Borna, K.
Osaka J. Math.
46 (2009), 1047–1058

ON LINEAR RESOLUTION OF POWERS OF AN IDEAL

KEIVAN BORNA

(Received August 8, 2008)

Abstract

In this paper we give a generalization of a result of Herzog, Hibi, and Zheng providing an upper bound for regularity of powers of an ideal. As the main result of the paper, we give a simple criterion in terms of Rees algebra of a given ideal to show that high enough powers of this ideal have linear resolution. We apply the criterion to two important ideals J, J_1 for which we show that J^k, and J_1^k have linear resolution if and only if $k \neq 2$. The procedures we include in this work is encoded in computer algebra package CoCoA [3].

1. Introduction

Let $S = K[x_1, \dots, x_r]$ and let

$$\mathbb{F}\colon \ \cdots \to F_i \to F_{i-1} \to \cdots$$

be a graded complex of free S-modules, with $F_i = \sum_j S(-a_{i,j})$. The Castelnuovo-Mumford regularity, or simply regularity, of $\mathbb{F}$ is the supremum of the numbers $a_{i,j} - i$. The regularity of a finitely generated graded S-module M is the regularity of a minimal graded free resolution of M. We will write $\operatorname{reg}(M)$ for this number. The regularity of an ideal is an important measure of how complicated the ideal is. The above definition of regularity shows how the regularity of a module governs the degrees appearing in a minimal resolution. As Eisenbud mentions in [8] Mumford defined the regularity of a coherent sheaf on projective space in order to generalize a classic argument of Castelnuovo. Mumford's definition [12] is given in terms of sheaf cohomology. The definition for modules, which extends that for sheaves, and the equivalence with the condition on the resolution used above definition, come from Eisenbud and Goto [9]. Alternate formulations in terms of Tor, Ext and local cohomology are given in the following. Let I be a graded ideal, $\mathfrak{m} = (x_1, \dots, x_r)$ the maximal ideal of S, and $n = \dim(S/I)$. Let

$$a_i(S/I) = \max\{t;\ H^i_{\mathfrak{m}}(S/I)_t \neq 0\}, \quad 0 \leq i \leq n,$$

2000 Mathematics Subject Classification. Primary 13D02; Secondary 13P10.
The author is grateful to Dipartimento di Matematica, Universitá di Genova, Italia.

where $H^i_{\mathfrak{m}}(S/I)$ is the i-th local cohomology module with the support in $\mathfrak{m}$ (with the convention $\max\emptyset = -\infty$). Then the regularity is the number

$$\operatorname{reg}(S/I) = \max\{a_i(S/I) + i\,;\ 0 \le i \le n\}.$$

Note that $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. We may also compute $\operatorname{reg}(I)$ in terms of Tor by the formula

$$\operatorname{reg}(I) = \max_k\{t_k(I) - k\},$$

where $t_p(I) := \max\{\text{degree of the minimal } p\text{-th syzygies of } I\}$. Simply this definition may be rewritten as

$$\begin{aligned}\operatorname{reg}(I) &= \max_{i,j}\{j - i\,;\ \operatorname{Tor}_i(I, k)_j \neq 0\}\\ &= \max_{i,j}\{j - i\,;\ \beta_{i,j}(I) \neq 0\}.\end{aligned}$$

Anyway, from local duality one see that the two ways of expressing the regularity are also connected termwise by the inequality $t_k(I) - k \ge a_i(S/I) + n - k$. Regularity is a kind of universal bound for important invariants of graded algebras, such as the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules. One has often tried to find upper bounds for the Castelnuovo-Mumford regularity in terms of simpler invariants which reflect the complexity of a graded algebra like dimension and multiplicity. Clearly $t_0(I^k) \le kt_0(I)$ and one may expect to have the same inequality for regularity, that is, $\operatorname{reg}(I^k) \le k\operatorname{reg}(I)$. Unfortunately this is not true in general. However, in [6] Cutkosky, Herzog, and Trung and in [11] Kodiyalam studied the asymptotic behavior of the Castelnuovo-Mumford regularity and independently showed that the regularity of I^k is a linear function for large k, i.e.,

$$\operatorname{reg}(I^k) = a(I)k + b(I), \quad \forall k \ge c(I). \tag{1.1}$$

Now assume that I is an equigenerated ideal, that is, generated by forms of the same degree d. Then one has $a(I) = d$ and hence, $\operatorname{reg}(I^{k+1}) - \operatorname{reg}(I^k) = d$ for all $k \ge c(I)$. Hence we have

$$\operatorname{reg}(I^k) = (k - c(I))d + \operatorname{reg}(I^{c(I)}), \quad \forall k \ge c(I). \tag{1.2}$$

