar{\mathfrak{c}}$ of the positive Weyl chamber $\mathfrak{c} := \{v \in \mathfrak{a} \mid \lambda_1(v) \geq 0,\ \lambda_2(v) \geq 0\}$, we once again explicitly describe the relation to their isotropy angle: $(\lambda_4^\sharp, (1/\sqrt{3})\lambda_2^\sharp)$ is an orthonormal basis of $\mathfrak{a}$ so that with $v_t := \cos(t)\lambda_4^\sharp + \sin(t)(1/\sqrt{3})\lambda_2^\sharp$ we have

$$\bar{\mathfrak{c}} = \left\{ s \cdot v_t \mid t \in \left[0, \frac{\pi}{6}\right],\ s \in \mathbb{R}_{\geq 0} \right\}, \tag{39}$$

and because the Weyl chamber $\mathfrak{c}$ is bordered by the two vectors $v_0 = \lambda_4^\sharp$ with $\varphi(v_0) = 0$ and $v_{\pi/6} = (1/\sqrt{3})\lambda_6^\sharp$ with $\varphi(v_{\pi/6}) = \pi/6$, we have

$$\varphi(s \cdot v_t) = t \quad \text{for all} \quad t \in \left[0, \frac{\pi}{6}\right],\ s \in \mathbb{R}_+. \tag{40}$$

Further we note the following simple fact on the adjoint action of G_2:

Proposition 5.1. *Let λ be a short root, and λ' be a long root of G_2. Then the adjoint action of the maximal torus $T := \exp(\mathfrak{a})$ on $\mathfrak{g}_2$ leaves $\mathfrak{a}$ pointwise fixed, and acts "jointly transitively" on the unit spheres in the root spaces $V_\lambda(\mathbb{C})$ and $V_{\lambda'}(\mathbb{C})$ in the sense that for any given $c_1, c_2, c_1', c_2' \in \mathbb{C}$ with $|c_1| = |c_2|$ and $|c_1'| = |c_2'|$ there exists $g \in T$ with $\mathrm{Ad}(g)V_\lambda(c_1) = V_\lambda(c_2)$ and $\mathrm{Ad}(g)V_{\lambda'}(c_1') = V_{\lambda'}(c_2')$.*

5.2. Lie triple systems in G_2. We continue to use the notations of the preceding section.

Theorem 5.2. *The linear subspaces $\mathfrak{m}' \subset \mathfrak{g}_2$ given in the following are Lie triple systems, and every Lie triple system $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{g}_2$ is congruent under the adjoint action to one of them.*

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/6]$:
  $\mathfrak{m}' = \mathbb{R}(\cos(t)\lambda_4^\sharp + \sin(t)(1/\sqrt{3})\lambda_2^\sharp)$ (*see* Equation (40)).
- ($\mathbb{S}$, $\varphi = 0$, l) *with* $l \in \{2, 3\}$:
  $\mathfrak{m}'$ *is an l-dimensional linear subspace of* $\mathbb{R}\lambda_1^\sharp \oplus \mathfrak{m}_{\lambda_1}$.
- ($\mathbb{S}$, $\varphi = \arctan(1/3\sqrt{3})$, l) *with* $l \in \{2, 3\}$:
  $\mathfrak{m}'$ *is an l-dimensional subspace of* $\mathrm{span}\{9\lambda_1^\sharp + 5\lambda_2^\sharp, V_{\lambda_1}(1) + V_{\lambda_2}((1/3)\sqrt{5}), V_{\lambda_1}(i) + V_{\lambda_2}((1/3)\sqrt{5}i)\}$.
- ($\mathbb{S}$, $\varphi = \pi/6$, l) *with* $l \in \{2, 3\}$:
  $\mathfrak{m}'$ *is an l-dimensional linear subspace of* $\mathbb{R}\lambda_6^\sharp \oplus \mathfrak{m}_{\lambda_6}$.

- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l), \mathrm{G})$ *with* $l \in \{2, 3\}$:
  $\mathfrak{m}'$ *is an* l*-dimensional subspace of* $\mathrm{span}\{\lambda_6^\sharp, V_{\lambda_2}(1) + V_{\lambda_4}(\sqrt{3}), V_{\lambda_2}(i) - V_{\lambda_4}(\sqrt{3}i)\}$.
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l), \mathrm{A}_2)$ *with* $l \in \{2, 3\}$:
  $\mathfrak{m}'$ *is an* l*-dimensional subspace of* $\mathrm{span}\{\lambda_6^\sharp, V_{\lambda_2}(1) + V_{\lambda_5}(1), V_{\lambda_2}(i) + V_{\lambda_5}(i)\}$.
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$:
  $\mathfrak{m}' = \mathrm{span}\{\lambda_6^\sharp, V_{\lambda_2}(1) + V_{\lambda_5}(1), V_{\lambda_2}(i) - V_{\lambda_3}(\sqrt{3}i) - V_{\lambda_4}(\sqrt{3}i) + V_{\lambda_5}(i)\}$.
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{G})$:
  $\mathfrak{m}' = \mathrm{span}\{\lambda_6^\sharp, V_{\lambda_2}(1) + V_{\lambda_4}(\sqrt{3}), V_{\lambda_3}(\sqrt{3}i) + V_{\lambda_5}(i), V_{\lambda_6}(1)\}$.
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{A}_2)$:
  $\mathfrak{m}' = \mathrm{span}\{\lambda_6^\sharp, V_{\lambda_2}(1) + V_{\lambda_5}(1), V_{\lambda_2}(i) - V_{\lambda_5}(i), V_{\lambda_6}(1)\}$.
- $(\mathbb{S} \times \mathbb{S}, l, l')$ *with* $l, l' \le 3$:
  $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1} \oplus \mathfrak{m}'_{\lambda_6}$, *where* $\mathfrak{m}'_{\lambda_1} \subset V_{\lambda_1}(\mathbb{C})$ *and* $\mathfrak{m}'_{\lambda_6} \subset V_{\lambda_6}(\mathbb{C})$ *are linear subspaces of dimension* $l - 1$ *resp.* $l' - 1$.
- (AI):
  $\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_2}(\mathbb{R}) \oplus V_{\lambda_5}(\mathbb{R}) \oplus V_{\lambda_6}(i\mathbb{R})$.
- (A_2):
  $\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_2}(\mathbb{C}) \oplus V_{\lambda_5}(\mathbb{C}) \oplus V_{\lambda_6}(\mathbb{C})$.
- (G):
  $\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_1}(\mathbb{R}) \oplus V_{\lambda_2}(\mathbb{R}) \oplus V_{\lambda_3}(i\mathbb{R}) \oplus V_{\lambda_4}(\mathbb{R}) \oplus V_{\lambda_5}(i\mathbb{R}) \oplus V_{\lambda_6}(\mathbb{R})$.

We call the full name (Geo, $\varphi = t$) *etc. given in the above table the type of the Lie triple systems which are congruent under the adjoint action to the space given in that entry.*[10] *Then no Lie triple system is of more than one type.*

The Lie triple systems $\mathfrak{m}'$ *of the various types have the properties given in the following table. The column "isometry type" again gives the isometry type of the totally geodesic submanifolds corresponding to the Lie triple systems of the respective type in abbreviated form, for the details see* Section 5.3.

type of $\mathfrak{m}'$	$\dim(\mathfrak{m}')$	$\mathrm{rk}(\mathfrak{m}')$	$\mathfrak{m}'$ *Lie subalgebra*	$\mathfrak{m}'$ *maximal*	*isometry type*
(Geo, $\varphi = t$)	1	1	*yes*	*no*	$\mathbb{R}$ *or* $\mathbb{S}^1$
$(\mathbb{S}, \varphi = 0, l)$	l	1	*for* $l = 3$	*no*	$\mathbb{S}^2_{r=1}$
$(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), l)$	l	1	*for* $l = 3$	*for* $l = 3$	$\mathbb{S}^l_{r=(2/3)\sqrt{21}}$
$(\mathbb{S}, \varphi = \pi/6, l)$	l	1	*for* $l = 3$	*no*	$\mathbb{S}^l_{r=1/\sqrt{3}}$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l), \mathrm{G})$	l	1	*no*	*no*	$\mathbb{R}\mathrm{P}^l$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l), \mathrm{A}_2)$	l	1	*for* $l = 3$	*no*	$\mathbb{R}\mathrm{P}^l$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$	3	1	*no*	*yes*	$\mathbb{R}\mathrm{P}^3$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{G})$	4	1	*no*	*no*	$\mathbb{C}\mathrm{P}^2$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{A}_2)$	4	1	*no*	*no*	$\mathbb{C}\mathrm{P}^2$
$(\mathbb{S} \times \mathbb{S}, l, l')$	$l + l'$	2	*for* $l, l' \in \{1, 3\}$	*for* $l = l' = 3$	$(\mathbb{S}^l_{r=1} \times \mathbb{S}^{l'}_{r=1/\sqrt{3}})/\mathbb{Z}_2$
(AI)	5	2	*no*	*no*	$\mathrm{SU}(3)/\mathrm{SO}(3)$
(A_2)	8	2	*yes*	*yes*	$\mathrm{SU}(3)$
(G)	8	2	*no*	*yes*	$G_2/\mathrm{SO}(4)$

[10] Notice that in this case, the types $(\mathbb{S} \times \mathbb{S}, l, l')$ and $(\mathbb{S} \times \mathbb{S}, l', l)$ with $l \neq l'$ are not equivalent, because the two irreducible components of the Lie triple systems of this type correspond to spheres of different radius, see Section 5.3.

