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DEGREES OF MAPS BETWEEN GRASSMANN MANIFOLDS

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Abstract

Let $f : \mathbb{G}_{n,k} \to \mathbb{G}_{m,l}$ be any continuous map between two *distinct* complex (resp. quaternionic) Grassmann manifolds of the same dimension. We show that the degree of f is zero provided n, m are sufficiently large and $l \geq 2$. If the degree of f is ± 1, we show that $(m, l) = (n, k)$ and f is a homotopy equivalence. Also, we prove that the image under f^* of every element of a set of algebra generators of $H^*(\mathbb{G}_{m,l}; \mathbb{Q})$ is determined up to a sign, $\pm$, by the degree of f, provided this degree is non-zero.

1. Introduction

The purpose of this paper is to study degrees of maps between two distinct complex (resp. quaternionic) Grassmann manifolds. It can be viewed as a continuation of the paper [14] where the case of oriented (real) Grassmann manifolds was settled completely. The same problem in the case of complex and quaternionic Grassmann manifolds was initiated and settled in [14] in half the cases. The problem can be formulated purely algebraically in terms of algebra homomorphism between the cohomology algebras of the complex Grassmann manifolds concerned. These algebras have additional structures, arising from Poincaré duality and the hard Lefschetz theorem. Our results are obtained by exploiting these properties. In view of the fact that the integral cohomology ring of a quaternionic Grassmann manifold is isomorphic to that of the corresponding complex Grassmann manifold via a degree doubling isomorphism, and since our proofs involve mostly analyzing the algebra-homomorphisms between the cohomology algebras of the Grassmann manifolds, we will only need to consider the case of complex Grassmann manifolds. (In the course of our proof of Theorem 1.3, simply-connectedness of the complex Grassmann manifold will be used; the same property also holds for the quaternionic Grassmann manifolds.) For this reason, we need only to consider the case of complex Grassmann manifolds.

Let $\mathbb{F}$ denote the field $\mathbb{C}$ of complex numbers or the skew-field $\mathbb{H}$ of quaternions. We denote by $\mathbb{F}\mathbb{G}_{n,k}$ the $\mathbb{F}$-Grassmann manifold of k-dimensional left $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$. Let $d := \dim_{\mathbb{R}} \mathbb{F}$. Since we will mostly deal with complex Grassmann manifolds, we shall write $\mathbb{G}_{n,k}$ instead of $\mathbb{C}\mathbb{G}_{n,k}$; the phrase 'Grassmann manifold', without further qualification, will always mean a complex Grassmann manifold.

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Using the usual 'hermitian' metric on $\mathbb{F}^n$, one obtains a diffeomorphism $\perp: \mathbb{F}\mathbb{G}_{n,k} \cong \mathbb{F}\mathbb{G}_{n,n-k}$. For this reason, it suffices to consider only those $\mathbb{F}$-Grassmann manifolds $\mathbb{F}\mathbb{G}_{n,k}$ with $1 \leq k \leq [n/2]$. Let $1 \leq l \leq [m/2]$ be another $\mathbb{F}$-Grassmann manifold having the same dimension as $\mathbb{F}\mathbb{G}_{n,k}$ so that $\dim_{\mathbb{F}} \mathbb{F}\mathbb{G}_{n,k} = k(n-k) = l(m-l) =: N$.

Complex Grassmann manifolds admit a natural orientation arising from the fact they have a natural complex structure. Although the quaternionic Grassmann manifolds do not admit even almost complex structures (cf. [11]), they are simply connected and hence orientable.

Let $f: \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map. It was observed in [14] that when $1 \leq k < l \leq [m/2]$, the degree of f is zero. When $l = 1$, one has $N = m - 1$ and $\mathbb{F}\mathbb{G}_{m,l}$ is just the $\mathbb{F}$-projective space $\mathbb{F}\mathbb{P}^N$. The set of homotopy classes of maps $f: \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{P}^N$ are in bijection with homomorphisms of abelian groups $\mathbb{Z} \cong H^d(\mathbb{F}\mathbb{P}^N; \mathbb{Z}) \to H^d(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$ where $d = \dim_{\mathbb{R}} \mathbb{F}$, via the induced homomorphism. Furthermore the degree of f is determined by $f^*: H^d(\mathbb{F}\mathbb{P}^N; \mathbb{Z}) \to H^d(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Z})$. (See [14] for details.)

We now state the main results of this paper.

Theorem 1.1. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ and let $d = \dim_{\mathbb{R}} \mathbb{F}$. Let $f: \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map between two $\mathbb{F}$-Grassmann manifolds of the same dimension. Then, there exist algebra generators $u_i \in H^{di}(\mathbb{F}\mathbb{G}_{m,l}; \mathbb{Q})$, $1 \leq i \leq l$, such that the image $f^*(u_i) \in H^{di}(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Q})$, $1 \leq i \leq l$, is determined up to a sign $\pm$ by the degree of f, provided this degree is non-zero.*

Theorem 1.2. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Fix integers $2 \leq l < k$. Let $m, n \geq 2k$ be positive integers such that $k(n-k) = l(m-l)$ and $f: \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{G}_{m,l}$ any continuous map. Then, degree of f is zero if $(l^2-1)(k^2-1)((m-l)^2-1)((n-k)^2-1)$ is not a perfect square. In particular, degree of f is zero for n sufficiently large.*

Theorem 1.3. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Suppose that $k(n-k) = l(m-l)$, and $1 \leq l \leq [m/2]$, $1 \leq k \leq [n/2]$. If $f: \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{G}_{m,l}$ is a map of degree ± 1, then $(m, l) = (n, k)$ and f is a homotopy equivalence.*

Our proofs make use of the notion of degrees of Schubert varieties, extended to cohomology classes. Theorem 1.3, which is an analogue in the topological realm of a result of K.H. Paranajape and V. Srinivas [13], is proved using Whitehead's theorem. Proof of Theorem 1.1 uses some properties of the cohomology of the complex Grassmann manifolds arising from Hodge theory. (See Proposition 3.2.) Theorem 1.2 is proved by reducing it to a diophantine problem and appealing to Siegel's Theorem on solutions of certain polynomial equation of the form $y^2 = F(x)$. In our situation, $F(x)$ will be of degree 4 over $\mathbb{Q}$ having distinct zeros.

We now highlight the following conjecture made in [14]. Theorem 1.2 provides the strongest evidence in support of the conjecture.

Conjecture. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ and let $2 \le l < k \le n/2 < m/2$ where $k, l, m, n \in \mathbb{N}$. Assume that $k(n-k) = l(m-l)$. Let $f\colon \mathbb{F}\mathbb{G}_{n,k} \to \mathbb{F}\mathbb{G}_{m,l}$ be* any *continuous map. The degree of f is zero.*

The paper is organized as follows. In §2 we recall basic and well-known facts concerning the cohomology algebra of the complex Grassmann manifolds. We shall consider continuous maps from a cohomologically Kähler manifold and establish some important properties in §3. They will be used in the course of our proofs. We prove the above theorems in §4.

2. Cohomology of Grassmann manifolds

There are at least two well-known descriptions of the cohomology ring of a complex Grassmann manifold $\mathbb{G}_{n,k}$. We recall both of them.

Let $\gamma_{n,k}$ be the 'tautological' bundle over $\mathbb{G}_{n,k}$ whose fibre over a point $V \in \mathbb{G}_{n,k}$ is the k-dimensional complex vector space V. Evidently $\gamma_{n,k}$ is a rank k-subbundle of the rank n trivial bundle $\mathcal{E}^n$ with projection $pr_1\colon \mathbb{G}_{n,k} \times \mathbb{C}^n \to \mathbb{G}_{n,k}$. The quotient bundle $\mathcal{E}^n/\gamma_{n,k}$ is isomorphic to the orthogonal complement $\gamma_{n,k}^{\perp}$ in $\mathcal{E}^n$ (with respect to a hermitian metric on $\mathbb{C}^n$) of the bundle $\gamma_{n,k}$. Let $c_i(\gamma_{n,k}) \in H^{2i}(\mathbb{G}_{n,k}; \mathbb{Z})$, be the i-th Chern class of $\gamma_{n,k}$, $1 \le i \le k$. Denoting the total Chern class of a vector bundle η by $c(\eta)$ we see that $c(\gamma_{n,k}) \,.\, c(\gamma_{n,k}^{\perp}) = 1$.

Let $c_1, \ldots, c_k$ denote the elementary symmetric polynomials in k indeterminates $x_1, \ldots, x_k$. Define $h_j = h_j(c_1, \ldots, c_k)$ by the identity

$$\prod_{1 \le i \le k} (1 + x_i t)^{-1} = \sum_{j \ge 0} h_j t^j.$$

Thus $c_j(\gamma_{n,k}^{\perp}) = h_j(c_1(\gamma_{n,k}), c_2(\gamma_{n,k}), \ldots, c_k(\gamma_{n,k}))$, $1 \le j \le n-k$. (See [12].)