One says that the regularity of the powers of I jumps at place k if $\operatorname{reg}(I^k) - \operatorname{reg}(I^{k-1}) > d$. In [4] the author gives several examples of ideals generated in degree d ($d = 2, 3$), with linear resolution (i.e., $\operatorname{reg}(I) = d$), and such that the regularity of the powers of I jumps at place 2, i.e., such that $\operatorname{reg}(I^2) > 2d$. As it is indicated in [4], the first example of such an ideal was given by Terai. Throughout this paper we use J for this ideal. Geometrically speaking, this is an example of Reisner which corresponds to the

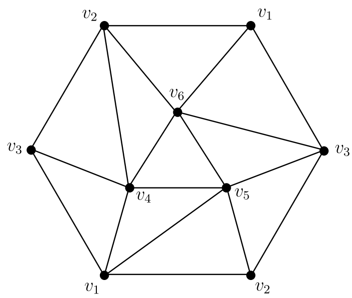

Fig. 1. The ideal of triangulation of the real projective plane $\mathbb{P}^2$.

(simplicial complex of a) triangulation of the real projective plane $\mathbb{P}^2$; see Fig. 1 and [2] for more details. Let $R := K[x_1, \dots, x_6]$ one has

$$J = (x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5,\ x_1x_4x_6,\ x_1x_5x_6,\ x_2x_3x_6,\ x_2x_4x_5,\ x_2x_5x_6,\ x_3x_4x_5,\ x_3x_4x_6). \tag{1.3}$$

It is known that J is a square-free monomial ideal whose Betti numbers, regularity and projective dimension depend on the characteristic of the base field. Indeed whenever $\mathrm{char}(K) \neq 2$, R/J is Cohen-Macaulay (and otherwise not), moreover one has $\mathrm{reg}(J) = 3$ and $\mathrm{reg}(J^2) = 7$ (which is of course $> 2 \times 3$). If $\mathrm{char}(K) = 2$, then J itself has no linear resolution. So the following natural question arises:

QUESTION A. How it goes on for the regularity of powers of J?

By the help of (1.1) we are able to write $\mathrm{reg}(J^k) = 3k + b(J), \forall k \geq c(J)$. But what are $b(J)$ and $c(J)$? In this paper we give an answer to this question and prove that J^k has linear resolution (in $\mathrm{char}(K) = 0$) $\forall k \neq 2$, that is, $b(J) = 0$ and $c(J) = 3$. That is

$$\mathrm{reg}(J^k) = 3k, \quad \forall k \neq 2.$$

To answer Question A we develop a general strategy and to this end we need to follow the literature a little bit. In [13] Römer proved that

$$\mathrm{reg}(I^n) \leq nd + \mathrm{reg}_x(R(I)), \tag{1.4}$$

where $R(I)$ is the Rees ring of I, which is naturally bigraded, and reg_x refers to the x-regularity of $R(I)$, that is,

$$\mathrm{reg}_x(R(I)) = \max\{b - i\,;\ \mathrm{Tor}_i(R(I), K)_{(b,d)} = 0\},$$

as defined by Aramova, Crona and De Negri [1]. In Section 2 we study Rees rings and their bigraded structure in more details. It follows from (1.4) that if $\mathrm{reg}_x(R(I)) = 0$,

Table 1. Count of elements of $\mathrm{in}(P)$ with $\deg_x > 1$ for the ideal of (1.3).

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	(1, 2) : 2, (2, 2) : 2	(1, 2) : 2, (2, 2) : 1
Lex	(1, 2) : 2, (2, 2) : 1	(1, 2) : 2, (2, 2) : 1

then each power of I admits a linear resolution. Based on Römer's formula, in [10, Theorem 1.1 and Corollary 1.2] Herzog, Hibi and Zheng showed the following:

Theorem 1.1. *Let $I \subseteq K[x_1,\dots,x_n] := S$ be an equigenerated graded ideal. Let m be the number of generators of I and let $T := S[t_1,\dots,t_m]$, and let $R(I) = T/P$ be the Rees algebra associated to I. If for some term order $<$ on T, P has a Gröbner basis G whose elements are at most linear in the variables $x_1,\dots,x_n$, that is $\deg_x(f) \le 1$ for all $f \in G$, then each power of I has a linear resolution.*

Throughout this paper we simply write $S = K[\underline{x}]$ and $T = S[\underline{t}]$. One can easily see that for J, (1.3), one has at least 3 elements in $\mathrm{in}(P)$ with $\deg_x > 1$, no matter if we take initial ideal w.r.t. term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in either Lex or DegRevLex order as it is reported in Table 1. Note that for example if one starts in DegRevLex order and $\underline{x} > \underline{t}$ then there is 4 elements in $\mathrm{in}(P)$ which have x-degree > 1 (= 2 actually) and among them 2 term has t-degree 1 and 2 term is in t-degree 2.