REMARK 5.3. The maximal totally geodesic submanifolds of G_2 of types ($\mathbb{S}$, $\varphi = \arctan(1/3\sqrt{3})$, 3) and ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), max), which are isometric to a 3-sphere of radius $(2/3)\sqrt{21}$ and a 3-dimensional real projective space of constant sectional curvature $3/4$ respectively, are missing from the classification by Chen and Nagano in Table VIII of [5]. The submanifolds of type ($\mathbb{S}$, $\varphi = \arctan(1/3\sqrt{3})$, 3) are once again in a "skew" position in the ambient manifold G_2 in the sense that their geodesic diameter $(2/3)\sqrt{21}\pi$ is strictly larger than the geodesic diameter $(2/3)\sqrt{3}\pi$ of G_2.

It should also be noted that with the three types of totally geodesic submanifolds ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), G), ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), A_2) and ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), max), and likewise with the two types ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{C}$, 2), G) and ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{C}$, 2), A_2), we have examples of totally geodesic submanifolds which are isometric to one another, but which are not congruent under the isometry action of the ambient space.

Proof of Theorem 5.2. Once again, it is easily checked that the spaces defined in the theorem are Lie triple systems, and thus the spaces which are conjugate to one of them under the adjoint action also are. It is also easily seen that the information in the table on the dimension and the rank of the Lie triple systems and regarding the question which of them are Lie subalgebras of $\mathfrak{g}_2$ is correct. The information on the isometry type of the totally geodesic submanifolds corresponding to the various types of Lie triple systems will be proved in Section 5.3.

For the fact that no Lie triple system is of more than one type: Notice that, with the exception of the types ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, l), $*$), no two types of Lie triple systems correspond to the same isometry type of totally geodesic submanifold, therefore none of these Lie triple systems can be of more than one type. To show that the various types of the form ($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, l), $*$) are also separate, we determine for a Lie triple system $\mathfrak{m}'$ of each of these types the type of the smallest Lie triple system $\hat{\mathfrak{m}}'$ of rank 2 which contains $\mathfrak{m}'$ (using the `satake` package as usual). We obtain the following result:

(41)

type of $\mathfrak{m}'$	type of $\hat{\mathfrak{m}}'$
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 2), G)	($\mathbb{S} \times \mathbb{S}$, 2, 2)
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 2), A_2)	(AI)
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), G)	($\mathbb{S} \times \mathbb{S}$, 3, 3)
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), A_2)	(A_2)
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), max)	$\hat{\mathfrak{m}}' = \mathfrak{g}_2$
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{C}$, 2), G)	(G)
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{C}$, 2), A_2)	(A_2)

We see that in each series of types $(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l), *)$ with fixed $(\mathbb{K}, l)$, the Lie triple systems $\hat{\mathfrak{m}}'$ corresponding to the Lie triple systems $\mathfrak{m}'$ of these types are of different type. Because we already know that no Lie triple system of rank 2 can be of more than one type, it follows that also among the types $(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l), *)$ no Lie triple system can be of more than one type.

We next show that the information on the maximality of the Lie triple systems is correct. For this purpose, we presume that the list of Lie triple systems given in the theorem is in fact complete; this will be proved in the remainder of the present section.

That the Lie triple systems which are claimed to be maximal in the table indeed are: This is clear for the types (A_2) and (G) because there are no Lie triple systems of greater dimension. For the type $(\mathbb{S}\times\mathbb{S}, 3, 3)$, we note that if it were not maximal, it could only be included in a Lie triple system of type (A_2) or (G) for dimension reasons. However, the Lie triple systems of type $(\mathbb{S}\times\mathbb{S}, 3, 3)$ have two orthogonal roots of multiplicity 2, whereas the systems of type (A_2) do not have a pair of orthogonal roots, and in the systems of type (G), all roots have multiplicity 1. So such an inclusion is not in fact possible, and hence the Lie triple systems of type $(\mathbb{S}\times\mathbb{S}, 3, 3)$ are maximal. For the type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 3)$: Let $\hat{\mathfrak{m}}' \subset \mathfrak{g}_2$ be a Lie triple system with $\mathfrak{m}' \subsetneq \hat{\mathfrak{m}}'$. If $\hat{\mathfrak{m}}'$ were of rank 1, then it would need to have the same isotropy angle $\varphi = \arctan(1/3\sqrt{3})$ and a strictly greater dimension than $\mathfrak{m}'$, but no such Lie triple system exists. So $\hat{\mathfrak{m}}'$ is of rank 2. It now follows from the description of the type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 3)$ that $\hat{\mathfrak{m}}'$ has two roots at an angle of $5\pi/6$ to each other and these two roots both have multiplicity 2. Therefore $\hat{\mathfrak{m}}' = \mathfrak{g}_2$ holds, and hence $\mathfrak{m}'$ is maximal. For the type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$: Again suppose that $\hat{\mathfrak{m}}' \subset \mathfrak{g}_2$ is a Lie triple system with $\mathfrak{m}' \subsetneq \hat{\mathfrak{m}}'$. Similarly as before, $\hat{\mathfrak{m}}'$ cannot be of rank 1. But we also know already from Table (41) that there is no rank 2 Lie triple system $\hat{\mathfrak{m}}'$ with $\mathfrak{m}' \subsetneq \hat{\mathfrak{m}}' \subsetneq \mathfrak{g}_2$ either. Therefore $\mathfrak{m}'$ is maximal.

That no Lie triple systems are maximal besides those mentioned above follows from the following table:

Every Lie triple system of type...	is contained in a Lie triple system of type...
$(\mathrm{Geo}, \varphi = t)$	$(\mathbb{S}\times\mathbb{S}, 1, 1)$
$(\mathbb{S}, \varphi = 0, l)$	$(\mathbb{S}\times\mathbb{S}, l, 1)$
$(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 2))$	(G)
$(\mathbb{S}, \varphi = \pi/6, l)$	$(\mathbb{S}\times\mathbb{S}, 1, l)$
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l), \mathrm{G})$	(G)
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, l), \mathrm{A}_2)$	(A_2)
$(\mathbb{S}\times\mathbb{S}, l, l')$ with $(l, l') \neq (3, 3)$	$(\mathbb{S}\times\mathbb{S}, 3, 3)$
(AI)	(A_2)

We now turn to the proof that the list of Lie triple systems of $\mathfrak{g}_2$ given in Theorem 5.2 is indeed complete. For this purpose, we let an arbitrary Lie triple system $\mathfrak{m}'$ of $\mathfrak{g}_2$, $\{0\} \neq \mathfrak{m}' \subsetneq \mathfrak{g}_2$, be given. Because the Lie algebra $\mathfrak{g}_2$ is of rank 2, the rank of $\mathfrak{m}'$ is either 1 or 2. We will handle these two cases separately in the sequel.

We first suppose that $\mathfrak{m}'$ is a Lie triple system of rank 2. Let us fix a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}'$; because of $\mathrm{rk}(\mathfrak{m}') = \mathrm{rk}(\mathfrak{g}_2)$, $\mathfrak{a}$ is then also a Cartan subalgebra of $\mathfrak{g}_2$. In relation to this situation, we again use the notations introduced in Sections 2 and 5.1. In particular, we consider the positive root system $\Delta_+ := \{\lambda_1, \dots, \lambda_6\}$ of the root system $\Delta := \Delta(\mathfrak{g}_2, \mathfrak{a})$ of $\mathfrak{g}_2$, and also the root system $\Delta' := \Delta(\mathfrak{m}', \mathfrak{a})$ of $\mathfrak{m}'$. By Proposition 2.1 (b), Δ' is a root subsystem of Δ, and therefore $\Delta'_+ := \Delta' \cap \Delta_+$ is a positive system of roots for Δ'. Moreover, in the root space decompositions of $\mathfrak{g}_2$ and $\mathfrak{m}'$

$$\mathfrak{g}_2 = \mathfrak{a} \oplus \bigoplus_{k=1}^{6} V_{\lambda_k}(\mathbb{C}) \quad \text{and} \quad \mathfrak{m}' = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta'_+} \mathfrak{m}'_\lambda, \tag{42}$$

the root space $\mathfrak{m}'_\lambda$ of $\mathfrak{m}'$ with respect to $\lambda \in \Delta'_+$ is related to the corresponding root space $V_\lambda(\mathbb{C})$ of $\mathfrak{g}_2$ by $\mathfrak{m}'_\lambda = V_\lambda(\mathbb{C}) \cap \mathfrak{m}'$.

Because the subset Δ' of Δ is invariant under its own Weyl group, we have (up to Weyl transformation) the following possibilities for Δ'_+, which we will treat individually in the sequel:

$$\Delta'_+ = \Delta_+, \quad \Delta'_+ = \{\lambda_2, \lambda_5, \lambda_6\}, \quad \Delta'_+ = \{\lambda_1, \lambda_3, \lambda_4\},$$
$$\Delta'_+ = \{\lambda_1, \lambda_6\}, \quad \Delta'_+ = \{\lambda_6\}, \quad \Delta'_+ = \{\lambda_1\} \quad \text{and} \quad \Delta'_+ = \varnothing.$$