Consider the ring $\mathbb{Z}[c_1, \ldots, c_k]/\mathcal{I}_{n,k}$ where degree of $c_i = 2i$, and $\mathcal{I}_{n,k}$ is the ideal $\langle h_j \mid j > n-k \rangle$. It can be shown that the elements h_j, $n-k+1 \le j \le n$, generate $\mathcal{I}_{n,k}$. The homomorphism of graded rings $\mathbb{Z}[c_1, \ldots, c_k] \to H^*(\mathbb{G}_{n,k}; \mathbb{Z})$ defined by $c_i \mapsto c_i(\gamma_{n,k})$ is surjective and has kernel $\mathcal{I}_{n,k}$ and hence we have an isomorphism $H^*(\mathbb{G}_{n,k}; \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k]/\mathcal{I}_{n,k}$. Henceforth we shall write c_i to mean $c_i(\gamma_{n,k}) \in H^*(\mathbb{G}_{n,k}; \mathbb{Z})$. We shall denote by $\overline{c}_j$ the element $c_j(\gamma_{n,k}^{\perp}) = h_j \in H^{2j}(\mathbb{G}_{n,k}; \mathbb{Z})$.

As an abelian group, $H^*(\mathbb{G}_{n,k}; \mathbb{Z})$ is free of rank $\binom{n}{k}$. A $\mathbb{Q}$-basis for $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$ is the set $\mathcal{C}_r$ of all monomials $c_1^{j_1} \cdots c_k^{j_k}$ where $j_i \le n-k$, $\forall i$, $\sum_{1 \le i \le k} i j_i = r$. In particular, c_k^{n-k} generates $H^{2N}(\mathbb{G}_{n,k}; \mathbb{Q}) \cong \mathbb{Q}$. If $\mathbf{j}$ denotes the sequence $j_1, \ldots, j_k$, we shall denote by $c^{\mathbf{j}}$ the monomial $c_1^{j_1} \cdots c_k^{j_k}$. If $k \le n/2$, the set $\overline{\mathcal{C}}_r := \{\overline{c}^{\mathbf{j}} \mid c^{\mathbf{j}} \in \mathcal{C}_r\}$ is also a basis for $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$ where $\overline{c}^{\mathbf{j}} := \overline{c}_1^{j_1} \cdots \overline{c}_k^{j_k}$.

Schubert calculus. Another, more classical description of the cohomology ring of the Grassmann manifold $\mathbb{G}_{n,k}$ is via the Schubert calculus. Recall that $\mathbb{G}_{n,k} = \mathrm{SL}(n, \mathbb{C})/P_k$

for the parabolic subgroup $P_k \subset \mathrm{SL}(n, \mathbb{C})$ which stabilizes $\mathbb{C}^k \subset \mathbb{C}^n$ spanned by $e_1, \ldots, e_k$; here e_i, $1 \le i \le n$, are the standard basis elements of $\mathbb{C}^n$. Denote by $B \subset \mathrm{SL}(n, \mathbb{C})$ the Borel subgroup of $\mathrm{SL}(n, \mathbb{C})$ which preserves the flag $\mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n$ and by $T \subset B$ the maximal torus which preserves the coordinate axes $\mathbb{C}e_j$, $1 \le j \le n$. Let $I(n, k)$ denote the set of all k element subsets of $\{1, 2, \ldots, n\}$; we regard elements of $I(n, k)$ as increasing sequences of positive integers $\mathbf{i} := i_1 < \cdots < i_k$ where $i_k \le n$. One has a partial order on $I(n, k)$ where, by definition, $\mathbf{i} \le \mathbf{j}$ if $i_p \le j_p$ for all p, $1 \le p \le k$. Let $\mathbf{i} \in I(n, k)$ and let $E_\mathbf{i} \in \mathbb{G}_{n,k}$ denote the vector subspace of $\mathbb{C}^n$ spanned by $\{e_j \mid j \in \mathbf{i}\}$. The fixed points for the action of $T \subset \mathrm{SL}(n)$ on $\mathbb{G}_{n,k}$ are precisely the $E_\mathbf{i}$, $\mathbf{i} \in I(n, k)$.

Schubert varieties in $\mathbb{G}_{n,k}$ are in bijection with the set $I(n, k)$. The B-orbit of the T-fixed point $E_\mathbf{i}$ is the Schubert cell corresponding to $\mathbf{i}$ and is isomorphic to the affine space of (complex) dimension $\sum_j (i_j - j) =: |\mathbf{i}|$; its closure, denoted $\Omega_\mathbf{i}$, is the Schubert variety corresponding to $\mathbf{i} \in I(n, k)$. It is the union of all Schubert cells corresponding to those $\mathbf{j} \in I(n, k)$ such that $\mathbf{j} \le \mathbf{i}$. Schubert cells yield a cell decomposition of $\mathbb{G}_{n,k}$. Since the cells have even (real) dimension, the class of Schubert varieties form a $\mathbb{Z}$-basis for the integral homology of $\mathbb{G}_{n,k}$. Denote by $[\Omega_\mathbf{i}] \in H^{2(N-|\mathbf{i}|)}(\mathbb{G}_{n,k}; \mathbb{Z})$ the fundamental *dual* cohomology class determined by $\Omega_\mathbf{i}$. (Thus $[\mathbb{G}_{n,k}] \in H^0(\mathbb{G}_{n,k}; \mathbb{Z})$ is the identity element of the cohomology ring.) We shall denote the fundamental homology class of $\mathbb{G}_{n,k}$ by $\mu_{n,k} \in H^{2N}(\mathbb{G}_{n,k}; \mathbb{Z})$.

Schubert varieties corresponding to $(n - k + 1 - i, n - k + 2, \ldots, n) \in I(n, k)$, $0 \le i \le n - k$, are called *special* and will be denoted Ω_i. More generally, if $\nu = \nu_1 \ge \cdots \ge \nu_k \ge 0$ is a partition of an integer r, $0 \le r \le N$, with $\nu_1 \le n - k$, we obtain an element $\mathbf{i} := (n - k + 1 - \nu_1, n - k + 2 - \nu_2, \ldots, n - \nu_k) \in I(n, k)$ with $|\mathbf{i}| = N - r$. This association establishes a bijection between such partitions and $I(n, k)$, or, equivalently, the Schubert varieties $\Omega_\mathbf{i}$ in $\mathbb{G}_{n,k}$. It is sometimes convenient to denote the Schubert variety $\Omega_\mathbf{i}$ by Ω_ν where ν corresponds to $\mathbf{i}$. This is consistent with our notation for a special Schubert variety.

The special Schubert classes form a set of algebra generators of $H^*(\mathbb{G}_{n,k}; \mathbb{Z})$. Indeed, $[\Omega_i] = c_i(\gamma_{n,k}^\perp) = \overline{c_i}$, $1 \le i \le n - k$. The structure constants are determined by (i) the Pieri formula, which expresses the cup-product of an arbitrary Schubert class with a special Schubert class as a linear combination of with non-negative integral linear combination of Schubert classes, and, (ii) the Giambelli formula, which expresses an arbitrary Schubert class as a determinant in the special Schubert classes [2, Chapter 14].

The basis $\{[\Omega_\mathbf{i}] \mid \mathbf{i} \in I(n, k)\}$ is 'self-dual' under the Poincaré duality. That is, assume that $\mathbf{i}, \mathbf{j} \in I(n, k)$ are such that $|\mathbf{i}| + |\mathbf{j}| = N$. Then

$$\langle [\Omega_\mathbf{i}][\Omega_\mathbf{j}], \mu_{n,k} \rangle = \delta_{\mathbf{i}',\mathbf{j}},$$

where $\mathbf{i}' = (n + 1 - i_k, \ldots, n + 1 - i_1) \in I(n, k)$.

The *degree* of a Schubert variety $\Omega_{\mathbf{i}}$ of (complex) dimension r is defined as the integer $\langle [\Omega_{\mathbf{i}}]\bar{c}_1^r, \mu_{n,k}\rangle \in \mathbb{Z}$. It is well-known [8], [2] that

$$\deg(\Omega_{\mathbf{i}}) = \frac{r!\prod_{1\le t<s\le k}(i_s - i_t)}{(i_1-1)!\cdots(i_k-1)!}. \tag{1}$$

In particular

$$\deg(\mathbb{G}_{n,k}) = \langle \bar{c}_1^N, \mu_{n,k}\rangle = \frac{N!\,1!\cdots(k-1)!}{(n-k)!\cdots(n-1)!}. \tag{2}$$

More generally, $\deg([\Omega_{\mathbf{i}}][\Omega_{\mathbf{j}}]) := \langle [\Omega_{\mathbf{i}}][\Omega_{\mathbf{j}}]\bar{c}_1^q, \mu_{n,k}\rangle = q!\,|1/(i_r + j_{k+1-j} - n - 1)!|$ where $q = \dim(\Omega_{\mathbf{i}}) + \dim(\Omega_{\mathbf{j}}) - \dim \mathbb{G}_{n,k}$. (See [2, p. 274]. We caution the reader that our notations for Grassmann manifolds and Schubert varieties are different from those used in Fulton's book [2].)

One has the following geometric interpretation for the degree of a Schubert variety. More generally, given any algebraic imbedding $X \hookrightarrow \mathbb{P}^m$ of a projective variety X of dimension d in the complex projective space $\mathbb{P}^m$, the *degree* of X is the number of points in the intersection of X with d hyperplanes in general position. The degree of a Schubert variety defined above is the degree of the Plücker imbedding $\Omega_{\mathbf{j}} \subset \mathbb{G}_{n,k} \hookrightarrow \mathbb{P}(\Lambda^k(\mathbb{C}^n))$, defined as $U \mapsto \Lambda^k(U)$, where $\Lambda^k(U)$ denotes the k-th exterior power of the vector space U.