The main motivation for our work is to generalize Herzog, Hibi and Zheng's techniques in order to apply them to a wider class. Furthermore, we will indicate the least exponent k_0 for which I^k has linear resolution for all $k \ge k_0$. Indeed our generalization works for all ideals which admit the following condition:

Theorem 1.2. *Let $Q \subseteq S = K[x_1,\dots,x_r]$ be a graded ideal which is generated by m polynomials all of the same degree d, and let $I = \mathrm{in}(g(P))$ for some linear bi-transformation $g \in \mathrm{GL}_r(K) \times \mathrm{GL}_m(K)$. Write $I = G + B$ where G is generated by elements of $\deg_x \le 1$ and B is generated by elements of $\deg_x > 1$. If $I_{(k,j)} = G_{(k,j)}$ for all $k \ge k_0$ and for all $j \in \mathbb{Z}$, then Q^k has linear resolution for all $k \ge k_0$. In other words, $\mathrm{reg}(Q^k) = kd$ for all $k \ge k_0$.*

Another motivation for our paper is an example that Conca considered in [4].

EXAMPLE 1.3. Let J_1 be the ideal of 3-minors of a 4×4 symmetric matrix of linear forms in 6 variables, that is, 3-minors of

$$\begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & x_4 & 0 & x_6 \\ x_3 & x_5 & x_6 & 0 \end{bmatrix}.$$

Table 2. Count of elements of $\mathrm{in}(P_1)$ with $\deg_x > 1$ for J_1, (1.5).

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	(1, 2) : 6, (2, 2) : 5, (1, 3) : 1, (4, 2) : 1	(1, 2) : 6, (2, 2) : 3, (1, 3) : 1
Lex	(1, 2) : 6, (2, 2) : 3	(1, 2) : 6, (2, 2) : 5

As an ideal of $S = \mathbb{Q}[x_1, \dots, x_6]$ one has:

(1.5)

$$\begin{aligned} J_1 := (&2x_1x_2x_4,\ 2x_1x_3x_5,\ 2x_2x_3x_6,\ 2x_4x_5x_6,\ x_1x_3x_4 + x_1x_2x_5 - x_1^2x_6,\\ &x_3x_4x_6 + x_2x_5x_6 - x_1x_6^2,\ -x_2x_3x_4 + x_2^2x_5 - x_1x_2x_6,\ -x_3^2x_4 + x_2x_3x_5 + x_1x_3x_6,\\ &-x_3x_4^2 + x_2x_4x_5 + x_1x_4x_6,\ -x_3x_4x_5 + x_2x_5^2 - x_1x_5x_6). \end{aligned}$$

As Conca mentioned in his paper [4, Remark 3.6] and as we will show in this paper, the ideals J, J_1 are very closely related. For instance, we prove that

$$\mathrm{reg}(J_1^k) = 3k, \quad \forall k \neq 2.$$

Similar to the ideal of (1.3), one can easily check that $\mathrm{in}(P_1)$, where P_1 is the associated ideal to Rees ring of J_1, has at least 9 elements with $\deg_x > 1$, no matter if we take initial ideal w.r.t. term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in Lex or DegRevLex order; see Table 2 for more details.

We also show that J and J_1 and their powers have the same Hilbert series (HS for short) correspondingly:

$$\mathrm{HS}(S/J^k) = \mathrm{HS}(S/J_1^k), \quad \forall k.$$

Indeed we have computed the multigraded Hilbert series of the corresponding ideals to the Rees algebra of J and J_1 and observed that they are the same. As a result we conclude that all of the powers of J and J_1 have the same graded Betti numbers as well:

$$\beta_{i,j}(J^k) = \beta_{i,j}(J_1^k), \quad \forall i, j, \forall k.$$

2. Main results

Let K be a field, $I = (f_1, \dots, f_m)$ be a graded ideal of $S = K[x_1, \dots, x_r]$ generated in a single degree, say d. The Rees algebra of I is known to be

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1t, \dots, f_mt] \subseteq S[t].$$

Let $T = S[t_1, \dots, t_m]$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi \colon T \to R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \dots, r$ and $\varphi(y_j) = f_jt$ for $j =$