CASE $\Delta'_+ = \Delta_+$. In this case the Dynkin diagram with multiplicities of $\mathfrak{m}'$ is $\bullet^{n_1} \Lleftarrow \bullet^{n_2}$ with $n_1, n_2 \in \{1, 2\}$. From the classification of the irreducible Riemannian symmetric spaces (see for example [14], p. 119, 146), we see that $n_1 = n_2 =: n \in \{1, 2\}$ holds. If $n = 2$ holds, we have $\mathfrak{m}' = \mathfrak{g}_2$. If $n = 1$ holds, we may by virtue of Proposition 5.1 suppose without loss of generality that $\mathfrak{m}'_{\lambda_1} = V_{\lambda_1}(\mathbb{R})$ and $\mathfrak{m}'_{\lambda_2} = V_{\lambda_2}(\mathbb{R})$ holds. Then we can calculate the remaining root spaces of $\mathfrak{m}'$ one by one: We have $R(\lambda_1^\sharp, V_{\lambda_2}(1))V_{\lambda_1}(1) = 3\sqrt{3}/4 \cdot V_{\lambda_3}(i)$ and therefore $\mathfrak{m}'_{\lambda_3} = V_{\lambda_3}(i\mathbb{R})$. We have $R(\lambda_1^\sharp, V_{\lambda_3}(i))V_{\lambda_1}(1) = \sqrt{3}/4 \cdot V_{\lambda_2}(1) - 1/2 \cdot V_{\lambda_4}(1)$ and therefore $\mathfrak{m}'_{\lambda_4} = V_{\lambda_4}(\mathbb{R})$. We have $R(\lambda_1^\sharp, V_{\lambda_4}(1))V_{\lambda_1}(1) = 1/2 \cdot V_{\lambda_3}(i) - \sqrt{3}/4 \cdot V_{\lambda_5}(i)$ and therefore $\mathfrak{m}'_{\lambda_5} = V_{\lambda_5}(i\mathbb{R})$. Finally, we have $R(\lambda_1^\sharp, V_{\lambda_5}(i))V_{\lambda_2}(1) = 3\sqrt{3}/4 \cdot V_{\lambda_6}(1)$ and therefore $\mathfrak{m}'_{\lambda_6} = V_{\lambda_6}(\mathbb{R})$. Thus it follows from Equation (42) that

$$\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_1}(\mathbb{R}) \oplus V_{\lambda_2}(\mathbb{R}) \oplus V_{\lambda_3}(i\mathbb{R}) \oplus V_{\lambda_4}(\mathbb{R}) \oplus V_{\lambda_5}(i\mathbb{R}) \oplus V_{\lambda_6}(\mathbb{R})$$

holds, and therefore $\mathfrak{m}'$ is of type (G).

CASE $\Delta'_+ = \{\lambda_2, \lambda_5, \lambda_6\}$. In this case, the Dynkin diagram with multiplicities of $\mathfrak{m}'$ is $\bullet^n — \bullet^n$ with $n \in \{1, 2\}$. In the case $n = 2$ we have $\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_2}(\mathbb{C}) \oplus V_{\lambda_5}(\mathbb{C}) \oplus V_{\lambda_6}(\mathbb{C})$ and therefore $\mathfrak{m}'$ is of type (A_2). In the case $n = 1$ we may suppose without loss of generality $\mathfrak{m}'_{\lambda_2} = V_{\lambda_2}(\mathbb{R})$ and $\mathfrak{m}'_{\lambda_5} = V_{\lambda_5}(\mathbb{R})$; then we have $\mathfrak{m}' \ni R(\lambda_1^\sharp, V_{\lambda_2}(1))V_{\lambda_5}(1) = -(3\sqrt{3}/4)V_{\lambda_6}(i)$ and hence $\mathfrak{m}'_{\lambda_6} = V_{\lambda_6}(i\mathbb{R})$. Therefore we have $\mathfrak{m}' = \mathfrak{a} \oplus V_{\lambda_2}(\mathbb{R}) \oplus V_{\lambda_5}(\mathbb{R}) \oplus V_{\lambda_6}(i\mathbb{R})$, thus $\mathfrak{m}'$ is of type (AI).

CASE $\Delta'_+ = \{\lambda_1, \lambda_3, \lambda_4\}$. Assume that $\mathfrak{m}'$ is a Lie triple system with this root system. Then there exist $c, d \in \mathbb{C}^\times$ so that $V_{\lambda_1}(c), V_{\lambda_3}(d) \in \mathfrak{m}'$ holds. We have

$$\mathfrak{m}' \ni R(\lambda_1^\sharp, V_{\lambda_1}(c))V_{\lambda_3}(d) = \frac{\sqrt{3}}{2} V_{\lambda_2}(\bar{c}di) + V_{\lambda_4}(cdi)$$

and therefore in particular $\lambda_2 \in \Delta'_+$, contrary to the hypothesis $\Delta'_+ = \{\lambda_1, \lambda_3, \lambda_4\}$. This calculation shows that there do not exist any Lie triple systems $\mathfrak{m}'$ of $\mathfrak{g}_2$ with $\Delta'_+ = \{\lambda_1, \lambda_3, \lambda_4\}$.

CASES $\Delta'_+ \subset \{\lambda_1, \lambda_6\}$. In this case we have $\mathfrak{m}' = \mathfrak{a} \oplus \mathfrak{m}'_{\lambda_1} \oplus \mathfrak{m}'_{\lambda_6}$ by Equation (42), therefore $\mathfrak{m}'$ is of type $(\mathbb{S} \times \mathbb{S}, l, l')$ with $l := 1 + \dim(\mathfrak{m}'_{\lambda_1})$ and $l' := 1 + \dim(\mathfrak{m}'_{\lambda_6})$.

This completes the treatment of the case where $\mathfrak{m}'$ is of rank 2.

We now suppose that $\mathfrak{m}' \subset \mathfrak{g}_2$ is a Lie triple system of rank 1. We may suppose without loss of generality that $\mathfrak{m}'$ contains a unit vector H from the closure of the positive Weyl chamber $\mathfrak{c}$. By Equations (39) and (40), we then have with $\varphi_0 := \varphi(H) \in [0, \pi/6]$

$$H = \cos(\varphi_0)\lambda_4^\sharp + \sin(\varphi_0)\frac{1}{\sqrt{3}}\lambda_2^\sharp. \tag{43}$$

Because of $\mathrm{rk}(\mathfrak{m}') = 1$, $\mathfrak{a}' := \mathbb{R}H$ is a Cartan subalgebra of $\mathfrak{m}'$, and we have $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$. It follows from Proposition 2.1 (a) that the root systems Δ' and Δ of $\mathfrak{m}'$ resp. $\mathfrak{g}_2$ with respect to $\mathfrak{a}'$ resp. to $\mathfrak{a}$ are related by

$$\Delta' \subset \{\lambda(H)\alpha_0 \mid \lambda \in \Delta,\ \lambda(H) \neq 0\} \tag{44}$$

with the linear form $\alpha_0 \colon \mathfrak{a}' \to \mathbb{R}$, $tH \mapsto t$; moreover for $\mathfrak{m}'$ we have the root space decomposition

$$\mathfrak{m}' = \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{m}'_\alpha, \tag{45}$$

where for any root $\alpha \in \Delta'$, the corresponding root space $\mathfrak{m}'_\alpha$ is given by

$$\mathfrak{m}'_\alpha = \left(\bigoplus_{\substack{\lambda \in \Delta \\ \lambda(H) = \alpha(H)}} V_\lambda(\mathbb{C}) \right) \cap \mathfrak{m}'. \tag{46}$$

If $\Delta' = \varnothing$ holds, we have $\mathfrak{m}' = \mathbb{R}H$, and therefore $\mathfrak{m}'$ is then of type $(\mathrm{Geo}, \varphi = \varphi_0)$. Otherwise it follows from Proposition 2.3 that one of the following two conditions holds: Either H is proportional to a root vector $\lambda^\sharp$ with $\lambda \in \Delta$, or there exist two $\lambda, \mu \in \Delta$ $(\lambda \neq \mu)$ so that H is orthogonal to $\lambda^\sharp - \mu^\sharp$. Evaluating all possible values for λ and μ, we see that $\varphi_0 \in \{0, \arctan(1/3\sqrt{3}), \pi/6\}$ holds.

In the sequel we consider the three possible values for φ_0 individually.

CASE $\varphi_0 = 0$. In this case we have $H = \lambda_4^\sharp$ by Equation (43) and therefore

$$\lambda_1(H) = \frac{1}{2}, \quad \lambda_2(H) = 0, \quad \lambda_3(H) = \frac{1}{2}, \quad \lambda_4(H) = 1, \quad \lambda_5(H) = \frac{3}{2}, \quad \lambda_6(H) = \frac{3}{2}.$$

Thus we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha, \pm 3\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_3|\mathfrak{a}'$ by Equation (44), $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha} \oplus \mathfrak{m}'_{3\alpha}$ by Equation (45) and $\mathfrak{m}'_\alpha \subset V_{\lambda_1}(\mathbb{C}) \oplus V_{\lambda_3}(\mathbb{C})$, $\mathfrak{m}'_{2\alpha} \subset V_{\lambda_4}(\mathbb{C})$ and $\mathfrak{m}'_{3\alpha} \subset V_{\lambda_5}(\mathbb{C}) \oplus V_{\lambda_6}(\mathbb{C})$ by Equation (46).

We now show that actually $\alpha, 3\alpha \notin \Delta'_+$ holds.

Indeed, let $v \in \mathfrak{m}'_\alpha$ be given. Then there exist $c, d \in \mathbb{C}$ so that $v = V_{\lambda_1}(c) + V_{\lambda_3}(d)$ holds. We have $|c| = |d|$ because of Proposition 2.4 and the fact that $\alpha^\sharp = (1/2)\lambda_1^\sharp + (1/2)\lambda_3^\sharp$ holds. Next we notice that because of $\lambda_2(H) = 0$, the $V_{\lambda_2}(\mathbb{C})$-component of every vector in $\mathfrak{m}'$ must vanish. However, the $V_{\lambda_2}(\mathbb{C})$-component of $R(H, v)v \in \mathfrak{m}'$ equals $(\sqrt{3}/2)V_{\lambda_2}(\bar{c}di)$, and so we conclude $\bar{c}d = 0$. Because of $|c| = |d|$, it follows that we have $c = d = 0$ and hence $v = 0$. Thus we have $\mathfrak{m}'_\alpha = \{0\}$ and hence $\alpha \notin \Delta'$.

A similar calculation also shows $3\alpha \notin \Delta'$, and therefore we have $\Delta' = \{\pm 2\alpha\}$ and hence $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_{2\alpha}$ with a linear subspace $\{0\} \neq \mathfrak{m}'_{2\alpha} \subset V_{\lambda_4}(\mathbb{C})$. It follows that $\mathfrak{m}'$ is of type $(\mathbb{S}, \varphi = 0, l)$ with $l := 1 + n'_{2\alpha}$.