Cohomology of quaternionic Grassmann manifolds. In the case of quaternionic Grassmann manifold $\mathbb{H}\mathbb{G}_{n,k}$, one has a Schubert cell decomposition with cells only in dimensions $4j$, $0 \le j \le N$, labeled by the same set $I(n,k)$ as in the case of the complex Grassmann manifold $\mathbb{C}\mathbb{G}_{n,k}$. Furthermore, denoting the quaternionic Schubert variety corresponding to $\mathbf{i} \in I(n,k)$ by $\Omega_{\mathbf{i}}^{\mathbb{H}}$, the structure constants defining the integral cohomology algebra of $\mathbb{H}\mathbb{G}_{n,k}$ for the basis $\{\Omega_{\mathbf{i}}^{\mathbb{H}}\}$ are identical to those in the case of $\mathbb{C}\mathbb{G}_{n,k}$. Thus, the association $[\Omega_{\mathbf{i}}] \mapsto [\Omega_{\mathbf{i}}^{\mathbb{H}}]$ defines an isomorphism of *rings* $H^*(\mathbb{C}\mathbb{G}_{n,k};\mathbb{Z}) \to H^*(\mathbb{H}\mathbb{G}_{n,k};\mathbb{Z})$ which doubles the degree. In particular one has the *identical* formula, namely (1), for the degrees of quaternionic Schubert classes. The orientation on $\mathbb{H}\mathbb{G}_{n,k}$ is chosen so that the image of the positive generator of $H^{2N}(\mathbb{C}\mathbb{G}_{n,k};\mathbb{Z})$ under the above isomorphism is positive.

3. Maps from cohomologically Kähler manifolds

In this section the symbol d will have a different meaning from what it did in §1.

Let $f: X \to Y$ be any continuous map between two compact connected oriented manifolds of the same dimension. It is well-known that if f^* has non-zero degree, then the induced map $f^*: H^r(Y;\mathbb{Z}) \to H^r(X;\mathbb{Z})$ is split-injective for all r. In particular, $f^*: H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is a monomorphism of *rings*.

Recall that a compact connected orientable smooth manifold X is called *c-symplectic* (or cohomologically symplectic) if there exists an element $\omega \in H^2(X;\mathbb{R})$, called a

c-symplectic class, such that $\omega^d \in H^{2d}(X;\mathbb{R}) \cong \mathbb{R}$ is non-zero where $d = (1/2)\dim_{\mathbb{R}} X$. If ω is a c-symplectic class in X, then (X, ω) is said to satisfy the *weak Lefschetz* (respectively *hard Lefschetz*) condition if $\cup\omega^{d-1}\colon H^1(X;\mathbb{R}) \to H^{2d-1}(X;\mathbb{R})$ (respectively $\cup\omega^i\colon H^{d-i}(X;\mathbb{R}) \to H^{d+i}(X;\mathbb{R})$, $1 \le i \le d$) is an isomorphism. If (X, ω) satisfies the hard Lefschetz condition, then X is called c-Kähler or cohomologically Kähler. If (X,ω) is c-Kähler, and if ω is in the image of the natural map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$, we call X c-Hodge. Note that if (X, ω) is c-Kähler and if $H^2(X;\mathbb{R}) \cong \mathbb{R}$, then $(X, t\omega)$ is c-Hodge for some $t \in \mathbb{R}$.

Clearly Kähler manifolds are c-Kähler and smooth projective varieties over $\mathbb{C}$ are c-Hodge. It is known that $\mathbb{P}^2\#\mathbb{P}^2$ is c-symplectic but not symplectic (hence not Kähler) since it is known that it does not admit even an almost complex structure. It is also c-Kähler. Examples of c-symplectic manifolds which satisfy the weak Lefschetz condition but not c-Kähler are also known (cf. [10]).

Any c-symplectic manifold (X, ω) is naturally oriented; the fundamental class of X will be denoted by $\mu_X \in H_{2d}(X;\mathbb{Z}) \cong \mathbb{Z}$.

Let (X, ω) be a c-Kähler manifold of dimension $2d$. Let $1 \le r \le d$. One has a bilinear form $(\cdot,\cdot)_\omega$ (or simply $(\cdot,\cdot)$ when there is no danger of confusion) on $H^r(X;\mathbb{R})$ defined as $(\alpha, \beta)_\omega = \langle \alpha\beta\omega^{d-r}, \mu_X\rangle$, $\alpha, \beta \in H^r(X;\mathbb{R})$. When (X, ω) is c-Hodge, the above form is rational, that is, it restricts to a bilinear form $H^r(X;\mathbb{Q}) \times H^r(X;\mathbb{Q}) \to \mathbb{Q}$. It will be important for us to consider the bilinear form on the rational vector space $H^r(X;\mathbb{Q})$ rather than on the real vector space $H^r(X;\mathbb{R})$. The bilinear form $(\,\cdot\,,\,\cdot\,)$ is symmetric (resp. skew symmetric) if r is even (resp. odd). Note that the above form is non-degenerate for all r. This follows from Poincaré duality and the hard Lefschetz condition that $\beta \mapsto \beta \cup \omega^{d-r}$ is an isomorphism $H^r(X;\mathbb{Q}) \to H^{2d-r}(X;\mathbb{Q})$. Further, if $r \le d$, the monomorphism $\cup\omega\colon H^{r-2}(X;\mathbb{Q}) \to H^r(X;\mathbb{Q})$ is an isometric imbedding, i.e., $(\alpha, \beta) = (\alpha\omega, \beta\omega)$ for all $\alpha, \beta \in H^{r-2}(X;\mathbb{Q})$.

As in the case of Kähler manifolds (cf. [7], [16], [6]), one obtains an orthogonal decomposition of the real cohomology groups of a c-Kähler manifold (X, ω). The decomposition, which preserves the rational structure when (X,ω) is c-Hodge, is obtained as follows: Let $1 \le r \le d$. Let $\mathcal{V}^r_\omega$, or more briefly $\mathcal{V}^r$ when ω is clear from the context, be the kernel of the homomorphism $\cup\omega^{d-r+1}\colon H^r(X;\mathbb{R}) \to H^{2d-r+2}(X;\mathbb{R})$. An element of $\mathcal{V}^r$ will be called a *primitive class*. One has the *Lefschetz decomposition*

$$H^r(X;\mathbb{R}) = \bigoplus_{0 \le q \le [r/2]} \omega^q \mathcal{V}^{r-2q}. \tag{3}$$

We have the following lemma.

Lemma 3.1. *Suppose that (X, ω) is a c-Hodge manifold of dimension $2d$ with second Betti number equal to 1. Let $f\colon X \to Y$ be any continuous map of non-zero degree where Y is a compact manifold with non-vanishing second Betti number. Then*:
(i) $(\,\cdot\,,\,\cdot\,)_{t\omega} = t^{d-r}(\,\cdot\,,\,\cdot\,)_\omega$ *on* $H^r(X;\mathbb{Q})$ *for* $t \in \mathbb{Q}$, $t \neq 0$.

(ii) (Y, φ) *is c-Hodge where* $\varphi \in H^2(Y;\mathbb{Q})$ *is the unique class such that* $f^*(\varphi) = \omega$. *Furthermore,* f^* *preserves the Lefschetz decomposition* (3), *that is,* $f^*(\mathcal{V}^r_\varphi) \subset \mathcal{V}^r_\omega$ *for* $r \le d$.
(iii) *If* $\alpha, \beta \in H^r(Y;\mathbb{Q})$, *then* $(f^*(\alpha), f^*(\beta))_\omega = \deg(f)(\alpha,\beta)_\varphi$. *In particular, degree of* f *equals* $\langle \omega^d, \mu_X\rangle / \langle \varphi^d, \mu_Y\rangle$.

Proof. (i) This is trivial.

(ii) Let $\dim(X) = 2d$. Since $\deg(f) \ne 0$, $f^*\colon H^i(Y;\mathbb{Q}) \to H^i(X;\mathbb{Q})$ is a monomorphism for all $i \le 2d$. Comparing the second Betti numbers of X and Y we conclude that $f^*\colon H^2(Y;\mathbb{Q}) \to H^2(X;\mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism. Let $\varphi \in H^2(Y;\mathbb{Q})$ be the unique class such that $f^*(\varphi) = \omega$. Since f^* is a homomorphism of *rings*, we have $0 \ne \omega^d = (f^*(\varphi))^d = f^*(\varphi^d)$ and so $\varphi^d \ne 0$.

Let $r \le d$ be a positive integer. One has a commuting diagram:

$$\begin{array}{ccc} H^r(Y;\mathbb{Q}) & \xrightarrow{\cup\varphi^{d-r}} & H^{2d-r}(Y;\mathbb{Q}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^r(X;\mathbb{Q}) & \xrightarrow{\cup\omega^{d-r}} & H^{2d-r}(X;\mathbb{Q}). \end{array}$$

The vertical maps are monomorphisms since $\deg(f) \ne 0$. By our hypothesis on X, the homomorphism $\cup\omega^{d-r}$ in the above diagram is an isomorphism. This implies that $\cup\varphi^{d-r}$ is a monomorphism. Since, by Poincaré duality, the vector spaces $H^r(Y;\mathbb{Q})$ and $H^{2d-r}(Y;\mathbb{Q})$ have the same dimension, $\cup\varphi^{d-r}$ is an *isomorphism* and so $(Y;\varphi)$ is c-Hodge. It is clear that $f^*(\mathcal{V}^r_\varphi) \subset \mathcal{V}^r_\omega$.