CASE $\varphi_0 = \arctan(1/3\sqrt{3})$. In this case we have $H = (\sqrt{21}/42)(9\lambda_4^\sharp + \lambda_2^\sharp)$ by Equation (43) and therefore

$$\begin{aligned} \lambda_1(H) &= \frac{\sqrt{21}}{14} \cdot 1, \quad \lambda_2(H) = \frac{\sqrt{21}}{14} \cdot 1, \quad \lambda_3(H) = \frac{\sqrt{21}}{14} \cdot 2, \\ \lambda_4(H) &= \frac{\sqrt{21}}{14} \cdot 3, \quad \lambda_5(H) = \frac{\sqrt{21}}{14} \cdot 4, \quad \lambda_6(H) = \frac{\sqrt{21}}{14} \cdot 5. \end{aligned}$$

In the present case we have $\lambda^\sharp \notin \mathfrak{a}'$ for every $\lambda \in \Delta$, therefore $\mathfrak{m}'$ can only have composite roots (see Definition 2.2) by Proposition 2.3 (a). This fact, together with the above values of $\lambda(H)$ and Equation (44), shows that we have $\Delta' = \{\pm\alpha\}$ with $\alpha := \lambda_1|\mathfrak{a}' = \lambda_2|\mathfrak{a}'$. Moreover, we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ by Equation (45) and $\mathfrak{m}'_\alpha \subset V_{\lambda_1}(\mathbb{C}) \oplus V_{\lambda_2}(\mathbb{C})$ by Equation (46).

Let $v \in \mathfrak{m}'_\alpha$ be given, say $v = V_{\lambda_1}(c_1) + V_{\lambda_2}(c_2)$ with $c_1, c_2 \in \mathbb{C}$. We have

$$\alpha^\sharp = \alpha(H) \cdot H = \lambda_1(H) \cdot H = \frac{\sqrt{21}}{14} \cdot H = \frac{1}{28}(9\lambda_4^\sharp + \lambda_2^\sharp) = \frac{9}{14}\lambda_1^\sharp + \frac{5}{14}\lambda_2^\sharp,$$

and therefore Proposition 2.4 shows that we have $|c_2| = \sqrt{(5/14)/(9/14)}|c_1| = (1/3)\sqrt{5}|c_1|$.

By Proposition 5.1 we may therefore suppose without loss of generality that $v_0 := V_{\lambda_1}(1) + V_{\lambda_2}((1/3)\sqrt{5}) \in \mathfrak{m}'_\alpha$ holds. Then we have

$$\mathfrak{m}' \ni R(v_0, v)H = -\frac{\sqrt{7}}{14} V_{\lambda_3}(i(3c_2 - \sqrt{5}c_1)). \tag{47}$$

Because of $\lambda_3|\mathfrak{a}' = 2\alpha \notin \Delta'$, the vector (47) must vanish, and thus we have $3c_2 = \sqrt{5}c_1$. This shows that $\mathfrak{m}'_\alpha$ is a linear subspace of $\{V_{\lambda_1}(c) + V_{\lambda_2}((\sqrt{5}/3)c) \mid c \in \mathbb{C}\}$, and therefore $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$ is of type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), l)$ with $l := 1 + n'_\alpha$.

CASE $\varphi_0 = \pi/6$. In this case we have $H = (1/\sqrt{3})\lambda_6^\sharp$ by Equation (43) and therefore

$$\lambda_1(H) = 0, \quad \lambda_2(H) = \frac{\sqrt{3}}{2}, \quad \lambda_3(H) = \frac{\sqrt{3}}{2},$$
$$\lambda_4(H) = \frac{\sqrt{3}}{2}, \quad \lambda_5(H) = \frac{\sqrt{3}}{2}, \quad \lambda_6(H) = \sqrt{3}.$$

Thus we have $\Delta' \subset \{\pm\alpha, \pm 2\alpha\}$ with $\alpha := \lambda_2|\mathfrak{a}' = \lambda_3|\mathfrak{a}' = \lambda_4|\mathfrak{a}' = \lambda_5|\mathfrak{a}'$ by Equation (44), $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha}$ by Equation (45) and $\mathfrak{m}'_\alpha \subset V_{\lambda_2}(\mathbb{C}) \oplus V_{\lambda_3}(\mathbb{C}) \oplus V_{\lambda_4}(\mathbb{C}) \oplus V_{\lambda_5}(\mathbb{C})$ and $\mathfrak{m}'_{2\alpha} \subset V_{\lambda_6}(\mathbb{C})$ by Equation (46). For the sake of brevity, we put $V_\alpha(c_2, c_3, c_4, c_5) := \sum_{k=2}^{5} V_{\lambda_k}(c_k)$ for $c_2, \ldots, c_5 \in \mathbb{C}$ in the sequel.

Let us first consider the case $\alpha \notin \Delta'$ and therefore $\Delta' = \{\pm 2\alpha\}$. Then we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_{2\alpha}$, and therefore $\mathfrak{m}'$ then is of type $(\mathbb{S}, \varphi = \pi/6, l)$ with $l := 1 + n'_{2\alpha}$.

So we now suppose $\alpha \in \Delta'$. Then we fix $v_0 \in \mathfrak{m}'_\alpha \setminus \{0\}$, say $v_0 = V_\alpha(c_2, \ldots, c_5)$ with $c_2, \ldots, c_5 \in \mathbb{C}$. For $d \in \mathbb{C}$ we have

$$\operatorname{ad}(V_{\lambda_6}(d))v_0 = \frac{\sqrt{3}}{2} V_\alpha(\overline{c_5}d, -\overline{c_4}d, \overline{c_3}d, -\overline{c_2}d),$$

whence it follows that the Lie subgroup $\exp(\mathbb{R}\lambda_6^\sharp \oplus V_{\lambda_6}(\mathbb{C}))$ acts via its adjoint representation on the coordinates $(c_3, c_4) \in \mathbb{C}^2$ of $V_\alpha(c_2, \ldots, c_5)$ as the conjugate of the canonical action of $SU(2)$ on $\mathbb{C}^2$. Therefore, by application of a suitable element of this Lie subgroup, we can pass from $\mathfrak{m}'$ to another Lie triple system of the same type, for which $c_3 = 0$ holds. Furthermore, by Proposition 5.1 we may suppose without loss of generality that $c_2 = t_2$ and $c_4 = t_4$ with $t_2, t_4 \in \mathbb{R}_{\geq 0}$ holds. Thus we have $v_0 = V_\alpha(t_2, 0, t_4, c_5)$.

We now calculate

$$\begin{aligned}\mathfrak{m}' \ni R(H, v_0)v_0 = \frac{\sqrt{3}}{2}&((2t_4^2 + 3|c_5|^2)\lambda_1^\sharp + (t_2^2 + t_4^2 + |c_5|^2)\lambda_2^\sharp) \\ &+ V_{\lambda_1}\left(-\frac{3}{2}t_4 c_5 i\right).\end{aligned} \tag{48}$$

The $\mathfrak{a}$-component of this vector lies in $\mathbb{R}H = \mathbb{R}(3\lambda_1^\sharp + 2\lambda_2^\sharp)$ (because of $\operatorname{rk}(\mathfrak{m}') = 1$), therefore we have

$$2 \cdot (2t_4^2 + 3|c_5|^2) = 3 \cdot (t_2^2 + t_4^2 + |c_5|^2)$$

and thus

$$t_4^2 + 3|c_5|^2 = 3t_2^2. \tag{49}$$

From this equation it follows that $t_2 \neq 0$ (otherwise we would have $t_4 = c_5 = 0$ and hence $v_0 = 0$). By appropriately scaling v_0, we can therefore arrange $t_2 = 1$ and therefore $v_0 = V_\alpha(1, 0, t_4, c_5)$.

Moreover, the $V_{\lambda_1}(\mathbb{C})$-component of the vector (48) vanishes (because of $\lambda_1(H) = 0$), and thus we have $t_4 c_5 = 0$. Therefore we have either $t_4 = 0$ and then $|c_5| = 1$ by Equation (49), by application of Proposition 5.1 we can arrange $c_5 = 1$; or else $c_5 = 0$ and then $t_4 = \sqrt{3}$ by Equation (49). So we see that

$$\text{either} \quad v_0 = V_\alpha(1, 0, 0, 1) \quad \text{or} \quad v_0 = V_\alpha(1, 0, \sqrt{3}, 0)$$

holds. We will treat these two possible cases separately.

Let us first look at the case where $v_0 = V_\alpha(1, 0, 0, 1) \in \mathfrak{m}'_\alpha$ holds. If $n'_\alpha = 1$ (i.e. $\mathfrak{m}'_\alpha = \mathbb{R} v_0$) holds, we necessarily have $2\alpha \notin \Delta'$ (otherwise $n'_{2\alpha} + 1$ divides n'_α) and therefore $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0$ then is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 2), \mathrm{A}_2)$. Thus we now suppose $n'_\alpha \geq 2$ and let $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0$ be given, say $v = V_\alpha(c_2, \ldots, c_5)$ with $c_2, \ldots, c_5 \in \mathbb{C}$. We have

$$\begin{aligned} \mathfrak{m}' \ni R(H, v_0)v = {} & \frac{\sqrt{3}}{2}(3\,\mathrm{Re}(c_5)\lambda_1^\sharp + (\mathrm{Re}(c_2) + \mathrm{Re}(c_5))\lambda_2^\sharp) \\ & - \frac{3}{4} V_{\lambda_1}(i(c_3 + \overline{c_4})) + \frac{3}{4} V_{\lambda_6}(i(c_5 - c_2)). \end{aligned} \tag{50}$$

Again, the $\mathfrak{a}$-component of this vector lies in $\mathbb{R}H = \mathbb{R}(3\lambda_1^\sharp + 2\lambda_2^\sharp)$, whence we have $2 \cdot 3\,\mathrm{Re}(c_5) = 3 \cdot (\mathrm{Re}(c_2) + \mathrm{Re}(c_5))$, hence $\mathrm{Re}(c_2) = \mathrm{Re}(c_5)$. Because of our hypothesis $v \perp v_0$ it follows that $\mathrm{Re}(c_2) = \mathrm{Re}(c_5) = 0$ holds, thus we have $c_2 = i t_2$ and $c_5 = i t_5$ with $t_2, t_5 \in \mathbb{R}$. Moreover, the $V_{\lambda_1}(\mathbb{C})$-component of the vector (50) vanishes, and thus we have $c_4 = -\overline{c_3}$. Thus we see that any $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0$ is of the form $v = V_\alpha(i t_2, c_3, -\overline{c_3}, i t_5)$ with $t_2, t_5 \in \mathbb{R}$ and $c_3 \in \mathbb{C}$.

We now first consider the case where $2\alpha \notin \Delta'$, i.e. $\Delta' = \{\pm\alpha\}$. We shall show that in this situation, any $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0$ is a scalar multiple of one of the following four vectors:

$$v_{1,\star} := V_\alpha(i, 0, 0, i), \quad v_{1,k} := V_\alpha(i, -\sqrt{3} i e^{2k\pi i/3}, -\sqrt{3} i e^{4k\pi i/3}, i) \quad \text{for} \quad k \in \{0, 1, 2\}.$$

Because $\mathbb{R}v_{1,\star} \cup \mathbb{R}v_{1,0} \cup \mathbb{R}v_{1,1} \cup \mathbb{R}v_{1,2}$ does not contain any linear subspaces of dimension ≥ 2, it follows that $n'_\alpha = 2$ and $\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}v_{1,k}$ holds with some $k \in \{\star, 0, 1, 2\}$. If we have $k = \star$ here, then $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_{1,\star}$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \mathrm{A}_2)$. If $k \in \{0, 1, 2\}$ holds, then $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_{1,k}$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$: For $k = 0$ this is obvious. For $k \in \{1, 2\}$ we consider $Z := 4\lambda_1^\sharp + 2\lambda_2^\sharp \in \mathfrak{a}$. Then we have for $n \in \mathbb{N}$

$$\mathrm{ad}(Z)^n H = 0, \quad \mathrm{ad}(Z)^n v_0 = V_\alpha(0, 0, 0, (3i)^n),$$

and

$$\mathrm{ad}(Z)^n v_{1,0} = V_\alpha(0, -\sqrt{3} i \cdot i^n, -\sqrt{3} i (2i)^n, i(3i)^n),$$

hence for $t \in \mathbb{R}$

$$\mathrm{Ad}(\exp(tZ))H = H, \quad \mathrm{Ad}(\exp(tZ))v_0 = V_\alpha(1, 0, 0, e^{3it}),$$

and

$$\mathrm{Ad}(\exp(tZ))v_{1,0} = V_\alpha(i, -\sqrt{3}ie^{it}, -\sqrt{3}ie^{2it}, ie^{3it}).$$

It follows that with $g := \exp((2k\pi/3)Z) \in G_2$, the adjoint transformation $\mathrm{Ad}(g)$ transforms the prototype Lie triple system $\mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_{1,0}$ of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$ given in the theorem into our given $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_{1,k}$, hence $\mathfrak{m}'$ is of that type.

For the proof that v is a multiple of one of the $v_{1,k}$ ($k \in \{\star, 0, 1, 2\}$), we use the fact that v is of the form $v = V_\alpha(it_2, c_3, -\overline{c_3}, it_5)$ as was shown above. Because of $2\alpha \notin \Delta'$, in the present situation also the $V_{\lambda_6}(\mathbb{C})$-component of the vector (50) vanishes, whence it follows that $t_2 = t_5$ holds. Thus we have $v = V_\alpha(it, c, -\overline{c}, it)$ with some $t \in \mathbb{R}$ and $c \in \mathbb{C}$. If $c = 0$ holds, then v is a multiple of $v_{1,\star}$. Thus we now suppose $c \neq 0$. The $V_{\lambda_1}(\mathbb{C})$-component of $R(H, v)v \in \mathfrak{m}'$ equals $V_{\lambda_1}(-3tc + \sqrt{3}i\overline{c}^2)$; this has to vanish, and thus we have

$$i\overline{c}^2 = \sqrt{3}tc. \tag{51}$$

Because of $c \neq 0$ this equation implies also $t \neq 0$, and therefore we can ensure $t = 1$ by scaling v appropriately. Writing $c = re^{i\varphi}$ with $r > 0$ and $-\pi < \varphi \leq \pi$, we now derive from Equation (51) the equality $r = -\sqrt{3}ie^{3i\varphi}$, which implies $r = \sqrt{3}$ and $\varphi = 2(k-1)\pi/3 + \pi/6 = 2k\pi/3 - \pi/2$ with $k \in \{0, 1, 2\}$, and thus $c = -\sqrt{3}ie^{2k\pi i/3}$. It follows that $v = V_\alpha(i, c, -\overline{c}, i) = v_{1,k}$ holds.

Let us now consider the case $\Delta' = \{\pm\alpha, \pm 2\alpha\}$. From the classification of the Riemannian symmetric spaces of rank 1 we know that $n'_{2\alpha} \in \{1, 3, 7\}$ holds. Because of $n'_{2\alpha} \leq 2$, we have in the present situation in fact $n'_{2\alpha} = 1$. Without loss of generality we may suppose $\mathfrak{m}'_{2\alpha} = \mathbb{R}w$ with $w := V_{\lambda_6}(1)$. Therefore we have besides $H, w, v_0 \in \mathfrak{m}'$ also $R(H, w)v_0 = V_\alpha(-(3/2)i, 0, 0, (3/2)i) \in \mathfrak{m}'$, hence $v_1 := V_\alpha(i, 0, 0, -i) \in \mathfrak{m}'$.

We now show that in fact $\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}v_1$ holds. For this purpose, we let $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0, v_1$ be given. We need to show $v = 0$. As we saw above, because of the conditions $v \in \mathfrak{m}'_\alpha$ and $v \perp v_0$ we have $v = V_\alpha(it_2, c_3, -\overline{c_3}, it_5)$ with $t_2, t_5 \in \mathbb{R}$ and $c_3 \in \mathbb{C}$. A similar calculation based on the fact that also $v \perp v_1$ holds shows that $t_2 = t_5 = 0$ and therefore $v = V_\alpha(0, c_3, -\overline{c_3}, 0)$ holds. We now have that the $V_{\lambda_1}(\mathbb{C})$-component of $R(H, v)v$, which equals $V_{\lambda_1}(\sqrt{3}\overline{c_3}^2 i)$, vanishes, and thus we have $c_3 = 0$, hence $v = 0$. Therefore we have $\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}v_1$ and thus $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_1 \oplus \mathbb{R}w$ is of the type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{A}_2)$.

This completes the treatment of the case where $v_0 = V_\alpha(1, 0, 0, 1) \in \mathfrak{m}'_\alpha$ holds. We now turn to the other possible case for v_0, namely $v_0 = V_\alpha(1, 0, \sqrt{3}, 0) \in \mathfrak{m}'_\alpha$. If $n'_\alpha = 1$ holds, we again have $2\alpha \notin \mathfrak{m}'$, and therefore $\mathfrak{m}' = \mathbb{R}H \oplus \mathbb{R}v_0$ then is of the type ($\mathbb{P}$, $\varphi = \pi/6$, $(\mathbb{R}, 2)$, G). Thus we again suppose $n'_\alpha \geq 2$ in the sequel. Then we have for any $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0$, say $v = V_\alpha(c_2, c_3, c_4, c_5)$ with $c_2, \dots, c_5 \in \mathbb{C}$,

$$\begin{aligned} R(H, v)v_0 &= 3\,\mathrm{Re}(c_4)\lambda_1^\sharp + \left(\frac{\sqrt{3}}{2}\,\mathrm{Re}(c_2) + \frac{3}{2}\,\mathrm{Re}(c_4)\right)\lambda_2^\sharp \\ &\quad - V_{\lambda_1}\left(\frac{3}{4}ic_3 + \frac{3}{2}i\overline{c_3} + \frac{3\sqrt{3}}{4}ic_5\right) - V_{\lambda_6}\left(\frac{3\sqrt{3}}{4}ic_3 + \frac{3}{4}ic_5\right). \end{aligned} \tag{52}$$

Because this vector is again a member of $\mathfrak{m}'$, its $\mathfrak{a}$-component must be proportional to H, and thus we have

$$2 \cdot (3\,\mathrm{Re}(c_4)) = 3 \cdot \left(\frac{\sqrt{3}}{2}\,\mathrm{Re}(c_2) + \frac{3}{2}\,\mathrm{Re}(c_4)\right),$$

hence $\mathrm{Re}(c_4) = \sqrt{3}\,\mathrm{Re}(c_2)$. Because of the condition $v \perp v_0$, this in fact implies $\mathrm{Re}(c_2) = \mathrm{Re}(c_4) = 0$. Moreover, the $V_{\lambda_1}(\mathbb{C})$-component of the vector (52) vanishes, and thus we have

$$\frac{3}{4}ic_3 + \frac{3}{2}i\overline{c_3} + \frac{3\sqrt{3}}{4}ic_5 = 0,$$

hence

$$c_5 = -\frac{1}{\sqrt{3}}(c_3 + 2\overline{c_3}). \tag{53}$$

This shows that any $v \in \mathfrak{m}'_\alpha$ with $v \perp v_0$ is of the form $v = V_\alpha(it_2, c_3, it_4, -(1/\sqrt{3})(c_3 + 2\overline{c_3}))$ with $t_2, t_4 \in \mathbb{R}$ and $c_3 \in \mathbb{C}$.