(iii) Suppose that $\alpha, \beta \in H^r(Y;\mathbb{R})$. Then

$$\begin{aligned} (f^*(\alpha), f^*(\beta))_\omega &= \langle f^*(\alpha) f^*(\beta)\omega^{d-r}; \mu_X\rangle \\ &= \langle f^*(\alpha\beta) f^*(\varphi^{d-r}); \mu_X\rangle \\ &= \langle f^*(\alpha\beta\varphi^{d-r}); \mu_X\rangle \\ &= \langle \alpha\beta\varphi^{d-r}, f_*(\mu_X)\rangle \\ &= \deg(f)\langle \alpha\beta\varphi^{d-r}; \mu_Y\rangle \\ &= \deg(f)(\alpha,\beta)_\varphi. \end{aligned}$$

The formula for the degree of f follows from what has just been established by taking $\alpha = \beta = \varphi$. □

Observe that the summands in the Lefschetz decomposition (3) are mutually orthogonal with respect to the bilinear form $(\cdot,\cdot)$. Indeed, let $\alpha \in \mathcal{V}^{r-2p}$, $\beta \in \mathcal{V}^{r-2q}$, $p < q$. Thus $\alpha\omega^{n-r+2p+1} = 0$ and so $\alpha\omega^{n-r+p+q} = 0$. Therefore $(\omega^p\alpha, \omega^q\beta) = \langle \alpha\beta\omega^{n-r+p+q}, \mu_X\rangle = 0$. As observed earlier the form $(\cdot,\cdot)$ is non-degenerate. It follows that the form restricted to each summand in (3) is non-degenerate. In favourable situations, the form is either positive or negative definite as we shall see in Proposition 3.2 below.

We shall recall some basic results from Hodge theory and use several facts concerning harmonic forms, all of which can be found in [6, §15]. They will be needed in the proof of Proposition 3.2.

Suppose that X has been endowed with a Kähler metric with Kähler class $\omega \in H^2(X;\mathbb{R})$. Recall that one has the decomposition $H^r(X;\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X;\mathbb{C})$ where $H^{p,q}$ denotes the $\bar{\partial}$-cohomology. We identify the $H^{p,q}(X;\mathbb{C})$ with the space of harmonic forms (with respect to the Kähler metric) $B^{p,q}$ of type (p,q).

We shall follow the notations used in [6, §15.8]. One has the operators L and Λ on $H^{p,q}(X;\mathbb{C})$ where $L\colon H^{p,q}(X;\mathbb{C}) \to H^{p+1,q+1}(X;\mathbb{C})$ equals wedging with the Kähler class ω and $\Lambda\colon H^{p,q}(X;\mathbb{C}) \cong B^{p,q} \to B^{p-1,q-1} \cong H^{p-1,q-1}(X;\mathbb{C})$ is the operator $(-1)^{p+q}\#L\#$ on $B^{p,q}(X;\mathbb{C})$. The operator Λ is dual to L with respect to the hermitian scalar product denoted $(\cdot\,,\,\cdot)_*$:

$$(\alpha,\beta)_* := \int_X \alpha \wedge \#\beta \tag{4}$$

on $H^r(X;\mathbb{C}) = \bigoplus_{p+q=r} B^{p,q}$.

The kernel of Λ is denoted by $B_0^{p,q}$. One has the Hodge decomposition

$$H^{p,q}(X) = \bigoplus_{0\le k\le \min\{p,q\}} B_k^{p,q} \tag{5}$$

where $B_k^{p,q} := L^k(B_0^{p-k,q-k})$ is the space of all harmonic forms φ of type (p,q) and class k. Then the distinct summands in (5) are pairwise orthogonal with respect to $(\cdot\,,\,\cdot)_*$. Also, ΛL^k is a non-zero scalar multiple of L^{k-1} on $B_0^{p-k,q-k}$ for $p+q \le d$, $1 \le k \le \min\{p,q\}$.

Proposition 3.2. *Suppose that (X,ω) is a compact connected Kähler manifold such that $H^{p,q}(X;\mathbb{C}) = 0$ for $p \ne q$. Then the form $(-1)^{q+r}(\cdot\,,\,\cdot)_\omega$ restricted to $\omega^q\mathcal{V}^{2r-2q} \subset H^{2r}(X;\mathbb{R})$ is positive definite for $0 \le q \le r$, $1 \le r \le [d/2]$.*

Proof. First assume that $d = \dim_{\mathbb{C}} X$ is even, say $d = 2s$. In view of our hypothesis, all odd Betti numbers of X vanish and we have $B_k^{p,q} = 0$ for all $p \ne q$, $k \ge 0$, so that

$$H^{2r}(X;\mathbb{C}) = H^{r,r}(X;\mathbb{C}) = \bigoplus_{0\le k\le r} B_k^{r,r}. \tag{6}$$

The real cohomology group $H^{2r}(X;\mathbb{R}) \subset H^{2r}(X;\mathbb{C}) = H^{r,r}(X;\mathbb{C})$ has an orthogonal decomposition induced from (3):

$$H^{2r}(X;\mathbb{R}) = \bigoplus_{0\le k\le s} E_k^{r,r} \tag{7}$$

where $E_k^{p,p} = \{\alpha \in B_k^{p,p} \mid \alpha = \overline{\alpha}\}$. Now taking $r = s = d/2$ one has $\#\alpha = (-1)^{s+k}\alpha$ for $\alpha \in E_k^{s,s}$. In particular the bilinear form (4) equals $(-1)^{s+k}Q$ where $Q(\alpha,\beta) = \int_X \alpha\beta$. Therefore $(-1)^{s+k}Q$ restricted to each $E_k^{s,s}$ is positive definite.

We shall show in Lemma 3.3 below that $\omega^k\mathcal{V}^{d-2k} = E_k^{s,s}$. The proposition follows immediately from this since $(\alpha,\beta) = (\omega^{s-r}\alpha, \omega^{s-r}\beta)$ for $\alpha,\beta \in \omega^k\mathcal{V}^{2r-2k}$ as $d = 2s$, completing the proof in this case.

Now suppose that d is odd. Consider the Kähler manifold $Y = X \times \mathbb{P}^1$ where we put the Fubini-Study metric on $\mathbb{P}^1$ with Kähler class η being the 'positive' generator of $H^2(\mathbb{P}^1;\mathbb{Z}) \subset H^2(\mathbb{P}^1;\mathbb{R})$ and the product structure on Y so that the Kähler class of Y equals $\omega + \eta =: \varphi$. By Künneth theorem $H^*(Y;\mathbb{R}) = H^*(X;\mathbb{R}) \otimes H^*(\mathbb{P}^1;\mathbb{R})$. We shall identify the cohomology groups of X and $\mathbb{P}^1$ with their images in $H^*(Y;\mathbb{R})$ via the monomorphisms induced by the first and second projection respectively. Under these identifications, $H^{p,q}(Y;\mathbb{C}) = H^{p,q}(X;\mathbb{C}) \oplus H^{p-1,q-1}(X;\mathbb{C}) \otimes H^{1,1}(\mathbb{P}^1;\mathbb{C})$. In particular, $H^{p,q}(Y;\mathbb{C}) = 0$ unless $p = q$. By what has been proven already, the form $(-1)^{r+k}(\cdot,\cdot)$ is positive definite on $\varphi^k\mathcal{V}_\varphi^{2r-2k} \subset H^{2r}(Y;\mathbb{R})$.

Choose a base point in $\mathbb{P}^1$ and consider the inclusion map $j\colon X \hookrightarrow Y$. The imbedding j is dual to η. Also $j^*(\varphi) = \omega$. It follows that $j^*(\varphi^k\mathcal{V}_\varphi^{2r-2k}) \subset \omega^k\mathcal{V}_\omega^{2r-2k}$ for $0 \le k < r$, $1 \le r < d$. Since the kernel of $j^*\colon H^{2r}(Y;\mathbb{R}) \to H^{2r}(X;\mathbb{R})$ equals $H^{2r-2}(X;\mathbb{R}) \otimes H^2(\mathbb{P}^1;\mathbb{R})$, and maps $H^{2r}(X;\mathbb{R}) \subset H^{2r}(Y;\mathbb{R})$ isomorphically onto $H^{2r}(X;\mathbb{R})$, we must have $j^*(\varphi^k\mathcal{V}_\varphi^{2r-2k}) = \omega^k\mathcal{V}_\omega^{2r-2k}$.