Once again, we now first consider the case where $\Delta' = \{\pm\alpha\}$ and thus $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$. Then also the $V_{\lambda_6}(\mathbb{C})$-component of the vector (52) vanishes, and hence we have $-\sqrt{3}c_3 = c_5 = -(1/\sqrt{3})(c_3 + 2\overline{c_3})$, thus $c_3 = \overline{c_3}$. Therefore v then is of the form $v = V_\alpha(it_2, t_3, it_4, -\sqrt{3}t_3)$ with $t_2, t_3, t_4 \in \mathbb{R}$. We now calculate

$$R(H, v)v = \frac{1}{2}\sqrt{3}((10t_3^2 + 2t_4^2)\lambda_1^\sharp + (t_2^2 + 4t_3^2 + t_4^2)\lambda_2^\sharp) + V_{\lambda_1}\left(-\frac{3}{2}t_2t_3 + \frac{5}{2}\sqrt{3}t_3t_4\right).$$

Because this is an element of $\mathfrak{m}'$, its $\mathfrak{a}$-component is a multiple of H, whence it follows that

$$2 \cdot (10t_3^2 + 2t_4^2) = 3 \cdot (t_2^2 + 4t_3^2 + t_4^2)$$

and hence

$$8t_3^2 + t_4^2 = 3t_2^2 \tag{54}$$

holds, and its $V_{\lambda_1}(\mathbb{C})$-component vanishes, whence it follows that we have $(3/2)t_2t_3 = (5/2)\sqrt{3}t_3t_4$, hence

$$\text{either} \quad t_3 = 0 \quad \text{or} \quad t_4 = \frac{1}{5}\sqrt{3}t_2.$$

If $t_3 = 0$ holds, then Equation (54) shows that we have $t_4 = \pm\sqrt{3}t_2$. Otherwise we have $t_4 = (1/5)\sqrt{3}t_2$, and therefore by Equation (54) $t_3 = \pm(3/5)t_2$.

This consideration shows that we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha$, where either of the following two equations holds:

$$\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}V_\alpha(i, 0, \varepsilon\sqrt{3}i, 0), \tag{55}$$

or

$$\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}V_\alpha\left(i, \varepsilon\frac{3}{5}, \frac{1}{5}\sqrt{3}i, -\varepsilon\frac{3}{5}\sqrt{3}\right) \tag{56}$$

with $\varepsilon \in \{\pm 1\}$.

In either case $\mathfrak{m}'$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \mathrm{G})$: If $\mathfrak{m}'_\alpha$ is given by Equation (55), this is obvious. On the other hand, if $\mathfrak{m}'_\alpha$ is given by Equation (56) (without loss of generality with $\varepsilon = 1$), we note that $\mathfrak{m}'$ is contained in the linear space $\hat{\mathfrak{m}}'$ spanned by the vectors

$$\begin{gathered} 3\lambda_1^\sharp + 2\lambda_2^\sharp, \quad \lambda_1^\sharp + \frac{4}{3}\lambda_2^\sharp + V_{\lambda_1}(\sqrt{3}), \\ V_\alpha(1, 0, \sqrt{3}, 0), \quad V_\alpha(i, 0, \sqrt{3}i, 0), \\ V_\alpha\left(i, \frac{3}{5}, \frac{\sqrt{3}}{5}i, -\frac{3\sqrt{3}}{5}\right), \quad V_\alpha\left(1, -\frac{3}{5}i, \frac{\sqrt{3}}{5}, \frac{3\sqrt{3}}{5}i\right). \end{gathered}$$

One checks that $\hat{\mathfrak{m}}'$ is a Lie triple system of rank 2 and dimension 6, and therefore (by the preceding classification of the Lie triple systems of $\mathfrak{g}_2$ of rank 2) of type $(\mathbb{S} \times \mathbb{S}, 3, 3)$. Hence $\hat{\mathfrak{m}}'$ is congruent under the adjoint action to the standard Lie triple system of type $(\mathbb{S} \times \mathbb{S}, 3, 3)$ given in the theorem. $\mathfrak{m}'$ corresponds to the diagonal in the local sphere product corresponding to $\hat{\mathfrak{m}}'$, and is therefore congruent under the adjoint action to the diagonal in the standard Lie triple system of type $(\mathbb{S} \times \mathbb{S}, 3, 3)$, which is the standard Lie triple system of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \mathrm{G})$. Therefore also $\mathfrak{m}'$ itself is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \mathrm{G})$.

Let us finally turn our attention to the case where $\Delta' = \{\pm\alpha, \pm 2\alpha\}$ holds. From the classification of Riemannian symmetric spaces of rank 1, we again must have $n'_{2\alpha} = 1$, and we again suppose without loss of generality $\mathfrak{m}'_{2\alpha} = V_{\lambda_6}(\mathbb{R})$. We then have besides $v_0 \in \mathfrak{m}'_\alpha$ also $\mathfrak{m}'_\alpha \ni R(H, V_{\lambda_6}(1))v_0 = V_\alpha(0, (9/2)i, 0, (3\sqrt{3}/2)i)$ and therefore $v_1 := V_\alpha(0, \sqrt{3}i, 0, i) \in \mathfrak{m}'_\alpha$. Thus we have $\mathbb{R}v_0 \oplus \mathbb{R}v_1 \subset \mathfrak{m}'_\alpha$. Below, we will show that in fact $\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}v_1$ holds. Therefore we have $\mathfrak{m}' = \mathbb{R}H \oplus \mathfrak{m}'_\alpha \oplus \mathfrak{m}'_{2\alpha} = \mathbb{R}H \oplus \mathbb{R}v_0 \oplus \mathbb{R}v_1 \oplus \mathbb{R}V_{\lambda_6}(1)$, and hence $\mathfrak{m}'$ is of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{G})$.

For the proof of $\mathfrak{m}'_\alpha = \mathbb{R}v_0 \oplus \mathbb{R}v_1$ we let $v \in \mathfrak{m}'_\alpha$ be given, and suppose that v is orthogonal to v_0 and v_1. Then we are to show $v = 0$. Because of $v \perp v_0$ and $v \in \mathfrak{m}'_\alpha$ we saw before that $v = V_\alpha(it_2, c_3, it_4, -(1/\sqrt{3})(c_3 + 2\overline{c_3}))$ holds with $t_2, t_4 \in \mathbb{R}$ and $c_3 \in \mathbb{C}$, and by a similar argument based on the evaluation of $R(H, v)v_1$ and $v \perp v_1$, we see that

$$\mathrm{Im}(c_3) = 0 \quad \text{and} \quad t_2 = -\sqrt{3}t_4$$

holds, thus we have $v = V_\alpha(\sqrt{3}it_4, t_3, it_4, -\sqrt{3}t_3)$ with $t_3, t_4 \in \mathbb{R}$. We now calculate

$$R(H, v)v = \sqrt{3}((5t_3^2 + t_4^2)\lambda_1^\sharp + (2t_3^2 + 2t_4^2)\lambda_2^\sharp) + \sqrt{3}V_{\lambda_1}(t_3t_4).$$

The $V_{\lambda_1}(\mathbb{C})$-component of this vector vanishes, and therefore we have either $t_3 = 0$ or $t_4 = 0$. Also the $\mathfrak{a}$-component of that vector is a scalar multiple of H, which together with the fact that either of t_3 and t_4 is zero shows that in fact $t_3 = t_4 = 0$ and hence $v = 0$ holds.

This completes the classification of the Lie triple systems in $\mathfrak{g}_2$. □

5.3. Totally geodesic submanifolds in G_2. Once again, we describe totally geodesic isometric embeddings for the maximal Lie triple systems of G_2 to determine the global isometry type of the totally geodesic submanifolds of G_2. We obtain the results of the following table, using the same notations for the isometry types as in Section 3.3:

type of Lie triple system	corresponding global isometry type	properties
(Geo, $\varphi = t$)	$\mathbb{R}$ or $\mathbb{S}^1$	
($\mathbb{S}$, $\varphi = 0$, l)	$\mathbb{S}^l_{r=1}$	
($\mathbb{S}$, $\varphi = \arctan(1/3\sqrt{3})$, l)	$\mathbb{S}^l_{r=(2/3)\sqrt{21}}$	$l = 3$: maximal
($\mathbb{S}$, $\varphi = \pi/6$, l)	$\mathbb{S}^l_{r=1/\sqrt{3}}$	$l = 3$: Helgason sphere
($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{K}$, l), $*$)	$\mathbb{K}\mathrm{P}^l_{\varkappa=3/4}$	($\mathbb{P}$, $\varphi = \pi/6$, ($\mathbb{R}$, 3), max): maximal
($\mathbb{S} \times \mathbb{S}$, l, l')	$\left(\mathbb{S}^l_{r=1} \times \mathbb{S}^{l'}_{r=1/\sqrt{3}}\right)/\{\pm\mathrm{id}\}$	$l = l' = 3$: meridian, maximal
(AI)	$\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=\sqrt{3}}$	
(A_2)	$\mathrm{SU}(3)_{\mathrm{srr}=\sqrt{3}}$	maximal
(G)	$G_2/\mathrm{SO}(4)_{\mathrm{srr}=1}$	polar, maximal

Type (G). The totally geodesic embedding corresponding to this type is the Cartan embedding $f\colon G_2/\mathrm{SO}(4) \to G_2$ of the Riemannian symmetric space $G_2/\mathrm{SO}(4)$.

We describe the Cartan embedding for the general situation of a Riemannian symmetric space $M = G/K$. Let $\sigma\colon G \to G$ be the involutive automorphism which describes the symmetric structure of M. Then the map

$$f\colon G/K \to G, \quad g \cdot K \mapsto \sigma(g) \cdot g^{-1}$$

is called the *Cartan map* of M. Because of $\mathrm{Fix}(\sigma)_0 \subset K \subset \mathrm{Fix}(\sigma)$, f is a well-defined covering map onto its image; moreover f turns out to be totally geodesic. If M is a "bottom space", i.e. there exists no non-trivial symmetric covering map with total space M, we have $K = \mathrm{Fix}(\sigma)$ and therefore f is a totally geodesic embedding in this case. Then f is called the *Cartan embedding* of M.