Let $\alpha,\beta \in H^{2r}(X;\mathbb{R}) \subset H^{2r}(Y;\mathbb{R})$. Since $j\colon X \hookrightarrow Y$ is dual to η, we have $j_*(\mu_X) = \eta \cap \mu_Y$. Therefore,

$$\begin{aligned}
(j^*(\alpha), j^*(\beta))_\omega &= \langle j^*(\alpha\beta) j^*(\omega)^{d-2r}; \mu_X\rangle \\
&= \langle \alpha\beta\omega^{d-2r}, j_*(\mu_X)\rangle \\
&= \langle \alpha\beta\omega^{d-2r}, \eta \cap \mu_Y\rangle \\
&= \langle \alpha\beta\omega^{d-2r}\eta, \mu_Y\rangle.
\end{aligned}$$

Since $\eta^2 = 0$ we have $\varphi^{d-2r+1} = \omega^{d-2r+1} + (d-2r+1)\omega^{d-2r}\eta$. Furthermore, $\alpha\beta\omega^{d-2r+1} \in H^{2d+2}(X;\mathbb{R}) = 0$. Therefore, we conclude that $(j^*(\alpha), j^*(\beta))_\omega = (1/(d-2r+1))\langle\alpha\beta\varphi^{d-2r+1}, \mu_Y\rangle = (1/(d-2r+1))(\alpha,\beta)_\varphi$. This shows that the bilinear form $(\cdot\,,\,\cdot)_\omega$ on $H^{2r}(X;\mathbb{R})$ is a positive multiple of the form $(\cdot\,,\,\cdot)_\varphi$ on $H^{2r}(Y;\mathbb{R})$ restricted to $H^{2r}(X;\mathbb{R})$. It follows that the bilinear form $(-1)^{r+k}(\cdot\,,\,\cdot)$ on $H^{2r}(X;\mathbb{R})$ restricted to $\omega^k\mathcal{V}^{2r-2k}(X)$ is positive definite. □

We must now establish the following

Lemma 3.3. *With notations as above, assume that $d = 2s$ is even. Under the hypothesis of the above proposition, $E_k^{s-k,s-k}$ equals $\omega^k\mathcal{V}^{d-2k}$, $0 \leq k \leq s$.*

Proof. Since L preserves real forms, it suffices to show that $E_0^{r,r} = \mathcal{V}^{2r}$ when $r \leq s$. By definition $E_0^{r,r} = B_0^{r,r} \cap H^{2r}(X;\mathbb{R}) = \{\alpha \in H^{r,r}(X;\mathbb{C}) \mid \Lambda(\alpha) = 0,\ \alpha = \overline{\alpha}\}$.

Let $\alpha \in E_0^{r,r}$. Suppose that $p \geq 1$ is the largest integer such that $\omega^{d-2r+p}\alpha =: \theta$ is a *non-zero* real harmonic form of type $(d-r+p, d-r+p)$. Since $L^{d-2r+2p}\colon H^{r-p,r-p}(X;\mathbb{C}) \to H^{d-r+p,d-r+p}(X;\mathbb{C})$ is an isomorphism, and since ω is real there must be a real form $\beta \in H^{r-p,r-p}(X;\mathbb{R})$ such that $L^{d-2r+2p}(\beta) = \theta = L^{d-2r+p}(\alpha)$. Since p is the largest, using the decomposition (6) we see that $\beta \in B_0^{r-p,r-p}$. Applying Λ^{d-2r+p} both sides and (repeatedly) using $\Lambda L^q\beta$ is a *non-zero* multiple of $L^{q-1}\beta$ when $r-p+q < d$ we see that β is a non-zero multiple of $\Lambda^p\alpha = 0$. Thus $\beta = 0$ and hence $\theta = 0$, which contradicts our assumption. Therefore $L^{d-2r+1}(\alpha) = 0$ and so $\alpha \in \mathcal{V}_0^r$. On the other hand Λ maps $H^{2r}(X;\mathbb{C})$ onto $H^{2r-2}(X;\mathbb{C})$. A dimension argument shows that $E_0^{r,r} = \mathcal{V}^{2r}$. $\square$

EXAMPLE 3.4. The Grassmann manifold $\mathbb{G}_{n,k}$ has the structure of a Kähler manifold with Kähler class $\omega := \overline{c}_1 = [\Omega_1] \in H^2(\mathbb{G}_{n,k};\mathbb{Z})$. (This fact follows, for example, from the Plücker imbedding $\mathbb{G}_{n,k} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$.) The bilinear form $(\,\cdot\,,\,\cdot\,)$ is understood to be defined with respect to ω. An orthogonal basis for $\mathcal{V}_{n,k}^{2r} \subset H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$ can be obtained inductively using Gram-Schmidt orthogonalization process as follows. Recall from §2 the basis $\overline{\mathcal{C}}_r$ for $H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$. Clearly $\omega \cdot \overline{\mathcal{C}}_{r-1} = \overline{c}_1 \cdot \overline{\mathcal{C}}_{r-1} = \{\overline{c}^{\mathbf{j}} \in \overline{\mathcal{C}}_r \mid j_1 > 0\}$ is a basis for $\omega H^{2r-2}(\mathbb{G}_{n,k};\mathbb{Q})$. Therefore we see that the subspace spanned by $\overline{\mathcal{C}}_{r,0} := \{\overline{c}^{\mathbf{j}} \in \overline{\mathcal{C}}_r \mid j_1 = 0\}$ is complementary to $\bigoplus_{q>0} B_q^{r-q,r-q} \subset H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$. The required basis is obtained by taking the orthogonal projection of $\overline{\mathcal{C}}_{r,0}$ onto $\mathcal{V}^{2r}$. Indeed, inductively assume that an orthogonal basis $\{v_{\mathbf{j}}\}$ for $\omega H^{2r-2}(\mathbb{G}_{n,k};\mathbb{Q})$ that is compatible with the direct sum decomposition $\bigoplus_{q>0} B_q^{r-q,r-q}$ has been constructed. We need only apply the orthogonalization process to the (ordered) set $\{v_{\mathbf{j}}\} \cup \{\overline{c}^{\mathbf{j}} \in \overline{\mathcal{C}}_r \mid j_1 = 0\}$ with respect to an ordering of $\mathcal{C}_{r,0}$ where $\overline{c}_r$ is the *last* element. To be specific, we list the elements $\overline{c}^{\mathbf{j}}$ in the *decreasing* order with respect to the lexicographic order of the exponents. (For example, taking $n = 12$, $k = 6$, $r = 6$, the elements of $\overline{\mathcal{C}}_{6,0}$ are ordered as $\overline{c}_2^3$, $\overline{c}_2\overline{c}_4$, $\overline{c}_3^2$, $\overline{c}_6$.) We denote the basis element of $\mathcal{V}^{2r}$ obtained from $c^{\mathbf{j}} \in \overline{\mathcal{C}}_{r,0}$ by $v_{\mathbf{j}}$. Note that when $r \leq k$, the span of the set $\{v_{\mathbf{j}} \mid j_r = 0\} \subset H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$ equals the space $\mathcal{D}$ of all decomposable elements in $H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$ since, according to our assumption on the ordering of elements $\overline{c}^{\mathbf{j}}$, the element $\overline{c}_r$ is the *last* to occur and so v_r does not occur in any other $v_{\mathbf{j}}$. Thus $v_r - \overline{c}_r$ belongs to $\mathcal{D} \subset H^{2r}(\mathbb{G}_{n,k};\mathbb{Q})$ and $v_{\mathbf{j}} \in \mathcal{D}$ for all other $\mathbf{j}$, $|\mathbf{j}| = r$.

We illustrate this for $r = 2, 3$. (When $r = 1$, $\mathcal{V}^2 = 0$.) The element $v_2 = \overline{c}_2 - ((\overline{c}_2, \omega^2)/(\omega,\omega))\omega^2 = \overline{c}_2 - (\deg \overline{c}_2/\deg \mathbb{G}_{n,k})\omega^2 \in H^4(\mathbb{G}_{n,k};\mathbb{Q})$ is a basis for the one-dimensional space $\mathcal{V}^4$.

Similarly, v_3 is a basis for $\mathcal{V}^6$ where

$$\begin{aligned} v_3 &:= \bar{c}_3 - \frac{(\bar{c}_3,\, v_2\omega)}{(v_2\omega,\, v_2\omega)} v_2\omega - \frac{(\bar{c}_3,\, \omega^3)}{(\omega^3,\, \omega^3)}\omega^3 \\ &= \bar{c}_3 - \frac{\deg \bar{c}_3}{\deg \mathbb{G}_{n,k}}\omega^3 - \frac{\deg \mathbb{G}_{n,k} \deg(\bar{c}_3\bar{c}_2) - \deg \bar{c}_2 \deg \bar{c}_3}{\deg \mathbb{G}_{n,k} \deg(\bar{c}_2^2) - (\deg \bar{c}_2)^2} v_2\omega. \end{aligned}$$

This leads to

$$(v_3,\, v_3) = (v_3,\, \bar{c}_3) = \deg(\bar{c}_3^2) - \frac{(\deg \bar{c}_3)^2}{\deg \mathbb{G}_{n,k}} - \frac{\deg(\bar{c}_3\bar{c}_2) \deg \mathbb{G}_{n,k} - \deg \bar{c}_2 \deg \bar{c}_3}{\deg \mathbb{G}_{n,k} \deg(\bar{c}_2^2) - (\deg \bar{c}_2)^2} \deg(\bar{c}_3 v_2).$$

The following calculation will be used in the course of the proof of Theorem 1.2.

Lemma 3.5. *With the above notation,* $(v_2, v_2) = (\deg \mathbb{G}_{n,k})(k^2-1)((n-k)^2-1)/(2(N-1)^2(N-2)(N-3))$.

Proof. The proof involves straightforward but lengthy calculation which we work out below.

Since $(v_2,\, \bar{c}_1^2) = 0$, we get $(v_2,\, v_2) = (v_2,\, c_2) = (\bar{c}_2,\, \bar{c}_2) - (\deg \bar{c}_2/\deg \mathbb{G}_{n,k})(\bar{c}_2,\, \omega^2) = \deg \mathbb{G}_{n,k}(\deg(\bar{c}_2^2)/\deg \mathbb{G}_{n,k} - (\deg \bar{c}_2/\deg \mathbb{G}_{n,k})^2)$.