Type ($\mathbb{S} \times \mathbb{S}$, l, l') and the types of rank 1 contained in that type. For the construction of these types we consider the skew-field of quaternions $\mathbb{H}$ and the division algebra of octonions $\mathbb{O}$. $\mathbb{O}$ can be realized as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$, where the octonion multiplication is for any $x, y \in \mathbb{O}$, say $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_i, y_i \in \mathbb{H}$, given by the equation

$$x \cdot y = (x_1 y_1 - \overline{y_2} x_2,\, x_2 \overline{y_1} + y_2 x_1).$$

In this setting, the symplectic group Sp(1) is realized as the space of unit quaternions with the quaternion multiplication as group action (hence Sp(1) is isometric to a 3-sphere), and the Lie group G_2 is realized as the automorphism group of $\mathbb{O}$, i.e.

$$G_2 = \{g \in \mathrm{GL}(\mathbb{O}) \mid \forall x, y \in \mathbb{O} \colon g(x \cdot y) = g(x) \cdot g(y)\}.$$

In this setting a group homomorphism $\Phi\colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to G_2$ has been described by Yokota in [16], Section 1.3: For any $g_1, g_2 \in \mathrm{Sp}(1)$, $\Phi(g_1, g_2)$ is given by

$$\forall x = (x_1, x_2) \in \mathbb{O}\colon \quad \Phi(g_1, g_2)x = (g_1 x_1 g_1^{-1},\, g_2 x_2 g_1^{-1}).$$

Φ is in particular a totally geodesic map; one easily sees that $\ker(\Phi) = \{\pm(1, 1)\}$ holds, and therefore Φ is a two-fold covering map onto its image. The image is therefore a 6-dimensional totally geodesic submanifold of G_2 of rank 2 which is isometric to $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\{\pm(1, 1)\} \cong \big(\mathbb{S}^3_{r=1} \times \mathbb{S}^3_{r=1/\sqrt{3}}\big)/\{\pm(1, 1)\}$, and which turns out to be of type $(\mathbb{S} \times \mathbb{S}, 3, 3)$.

The totally geodesic submanifolds of type $(\mathbb{S}\times\mathbb{S}, l, l')$ correspond to the submanifolds $\big(\mathbb{S}^l_{r=1} \times \mathbb{S}^{l'}_{r=1/\sqrt{3}}\big)/\{\pm(1, 1)\}$ in this product, the totally geodesic submanifolds of type $(\mathbb{S}, \varphi = 0, l)$ resp. $(\mathbb{S}, \varphi = \pi/6, l')$ correspond to the factors $\mathbb{S}^l_{r=1}$ resp. $\mathbb{S}^{l'}_{r=1/\sqrt{3}}$ in that product, and the totally geodesic submanifolds of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, l), \mathrm{G})$ correspond to the diagonal $\{(x, (1/\sqrt{3})x) \mid x \in \mathbb{S}^l_{r=1}\}/\{\pm(1, 1)\}$ in that product.

Types (A_2) and (AI). We again realize G_2 as the automorphism group of $\mathbb{O}$. We fix an imaginary unit octonion i of $\mathbb{O}$, and consider the subgroup $H := \{g \in G_2 \mid g(i) = i\}$ of G_2. H is isomorphic to SU(3); as totally geodesic submanifold of G_2, this subgroup is of type (A_2). Consider the splitting $\mathbb{O} = V \oplus W$ of $\mathbb{O}$ with $V := \mathrm{span}_{\mathbb{R}}\{1, i\} \cong \mathbb{C}$ and $W := V^\perp$; V and W are complex subspaces of dimension 1 resp. 3 with respect to the complex structure induced by the element $i \in \mathbb{O}$. Then $H \cong \mathrm{SU}(3)$ acts trivially on V and in the canonical way on $W \cong \mathbb{C}^3$.

Fixing a real form $W_{\mathbb{R}}$ of W, we obtain the subgroup $H' := \{g \in H \mid g(W_{\mathbb{R}}) = W_{\mathbb{R}}\}$, which is isomorphic to SO(3). H/H' is a Riemannian symmetric space isomorphic to SU(3)/SO(3), and the image of the Cartan embedding $H/H' \to H \subset G_2$ is a totally geodesic submanifold of G_2 of type (AI).

Type ($\mathbb{S}$, φ = arctan($1/3\sqrt{3}$), 3). Let $\mathfrak{m}'$ be a Lie triple system of type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 3)$. It is apparent from the part of the proof of Theorem 5.2 which handled the classification for the case $\mathrm{rk}(\mathfrak{m}') = 1$, $\varphi_0 = \arctan(1/3\sqrt{3})$ that (with respect to a suitable choice of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}_2$ and of the positive root system Δ_+ corresponding to the root system Δ of $\mathfrak{g}_2$ with respect to $\mathfrak{a}$) the unit vector $H :=$

$(1/\sqrt{21})(9\lambda_1^\sharp + 5\lambda_2^\sharp)$ lies in $\mathfrak{m}'$, with respect to its Cartan subalgebra $\mathbb{R}H$ the Lie triple system $\mathfrak{m}'$ has only one positive root α, which is characterized by $\alpha(H) = \sqrt{21}/14$, hence we have $\|\alpha^\sharp\|^2 = 3/28 = 1/r^2$ with $r := (2/3)\sqrt{21}$.

It follows that the connected, complete totally geodesic submanifold $M' \subset G_2$ corresponding to $\mathfrak{m}'$ is a symmetric space of constant curvature $1/r^2$, and therefore isometric either to the sphere $\mathbb{S}^3_r$, or to the real projective space $\mathbb{R}\mathrm{P}^3_{\varkappa=1/r^2}$. To distinguish between these two cases, we calculate the length of a closed geodesic in M'.

To do so, we use the well-known fact (see [6], Theorem VII.8.5, p. 322) that the unit lattice $\mathfrak{a}_e := \{v \in \mathfrak{a} \mid \exp(v) = e\}$ is generated by the vectors $2X_\lambda$, where we put $X_\lambda := (2\pi/\|\lambda^\sharp\|^2)\lambda^\sharp \in \mathfrak{a}$, and λ runs through all the roots of G_2. In this specific situation, $\mathfrak{a}_e$ is generated by the vectors $2X_{\lambda_2} = (4\pi/3)\lambda_2^\sharp$ and $2X_{\lambda_5} = (4\pi/3)\lambda_5^\sharp$.

The length of the geodesic γ tangent to H equals the smallest $t > 0$ so that $tH \in \mathfrak{a}_e$ holds, i.e. so that there exist $k, l \in \mathbb{Z}$ with $tH = k \cdot (4\pi/3)\lambda_2^\sharp + l \cdot (4\pi/3)\lambda_5^\sharp$. Because we have $H = (1/\sqrt{21})(2\lambda_2^\sharp + 3\lambda_5^\sharp)$, that equation leads to the conditions

$$k = 2 \cdot \frac{3}{4\pi \cdot \sqrt{21}} t \quad \text{and} \quad l = 3 \cdot \frac{3}{4\pi \cdot \sqrt{21}} t.$$

Therefore, the smallest $t > 0$ such that $k, l \in \mathbb{Z}$ holds, is $t = 4\pi \cdot \sqrt{21}/3 = 2\pi r$, and hence the geodesic γ is closed and has the length $2\pi r$. It follows that the totally geodesic submanifold M' is isometric to the sphere $\mathbb{S}^3_r$.

Type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$. By the analogous arguments as for the type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 3)$, we see that the totally geodesic submanifold M' of G_2 corresponding to a Lie triple system of type $(\mathbb{P}, \varphi = \pi/6, (\mathbb{R}, 3), \max)$ is a 3-dimensional space of constant curvature $1/r^2$ with $r := 2/\sqrt{3}$, and that any geodesic of G_2 running in M' is closed with length πr. Hence M' is isometric to the real projective space $\mathbb{R}\mathrm{P}^3_{\varkappa=1/r^2=3/4}$.

5.4. Totally geodesic submanifolds in $G_2/\mathrm{SO}(4)$. Finally we derive from the classification of the Lie triple systems resp. totally geodesic submanifolds in G_2 the same classification in the totally geodesic submanifold $G_2/\mathrm{SO}(4)$ of G_2.

For this purpose, we consider the Lie group G_2 as a Riemannian symmetric space in the same way as in the Sections 5.1 and 5.2, and use the names for the types of Lie triple systems of $\mathfrak{g}_2$ as introduced in Theorem 5.2.

Further, we let $\mathfrak{m}_1$ be a Lie triple system of $\mathfrak{g}_2$ of type (G), i.e. $\mathfrak{m}_1$ corresponds to a totally geodesic submanifold which is isometric to $G_2/\mathrm{SO}(4)$.

Theorem 5.4. *Exactly the following types of Lie triple systems of G_2 have representatives which are contained in $\mathfrak{m}_1$:*

- (Geo, $\varphi = t$) *with* $t \in [0, \pi/6]$,
- $(\mathbb{S}, \varphi = 0, 2)$,
- $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 2)$,

- $(\mathbb{S}, \varphi = \pi/6, 2)$,
- $(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, 2), \mathrm{G})$ *with* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,
- (AI),
- $(\mathbb{S} \times \mathbb{S}, l, l')$ *with* $l, l' \leq 2$.

Among these, the Lie triple systems which are maximal in $\mathfrak{m}_1$ *are*: $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 2)$, $(\mathbb{P}, \varphi = \pi/6, (\mathbb{C}, 2), \mathrm{G})$, (AI) *and* $(\mathbb{S} \times \mathbb{S}, 2, 2)$.

Proof. Again similar to the proofs of Theorems 3.8 and 3.10. □

REMARK 5.5. The maximal totally geodesic submanifolds of $G_2/\mathrm{SO}(4)$ of type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 2)$, which are isometric to a 2-sphere of radius $(2/3)\sqrt{21}$, are missing from the classification by Chen and Nagano in Table VIII of [5]. They are in a similar "skew" position in $G_2/\mathrm{SO}(4)$ as the 3-spheres of type $(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 3)$ are in G_2, compare Remark 5.3.

We can infer the isometry type of the totally geodesic submanifolds corresponding to the Lie triple systems of $G_2/\mathrm{SO}(4)$ from the corresponding information on the totally geodesic submanifolds of G_2, given in Section 5.3:

type of Lie triple system	corresponding global isometry type	properties
$(\mathrm{Geo}, \varphi = t)$	$\mathbb{R}$ or $\mathbb{S}^1$	
$(\mathbb{S}, \varphi = 0, 2)$	$\mathbb{S}^2_{r=1}$	
$(\mathbb{S}, \varphi = \arctan(1/3\sqrt{3}), 2)$	$\mathbb{S}^2_{r=(2/3)\sqrt{21}}$	maximal
$(\mathbb{S}, \varphi = \pi/6, 2)$	$\mathbb{S}^2_{r=1/\sqrt{3}}$	Helgason sphere
$(\mathbb{P}, \varphi = \pi/6, (\mathbb{K}, 2), \mathrm{G})$	$\mathbb{K}\mathrm{P}^2_{\varkappa=3/4}$	$\mathbb{K} = \mathbb{C}$: maximal
(AI)	$\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=\sqrt{3}}$	maximal
$(\mathbb{S} \times \mathbb{S}, l, l')$	$\left(\mathbb{S}^l_{r=1} \times \mathbb{S}^{l'}_{r=1/\sqrt{3}}\right)/\{\pm\mathrm{id}\}$	$l = l' = 2$: polar, meridian, maximal

6. Summary

In the following table, we list the global isometry types of the maximal totally geodesic submanifolds of all the irreducible, simply connected Riemannian symmetric spaces M of rank 2, thereby combining information from the papers [7], [8] and [9] (Section 6), as well as the present paper.

We once again use the notations from Section 3.3 for describing the scaling factor of the invariant Riemannian metric on the symmetric spaces involved. For the three infinite families of Grassmann manifolds $G_2^+(\mathbb{R}^n)$, $G_2(\mathbb{C}^n)$ and $G_2(\mathbb{H}^n)$, we also use the notation $_{\mathrm{srr}=1*}$ to denote the invariant Riemannian metric scaled in such a way that the shortest root occurring *for large* n has length 1, disregarding the fact that this root might vanish for certain small values of n.

M	maximal totally geodesic submanifolds
$G_2^+(\mathbb{R}^{n+2})_{\mathrm{srr}=1*}$	$\mathbb{S}^n_{r=1}$, $G_2^+(\mathbb{R}^{n+1})_{\mathrm{srr}=1*}$, $(\mathbb{S}^l_{r=1}\times\mathbb{S}^{l'}_{r=1})/\mathbb{Z}_2$ with $l+l'=n$ *for* $n\geq 4$ *even*: $\mathbb{C}\mathrm{P}^{n/2}_{\varkappa=1/2}$ *for* $n=2$: $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2}\times\mathbb{R}\mathrm{P}^1_{\varkappa=1/2}$ *for* $n=3$: $\mathbb{S}^2_{r=\sqrt{5}}$
$G_2(\mathbb{C}^{n+2})_{\mathrm{srr}=1*}$	$\mathbb{C}\mathrm{P}^n_{\varkappa=1}$, $G_2(\mathbb{R}^{n+2})_{\mathrm{srr}=1*}$, $G_2(\mathbb{C}^{n+1})_{\mathrm{srr}=1*}$ $\mathbb{C}\mathrm{P}^l_{\varkappa=1}\times\mathbb{C}\mathrm{P}^{l'}_{\varkappa=1}$ with $l+l'=n$ *for* n *even*: $\mathbb{H}\mathrm{P}^{n/2}_{\varkappa=1/2}$ *for* $n=2$: $G_2^+(\mathbb{R}^5)_{\mathrm{srr}=\sqrt{2}}$, $\big(\mathbb{S}^3_{r=1/\sqrt{2}}\times\mathbb{S}^1_{r=1/\sqrt{2}}\big)/\mathbb{Z}_2$ *for* $n=4$: $\mathbb{C}\mathrm{P}^2_{\varkappa=1/5}$
$G_2(\mathbb{H}^{n+2})_{\mathrm{srr}=1*}$	$\mathbb{H}\mathrm{P}^n_{\varkappa=1}$, $G_2(\mathbb{H}^{n+1})_{\mathrm{srr}=1*}$, $G_2(\mathbb{C}^{n+2})_{\mathrm{srr}=1*}$ $\mathbb{H}\mathrm{P}^l_{\varkappa=1}\times\mathbb{H}\mathrm{P}^{l'}_{\varkappa=1}$ with $l+l'=n$ *for* $n=2$: $\big(\mathbb{S}^5_{r=1/\sqrt{2}}\times\mathbb{S}^1_{r=1/\sqrt{2}}\big)/\mathbb{Z}_2$, $\mathrm{Sp}(2)_{\mathrm{srr}=\sqrt{2}}$ *for* $n=4$: $\mathbb{S}^3_{r=2\sqrt{5}}$ *for* $n=5$: $\mathbb{H}\mathrm{P}^2_{\varkappa=1/5}$
$\mathrm{SU}(3)/\mathrm{SO}(3)_{srr=1}$	$\mathbb{R}\mathrm{P}^2_{\varkappa=1/4}$, $\big(\mathbb{S}^2_{r=1}\times\mathbb{S}^1_{r=\sqrt{3}}\big)/\mathbb{Z}_2$
$\mathrm{SU}(6)/\mathrm{Sp}(3)_{srr=1}$	$\mathbb{H}\mathrm{P}^2_{\varkappa=1/4}$, $\mathbb{C}\mathrm{P}^3_{\varkappa=1/4}$, $\mathrm{SU}(3)_{\mathrm{srr}=1}$, $\big(\mathbb{S}^5_{r=1}\times\mathbb{S}^1_{r=\sqrt{3}}\big)/\mathbb{Z}_2$
$\mathrm{SO}(10)/\mathrm{U}(5)_{srr=1}$	$\mathbb{C}\mathrm{P}^4_{\varkappa=1}$, $\mathbb{C}\mathrm{P}^3_{\varkappa=1}\times\mathbb{C}\mathrm{P}^1_{\varkappa=1}$, $G_2^+(\mathbb{R}^8)_{\mathrm{srr}=\sqrt{2}}$, $G_2(\mathbb{C}^5)_{\mathrm{srr}=1}$, $\mathrm{SO}(5)_{\mathrm{srr}=1}$
$E_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))_{srr=1}$	$\mathbb{O}\mathrm{P}^2_{\varkappa=1/2}$, $\mathbb{C}\mathrm{P}^5_{\varkappa=1}\times\mathbb{C}\mathrm{P}^1_{\varkappa=1}$, $G_2^+(\mathbb{R}^{10})_{\mathrm{srr}=\sqrt{2}}$, $G_2(\mathbb{C}^6)_{\mathrm{srr}=1}$, $(G_2(\mathbb{H}^4)/\mathbb{Z}_2)_{\mathrm{srr}=1}$, $\mathrm{SO}(10)/\mathrm{U}(5)_{\mathrm{srr}=1}$
$(E_6/F_4)_{srr=1}$	$\mathbb{O}\mathrm{P}^2_{\varkappa=1/4}$, $\mathbb{H}\mathrm{P}^3_{\varkappa=1/4}$, $((\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3)_{\mathrm{srr}=1}$, $\big(\mathbb{S}^9_{r=1}\times\mathbb{S}^1_{r=\sqrt{3}}\big)/\mathbb{Z}_4$
$G_2/\mathrm{SO}(4)_{srr=1}$	$\mathbb{S}^2_{r=(2/3)\sqrt{21}}$, $\mathbb{C}\mathrm{P}^2_{\varkappa=3/4}$, $\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=\sqrt{3}}$, $\big(\mathbb{S}^2_{r=1}\times\mathbb{S}^2_{r=1/\sqrt{3}}\big)/\mathbb{Z}_2$
$\mathrm{SU}(3)_{srr=1}$	$\mathbb{C}\mathrm{P}^2_{\varkappa=1/4}$, $\mathbb{R}\mathrm{P}^3_{\varkappa=1/4}$, $\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=1}$, $\big(\mathbb{S}^3_{r=1}\times\mathbb{S}^1_{r=\sqrt{3}}\big)/\mathbb{Z}_2$
$\mathrm{Sp}(2)_{srr=1}$	$\mathbb{S}^3_{r=\sqrt{5}}$, $\mathbb{H}\mathrm{P}^1_{\varkappa=1/2}$, $\mathbb{S}^3_{r=1/\sqrt{2}}\times\mathbb{S}^3_{r=1/\sqrt{2}}$, $G_2^+(\mathbb{R}^5)_{\mathrm{srr}=1}$
$(G_2)_{\mathrm{srr}=1}$	$\mathbb{S}^3_{r=(2/3)\sqrt{21}}$, $\mathbb{R}\mathrm{P}^3_{\varkappa=3/4}$, $\big(\mathbb{S}^3_{r=1}\times\mathbb{S}^3_{r=1/\sqrt{3}}\big)/\mathbb{Z}_2$, $\mathrm{SU}(3)_{\mathrm{srr}=\sqrt{3}}$, $G_2/\mathrm{SO}(4)_{\mathrm{srr}=1}$

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