Since $\bar{c}_2^2 = [\Omega_2]^2 = [\Omega_4] + [\Omega_{3,1}] + [\Omega_{2,2}]$, we see that $\deg \bar{c}_2^2/\deg \mathbb{G}_{n,k} = \deg \bar{c}_4/\deg \mathbb{G}_{n,k} + \deg \Omega_{3,1}/\deg \mathbb{G}_{n,k} + \deg \Omega_{2,2}/\deg \mathbb{G}_{n,k}$.

Now an explicit calculation yields, upon using $N = k(n-k)$:

$$\begin{aligned} \frac{\deg \bar{c}_4}{\deg \mathbb{G}_{n,k}} &= \frac{(n-k-1)(n-k-2)(n-k-3)(k+1)(k+2)(k+3)}{4!\,(N-1)(N-2)(N-3)}, \\ \frac{\deg \Omega_{3,1}}{\deg \mathbb{G}_{n,k}} &= \frac{(n-k+1)(n-k-1)(n-k-2)(k+2)(k+1)(k-1)}{2!\,4(N-1)(N-2)(N-3)}, \\ \frac{\deg \Omega_{2,2}}{\deg \mathbb{G}_{n,k}} &= \frac{N(k-1)(k+1)(n-k+1)(n-k-1)}{2!\,3\cdot 2(N-1)(N-2)(N-3)}, \\ \frac{\deg \bar{c}_2}{\deg \mathbb{G}_{n,k}} &= \frac{(k+1)(n-k-1)}{2!\,(N-1)}. \end{aligned}$$

Substituting these in the above expression for $(v_2,\, v_2)$ we get $(v_2,\, v_2) = ((k+1)(n-k-1)/(4!(N-1)^2(N-2)(N-3)))A$ where, again using $N = k(n-k)$ repeatedly,

$$\begin{aligned} A := (N-1)\{&(n-k-2)(k+2)(n-k-3)(k+3) \\ &+ 3(n-k-2)(k+2)(n-k+1)(k-1) + 2N(k-1)(n-k+1)\} \\ &- 6(N-2)(N-3)((n-k-1)(k+1))^2) \end{aligned}$$

$$\begin{aligned}
&= (N-1)\{(N+2(n-2k)-4)(N+3(n-2k)-9) \\
&\qquad\qquad + 3(N+2(n-2k)-4)(N-(n-2k)-1) + 2(N-(n-2k)-1)\} \\
&\quad - 6(N-2)(N-3)(N+(n-2k)-1) \\
&= 12(N-(n-2k)-1) \\
&= 12(k-1)(n-k+1).
\end{aligned}$$

Therefore, $(v_2, v_2) = (\deg \mathbb{G}_{n,k})(k^2-1)((n-k)^2-1)/(2(N-1)^2(N-2)(N-3))$. □

REMARK 3.6. Although quaternionic Grassmann manifolds are not c-Kähler, one could use the symplectic Pontrjagin class $\eta := e_1(\gamma_{n,k}) \in H^4(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Z})$ in the place of $\overline{c_1} \in H^2(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$ to define a pairing $(\,\cdot\,,\,\cdot\,)_\eta$ on $H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$ and the primitive classes $v_j \in H^{4j}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$. We define $\mathcal{V}^{4r} \subset H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$ to be the kernel of

$$\cup\eta^{N-2r+1} \colon H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q}) \to H^{4N-4r+4}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q}).$$

The form $(\,\cdot\,,\,\cdot\,)_\eta$ is definite when restricted to the space $\eta^q\mathcal{V}^{4r-4q} \subset H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$. The formula given in Lemma 3.5 holds without any change. These statements follow from the degree doubling isomorphism from the cohomology algebra of $\mathbb{G}_{n,k}$ to that of $\mathbb{H}\mathbb{G}_{n,k}$ which maps the i-th Chern class of the tautological complex k-plane bundle over $\mathbb{G}_{n,k}$ to the i-th symplectic Pontrjagin class of the tautological left $\mathbb{H}$-bundle over $\mathbb{H}\mathbb{G}_{n,k}$.

4. Proofs of main results

In this section we prove the main results of the paper, namely Theorems 1.1, 1.2 and 1.3. *We will only consider the case of complex Grassmann manifolds*. The proofs in the case of quaternionic Grassmann manifolds follow in view of the fact that the cohomology algebra of $\mathbb{H}\mathbb{G}_{n,k}$ is isomorphic to that of $\mathbb{C}\mathbb{G}_{n,k}$ via an isomorphism that doubles the degree.

Recall that complex Grassmann manifolds are smooth projective varieties and that Schubert subvarieties yield an *algebraic* cell decomposition. In particular their Chow ring is isomorphic to singular cohomology (with $\mathbb{Z}$-coefficients) via an isomorphism that doubles the degree. It follows that $H^{p,q}(\mathbb{G}_{n,k}; \mathbb{C}) = 0$ for $p \neq q$. Therefore results of the previous section hold for $\mathbb{G}_{n,k}$. The bilinear form $(\,\cdot\,,\,\cdot\,)$ is understood to be defined with respect to $\omega = \overline{c_1} \in H^2(\mathbb{G}_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$.

Lemma 4.1. *Let $f \colon \mathbb{G}_{n,k} \to \mathbb{G}_{m,l}$ be any continuous map where $k(n-k) = l(m-l) =: N$. Suppose that $f^*(c_1(\gamma_{m,l}^\perp)) = \lambda c_1(\gamma_{n,k}^\perp)$ where $\lambda \in \mathbb{Z}$. Then*

$$\deg(f) = \lambda^N \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}}.$$

Proof. This follows immediately from Lemma 3.1 (i) and (iii). □

Proof of Theorem 1.3. We may suppose that $\mathbb{F} = \mathbb{C}$ and that $l \leq k$; otherwise $k < l \leq [m/2]$ in which case $\deg(f) = 0$ for any f by [14, Theorem 2].

Suppose that $\deg(f) = \pm 1$ and that $l < k$. We have

$$\begin{aligned}\frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}} &= \frac{1!\cdots(k-1)!\,(m-l)!\cdots(m-1)!}{1!\cdots(l-1)!\,(n-k)!\cdots(n-1)!}\\ &= \frac{l!\cdots(k-1)!\,(m-l)!\cdots(m-1)!}{(n-k)!\cdots(n-1)!}\\ &= \left(\prod_{1\leq j\leq k-l}\frac{(l-1+j)!}{(n-k+j-1)!}\right)\left(\prod_{1\leq j\leq l}\frac{(m-j)!}{(n-j)!}\right).\end{aligned}$$

Note that after simplifying $(l+j-1)!/(n-k+j-1)!$ for each j in the first product, we are left with product of $(k-l)$ blocks of $(n-k-l)$ consecutive positive integers in the denominator, the *largest* to occur being $(n-l-1)$. Similar simplification in the second product yields a product of l blocks of $(m-n)$ consecutive integers in the numerator, the *smallest* to occur being $(n-l+1)$. Since $(k-l)(n-k-l) = l(m-n)$ we conclude that $\deg \mathbb{G}_{n,k} > \deg \mathbb{G}_{m,l}$.

In the notation of Lemma 4.1 above, we see that either $\deg(f) = 0$ or $|\deg(f)| > |\lambda|^N \geq 1$—a contradiction. Therefore $(m, l) = (n, k)$ if $\deg(f) = \pm 1$. Now $f^*\colon H^*(\mathbb{G}_{n,k};\mathbb{Z}) \to H^*(\mathbb{G}_{n,k};\mathbb{Z})$ induces an isomorphism. Since $\mathbb{G}_{n,k}$ is a simply connected CW complex, by Whitehead's theorem, f is a homotopy equivalence. □

REMARK 4.2. (i) The above is a topological analogue of the result of Paranjape and Srinivas [13] that any non-constant morphism $f\colon \mathbb{G}_{n,k} \to \mathbb{G}_{m,l}$ is an *isomorphism* of varieties provided the $\mathbb{G}_{m,l}$ is not the projective space. Our conclusion in the topological realm is weaker. Indeed it is known that there exist continuous self-maps of any complex and quaternionic Grassmann manifold which have large positive degrees. See [1] and also [15].

(ii) Endomorphisms of the cohomology algebra of $\mathbb{G}_{n,k}$ having non-zero degree have been classified by M. Hoffman [9]. These are either 'grading homomorphisms' defined by $c_i \mapsto \lambda^i c_i$, $1 \leq i \leq k$ for some λ or when $n = 2k$, the composition of a grading homomorphism with the homomorphism induced by the diffeomorphism $\perp\colon \mathbb{G}_{n,k} \to \mathbb{G}_{n,k}$ defined as $U \mapsto U^{\perp}$. If the degree of an endomorphism h of $H^*(\mathbb{G}_{n,k};\mathbb{Q})$ is zero, then $h(c_1) = 0$. Hoffman has conjectured in [9] that in this case h vanishes in positive dimensions. This conjecture has been established in [4] when $n > 2k^2 - 1$ and it is also known to hold when $k \leq 3$.

Recall from Example 3.4 the construction of the primitive classes $v_j \in H^{2j}(\mathbb{G}_{n,k};\mathbb{Q})$, $2 \leq j \leq k$. To avoid possible confusion, we shall denote the primitive classes in

$H^{2j}(\mathbb{G}_{m,l};\mathbb{Q})$ corresponding to $j=2,\ldots,l$ by u_j. Also $\mathcal{V}^{2r}_{m,l}\subset H^{2r}(\mathbb{G}_{m,l};\mathbb{Q})$ will denote the space of primitive classes. The following lemma is crucial for the proof of Theorem 1.1.

Lemma 4.3. *Suppose that* $f\colon \mathbb{G}_{n,k}\to\mathbb{G}_{m,l}$ *is a continuous map such that* $f^*(c_1(\gamma^\perp_{m,l}))=\lambda c_1(\gamma^\perp_{n,k})=\lambda\bar{c}_1$ *with* $\lambda\neq 0$. *Let* $2\le j\le l$. *Assume that* $k(n-k)=l(m-l)$. *Then, with the above notations,* $f^*(u_j)=\lambda_j v_j$ *where* $\lambda_j\in\mathbb{Q}$ *is such that*

$$\lambda_j^2=\lambda^{2j}\frac{\deg\mathbb{G}_{n,k}}{\deg\mathbb{G}_{m,l}}\frac{(u_j,u_j)}{(v_j,v_j)}$$

for $2\le j\le l$.

Proof. The degree of f equals $\lambda^N\deg\mathbb{G}_{n,k}/\deg\mathbb{G}_{m,l}\neq 0$ by Lemma 4.1.

Therefore $f^*\colon H^{2j}(\mathbb{G}_{m,l};\mathbb{Q})\to H^{2j}(\mathbb{G}_{n,k};\mathbb{Q})$ is an isomorphism and $f^*(\mathcal{V}^{2j}_{m,l})=\mathcal{V}^{2j}_{n,k}$, since f^* is a monomorphism and the dimensions are equal as $j\le l$. Note that f^* maps the space of decomposable elements $\mathcal{D}^{2j}_{m,l}\subset H^{2j}(\mathbb{G}_{m,l};\mathbb{Q})$ isomorphically onto $\mathcal{D}^{2j}_{n,k}$. Since $u_j\perp\mathcal{D}^{2j}_{m,l}\cap\mathcal{V}^{2j}_{m,l}$ we see that, by Lemma 3.1 (ii), $f^*(u_j)\perp\mathcal{D}^{2j}_{n,k}\cap\mathcal{V}^{2j}_{n,k}$. As the form $(\cdot\,,\,\cdot)$ on $\mathcal{V}^{2j}_{n,k}$ is definite by Proposition 3.2 and $\mathcal{V}^{2j}_{n,k}=\mathbb{Q}v_j\oplus(\mathcal{V}^{2j}_{n,k}\cap\mathcal{D}^{2j}_{n,k})$ is an orthogonal decomposition, we must have $f^*(u_j)=\lambda_j v_j$ for some $\lambda_j\in\mathbb{Q}$.

Recall that $\deg(f)=\lambda^N\deg\mathbb{G}_{n,k}/\deg\mathbb{G}_{m,l}$. Note that

$$\begin{aligned}\lambda^{N-2j}(f^*(u_j),f^*(u_j))&=(f^*(u_j),f^*(u_j))_{\lambda\bar{c}_1}\\&=\deg(f)(u_j,u_j)_\omega\\&=\lambda^N\frac{\deg\mathbb{G}_{n,k}}{\deg\mathbb{G}_{m,l}}(u_j,u_j)\end{aligned}$$

by Lemma 3.1. Thus $\lambda_j^2(v_j,v_j)=(f^*(u_j),f^*(u_j))=\lambda^{2j}(\deg\mathbb{G}_{n,k}/\deg\mathbb{G}_{m,l})(u_j,u_j)$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We need only consider the case $\mathbb{F}=\mathbb{C}$. Recall that the cohomology algebra $H^*(\mathbb{G}_{m,l};\mathbb{Z})$ is generated by $\bar{c}_1,\ldots,\bar{c}_l$ where $\bar{c}_j=c_j(\gamma^\perp_{m,l})$. Therefore $f^*\colon H^*(\mathbb{G}_{m,l};\mathbb{Z})\to H^*(\mathbb{G}_{n,k};\mathbb{Z})$ is determined by the images of $\bar{c}_j$, $1\le j\le l$.

As observed in Example 3.4, one has $u_j-\bar{c}_j\in\mathcal{D}^{2j}_{m,l}$, $2\le j\le l$. It follows easily by induction that each $\bar{c}_j$, $1\le j\le l$, can be expressed as a polynomial with rational coefficients in $\bar{c}_1,u_2,\ldots,u_l$. Therefore $\bar{c}_1=:u_1,u_2,\ldots,u_l$ generate $H^*(\mathbb{G}_{m,l};\mathbb{Q})$.

Lemma 4.1 implies that $f^*(u_1)=\lambda c_1(\gamma^\perp_{n,k})$ where λ^N—and hence λ up to a sign—is determined by the degree of f.

Now by Lemma 4.3, the image of u_j under f^* equals $\lambda_j v_j$ where λ_j is determined up to a sign by the degree of f, if $\deg(f)\neq 0$. □

Proof of Theorem 1.2. We assume, as we may, that $\mathbb{F} = \mathbb{C}$. We preserve the notations used in the above proof. Recall from Lemma 3.5 that $(v_2, v_2) = (\deg \mathbb{G}_{n,k})(k^2 - 1)((n-k)^2 - 1)/(2(N-1)^2(N-2)(N-3))$. Therefore, by Lemma 4.3 we have

$$\begin{aligned}\lambda_2^2 &= \lambda^4 \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}} \frac{(v_2, v_2)}{(u_2, u_2)}\\ &= \lambda^4 \left(\frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}}\right)^2 \frac{(k^2-1)((n-k)^2-1)}{(l^2-1)((m-l)^2-1)}\\ &= B^2(k^2-1)(l^2-1)((n-k)^2-1)((m-l)^2-1)\end{aligned}$$

where $B := \lambda^2 \deg \mathbb{G}_{n,k}/(\deg \mathbb{G}_{m,l}(l^2-1)((m-l)^2-1)) \in \mathbb{Q}$. It follows that $\deg(f) = 0$ unless $Q := (l^2-1)(k^2-1)((m-l)^2-1)((n-k)^2-1)$ is a perfect square. It remains to show that there are at most finitely many values for m, n for which the Q is a perfect square. This is proved in the following proposition.

Proposition 4.4. *Let $1 < a < b$ be positive integers. Then there are at most finitely many solutions in $\mathbb{Z}$ for the system of equations*

$$y^2 = Q(a, b, x, z), \quad az = bx, \tag{8}$$

where $Q(a, b, x, z) := (a^2-1)(b^2-1)(x^2-1)(z^2-1)$.

Proof. Let $r = \gcd(a, b)$ and write $a = rs$, $b = rt$ so that $tx = sz$. Then the system of equations (8) can be rewritten as $y^2 = F(x)$ where $F(x) := (1/s^2)(a^2-1)(b^2-1)(x^2-1)(t^2x^2-s^2)$. Note that $F(x) \in \mathbb{Q}[x]$ has distinct zeros in $\mathbb{Q}$. By a theorem of Siegel [5, Theorem D.8.3, p. 349] it follows that the equation $y^2 = F(x)$ has only finitely many solutions in the ring $R_S \subset K$ of S-integers where K is any number field and S any finite set of absolute valuations of K, including all archimedean valuations. In particular, taking $K = \mathbb{Q}$ and S the usual (archimedean) absolute value, we see that there are only finitely many integral solutions of (8). □

For the rest of the paper we shall only be concerned with the number theoretic question of $Q(a, b, c, d)$ being a perfect square.

REMARK 4.5. (i) We observe that there are *infinitely* many integers $1 < a < b < c < d$ such that $Q(a, b, c, d)$ is a perfect square. Indeed given a, b, let c be any positive integer such that $(a^2-1)(b^2-1)(c^2-1) = Pu^2$ where $P > 1$ is square free. Let (x, y) be any solution with $x \neq 0$ of the so called Pell's equation $y^2 = 1 + Px^2$. Then $d = |y|$ is a solution whenever $d > c$. Since the Pell's equation has infinitely many solutions, there are infinitely many such d.
(ii) Suppose that $(l^2-1)(k^2-1)(c^2-1) = x^2$ is a perfect square. (There exists such positive integers c—in fact infinitely many of them—for which this happens if and only

if $(l^2-1)(k^2-1)$ is *not* a perfect square.) Then there does *not* exist any $d>1$ such that $Q(l,k,c,d)$ is a perfect square. Assume further that $l \mid (kc)$—this can be arranged, for example, taking k to be a multiple of l—and set $n := c+k$, $m := kc/l$ so that $k(n-k) = l(m-l)$. Then $Q(l,k,n-k,m-l)$ is not a perfect square.
(iii) We illustrate below situations in which $Q(l,k,n-k,m-l)$ is not a perfect square (assuming that $k(n-k)=l(m-l)$) depending on congruence classes, modulo a suitable prime power, of the parameters involved.

(1) For an odd prime p, suppose that $k \equiv p^{2r-1} \pm 1 \bmod p^{2r}$ and none of the numbers $l, m-l, n-k$ is congruent to $\pm 1 \bmod p$. Then $p^{2r-1} \mid Q$ but $p^{2r} \nmid Q$.
(2) Suppose that $m \equiv l \equiv 5 \bmod 8$, and $k \equiv 7 \bmod 16$. Then $(m-l)^2-1$ is odd, $l^2-1 \equiv 8 \bmod 16$, $k^2-1 \equiv 16 \bmod 32$ and $l(m-l) = k(n-k)$ implies $(n-k)$ is even and so $(n-k)^2-1$ is odd. Thus $Q \equiv 2^7 \bmod 2^8$.
(3) Suppose that $l \equiv 0 \bmod 8$, $m \equiv l \bmod 2$, $k \equiv 3 \bmod 8$. Then $Q \equiv 8 \bmod 16$.

We conclude the paper with the following

Proposition 4.6. *Let $c>1$ and let $k=3$ or 7. Suppose that $Q(2,k,2c,kc)$ is a perfect square. Then there exist integers $\xi, \eta, v > 1$ such that $c = (1/2)(\xi^2\eta^2+1)$, $\xi^2\eta^2 - 3v^2 = -2$ and* (i) *$\xi^2 - 3\eta^2 = -2$ when $k=3$ and* (ii) *$\xi^2 - 7\eta^2 = -6$ when $k=7$.*

Proof. Assume that $k=7$ and that $Q := Q(2,7,2c,7c) = 3^2 2^4(2c-1)(2c+1)(7c-1)(7c+1)$ is a perfect square. There are several cases to consider depending on the gcd of the pairs of numbers involved. Write $(2c-1) = \alpha u^2$, $2c+1 = \beta v^2$, $7c-1 = \gamma x^2$, $7c+1 = \delta y^2$, where $\alpha, \beta, \gamma, \delta$ are square free integers. Since Q is a perfect square and since $\gcd(2c-1, 2c+1) = 1$, $\gcd(7c-1, 7c+1) = 1$ or 2, $\gcd(2c \pm 1, 7c \pm 1) = 1$, or 5, $\gcd(2c \pm 1, 7c \mp 1) = 1$, 3, or 9, the possible values for (α, β) are: $(1,1)$, $(1,5)$, $(1,3)$, $(3,1)$, $(5,1)$, $(1,15)$, $(15,1)$, $(5,3)$, $(3,5)$. The possible values for (γ, δ) are the same as for (α, β) as well as $(2\alpha, 2\beta)$.

Suppose $(\alpha, \beta) = (1,1)$. Since $(2c-1)+2 = (2c+1)$, we obtain $u^2+2 = v^2$ which has no solution. If $(\alpha,\beta) = (3,1)$, then $3u^2+2 = v^2$. This equation has no solution mod 3. Similar arguments show that if $(\alpha,\beta) = (5,1), (1,5), (1,15), (15,1), (5,3)$, there are no solutions for u, v. If $(\alpha,\beta) = (3,5)$, then $(\gamma,\delta) = (5,3)$ or $(10,6)$. If $(\gamma,\delta) = (5,3)$ again there is no solution mod 3 for the equation $5x^2+2 = 3y^2$. When $(\gamma,\delta) = (10,6)$ we obtain $10x^2+2 = 6y^2$. This has no solution mod 5.

It remains to consider the case $(\alpha,\beta) = (1,3)$. In this case we obtain the equation $u^2+2 = 3v^2$ which has solutions, for example, $(u,v) = (5,3)$. Now $(\alpha,\beta) = (1,3)$ implies $(\gamma,\delta) = (3,1)$ or $(6,2)$. If $(\gamma,\delta) = (3,1)$ then we obtain the equation $3x^2+2 = y^2$ which has no solution mod 3. So assume that $(\gamma,\delta) = (6,2)$. As $(\alpha,\delta) = (1,2)$ we obtain $4y^2 - 7u^2 = 9$, that is, $4y^2 - 7u^2 = 9$. Thus $(2y-3)(2y+3) = 7u^2$. Either $7 \mid (2y-3)$ or $7 \mid (2y+3)$. Say $7 \mid (2y-3)$ and write $(2y-3) = 7z$. Now $z(7z+6) = u^2$. Observe that $\gcd(z, 7z+6)$ divides 6. Since $\beta = 3$, $2c-1 = u^2$ is not divisible by 3. Also, u being odd, we must have $\gcd(z, 7z+6) = 1$. It follows that both z and $7z+6$ are perfect squares. This forces 6 to be a square mod 7—a contradiction. Finally, suppose that $7 \mid (2y+3)$. Then repeating the above argument we see that both $(2y-3) =: \eta^2$ and $(2y+3)/7 =: \xi^2$ are perfect squares. It follows that $7\xi^2 - 6 = \eta^2$ is a perfect square. Hence $2c-1 = u^2 = \xi^2\eta^2$. Since $2c+1 = 3v^2$, the proposition follows.

We now consider the case $k = 3$. We merely sketch the proof in this case. Let, if possible, $Q = 2^3 3(2c-1)(2c+1)(3c-1)(3c+1)$ be a perfect square. Write $2c-1 = \alpha u^2$, $2c+1 = \beta v^2$, $3c-1 = \gamma x^2$, $3c+1 = \delta y^2$, where $\alpha, \beta, \gamma, \delta$ are square free integers and u, v, x, y are positive integers. Arguing as in the case $k = 7$, following are the only possible values for $\alpha, \beta, \gamma, \delta$: $(\alpha, \beta) = (1, 3), (3, 1), (3, 5), (5, 3), (1, 15), (15, 1)$, and $(\gamma, \delta) = (1, 2), (2, 1), (2, 5), (5, 2), (1, 10), (10, 1)$. It can be seen that only the case $(\alpha, \beta, \gamma, \delta) = (1, 3, 2, 1)$ remains to be considered, the remaining possibilities leading to contradictions. Thus we have $2c-1 = u^2$, $2c+1 = 3v^2$, $3c-1 = 2x^2$ and $3c+1 = y^2$. Therefore, we have $4x^2 - 1 = 3u^2$, i.e., $(2x-1)(2x+1) = 3u^2$. Hence, $3 \mid (2x-1)$ or $3 \mid (2x+1)$.

Suppose that $3 \mid (2x-1)$. Write $3z = 2x-1$, $z \in \mathbb{Z}$. Since z is odd, we have $\gcd(z, 3z+2) = 1$. As $z(3z+2) = u^2$ we conclude that z and $3z+2$ have to be perfect squares. This implies that 2 is a quadratic residue mod 3—a contradiction. Therefore $3 \nmid (2x-1)$ and we must have $3 \mid (2x+1)$ and both z and $3z-2$ will have to be perfect squares. Write $z = \eta^2$ and $3z-2 = \xi^2$ so that $\xi^2 - 3\eta^2 = -2$ and $v^2 = u^2 + 2 = \xi^2\eta^2 + 2$. This completes the proof. $\square$

REMARK 4.7. (i) Let $K = \mathbb{Q}[\sqrt{7}]$ and let R be the ring of integers in K. If $\xi + \eta\sqrt{7} \in R$, then $\xi, \eta \in \mathbb{Z}$. Denote the multiplicative ring of units in R by U. Note that any element of U has norm 1. (This is because -1 is a quadratic non-residue mod 7.) Using Dirichlet Unit theorem U has rank 1; indeed U is generated by $\nu := (8+3\sqrt{7})$ and ± 1. The integers ξ, η as in the above proposition yield an element $\xi + \eta\sqrt{7}$ of norm -6 and the set $S \subset R$ of all elements of norm -6 is stable under the multiplication action by U. An easy argument shows that S is the union of orbits through $\lambda := 1+\sqrt{7}$, $\bar{\lambda} = 1 - \sqrt{7}$. Thus $S = \{\pm\lambda\nu^k, \pm\bar{\lambda}\nu^k \mid k \in \mathbb{Z}\}$.

Observe that if ξ, η are as in Proposition 4.6 (ii), then $\xi + \sqrt{7}\eta \in S$. Listing elements $\xi + \eta\sqrt{7} \in S$ with $\xi, \eta > 1$ in increasing order of η, the first three elements are $13+5\sqrt{7}$, $29+11\sqrt{7}$, $209+79\sqrt{7}$. Straightforward verification shows that when $\xi + \eta\sqrt{7}$ is any of these, then there does not exist an integer v such that $\xi^2\eta^2 + 2 = 3v^2$. Since the next term is $463+175\sqrt{7}$, we have the lower bound $2c > 175^2 \times 463^2 = 6565050625$ in order that $Q(2, 7, 2c, 7c)$ be a perfect square (assuming $c > 1$).

(ii) Now, let $K = \mathbb{Q}[\sqrt{3}]$ and let R be the ring of integers in K. Note that if $\xi + \eta\sqrt{3} \in R$, then $\xi, \eta \in \mathbb{Z}$. Denote the multiplicative ring of units in R by U, which is generated by $(2+\sqrt{3})$ and ± 1.

Suppose that $Q(2, 3, 2c, 3c)$ is a perfect square, $c > 1$. Then the integers ξ, η, as in the above proposition, yield an element $\xi + \eta\sqrt{3}$ of norm -2. The set $S \subset R$ of all elements of norm -2 is stable under the multiplication action by U. In fact it can be verified easily that $S = \{\pm(1+\sqrt{3})(2+\sqrt{3})^m \mid m \in \mathbb{Z}\}$.

Listing these with $\xi, \eta > 1$, in increasing order of η, the first five elements are $5 + 3\sqrt{3}$, $19 + 11\sqrt{3}$, $71 + 41\sqrt{3}$, $265 + 153\sqrt{3}$, $989 + 571\sqrt{3}$. If $\xi + \eta\sqrt{3}$ equals any of these, direct verification shows that there is no integer v satisfying the equation $\xi^2\eta^2 + 2 = 3v^2$. The next term of the sequence being $3691 + 2131\sqrt{3}$ we obtain the lower bound $2c > 2131^2 \times 3691^2 = 61866420601441$.

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