$1, \ldots, m$. So one can write $R(I) = T/P$. In this paper we consider T, and so $R(I)$, as a standard bigraded polynomial ring with $\deg(x_i) = (0, 1)$ and $\deg(t_j) = (1, 0)$. Indeed if we start with the natural bigraded structure $\deg(x_i) = (0, 1)$ and $\deg(f_j t) = (d, 1)$ then $R(I)_{(k,vd)} = (I^k)_{vd}$, but the standard bidegree normalizes the bigrading in the following sense:

$$R(I)_{(k,j)} = (I^k)_{kd+j}. \tag{2.1}$$

For each $k \in \mathbb{Z}$ we define a functor F_k from the category of bigraded T-modules to the category of graded S-modules with bigraded maps of degree zero. Let M be a bigraded T-module, define

$$F_k(M) = \bigoplus_{j \in \mathbb{Z}} M_{(k,j)},$$

obviously F_k is an exact functor and associates to each free $K[\underline{x}, \underline{t}]$-module a free $K[\underline{x}]$-module. Sometimes we simply write $M_{(k,\star)}$ instead of $F_k(M)$. Using (2.1) we get

$$[T/P]_{(k,\star)} = R(I)_{(k,\star)} = \bigoplus_{j \in \mathbb{Z}} R(I)_{(k,j)} = \bigoplus_{j \in \mathbb{Z}} (I^k)_{kd+j} = I^k(kd), \tag{2.2}$$

which provides the link between I and its Rees ring $R(I)$. In the sequel we need to know what is $F_k(T(-a, -b))$. For the convenience of reader we provide a proof.

REMARK 2.1. For each integer k we have

$$T(-a, -b)_{(k,\star)} = \begin{cases} 0, & \text{if } \quad k < a, \\ S(-b)^N, & \text{otherwise.} \end{cases} \tag{2.3}$$

Where $N := \#\{\underline{t}^\alpha;\ |\alpha| = k - a\} = \binom{m - 1 + k - a}{m - 1}$.

Proof.

$$\begin{aligned} T(-a, -b)_{(k,\star)} &= \bigoplus_{j \in \mathbb{Z}} T(-a, -b)_{(k,j)} = \bigoplus_{j \in \mathbb{Z}} T_{(k-a, j-b)} \\ &= \bigoplus_{j \in \mathbb{Z}} \langle \underline{\mathbf{t}}^\alpha \underline{\mathbf{x}}^\beta;\ |\alpha| = k - a,\ |\beta| = j - b \rangle, \end{aligned} \tag{2.4}$$

where the last equality is as vector spaces. From (2.4) the proof is immediate when $k < a$. Considering as an $S = K[\underline{x}]$-module the last module in (2.4) is free. Since $|\beta| = j - b$ could be any integer where j changes over $\mathbb{Z}$, a shift by $-b$ is required for the representation of the graded free module $T(-a, -b)_{(k,\star)}$ and finally the proposed N will take care of the required copies. $\square$

Note that in the spacial case $a = b = 0$, we have

$$T_{(k,\star)} = S^{\binom{m-1+k}{m-1}}. \tag{2.5}$$

As we mentioned in Introduction, Theorem 1.1 is subject to condition that $\mathrm{in}(P) = (u_1, \ldots, u_m)$ and $\deg_x(u_i) \leq 1$. So the natural way to generalize it is to change the upper bound for x-degree of u_i with some number t. As one may expect, we end up with $\mathrm{reg}(I^n) \leq nd + (t-1)\,\mathrm{pd}(T/\mathrm{in}(P))$. The proof is mainly as that of Theorem 1.1 but for the convenience of reader we bring it here.

Proposition 2.2. *Let $I \subseteq S$ be an equigenerated graded ideal and let $R(I) = T/P$. If $\mathrm{in}(P) = (u_1, \ldots, u_m)$ and $\deg_x(u_i) \leq t$, then $\mathrm{reg}(I^n) \leq nd + (t-1)\,\mathrm{pd}(T/\mathrm{in}(P))$.*

Proof. Let $C_\bullet$ be the Taylor resolution of $\mathrm{in}(P)$. The module C_i has the basis e_σ with $\sigma = j_1 < j_2 < \cdots < j_i \subseteq [m]$. Each basis element e_σ has the multidegree (a_σ, b_σ) where $x^{a_\sigma} . y^{b_\sigma} = \mathrm{lcm}\{u_{j_1}, \ldots, u_{j_m}\}$. It follows that $\deg_x(e_\sigma) \leq ti$ for all $e_\sigma \in C_i$. Since the shifts of $C_\bullet$ bound the shifts of a minimal multigraded resolution of $\mathrm{in}(P)$, we conclude that

$$\begin{aligned}\mathrm{reg}_x(T/P) \leq \mathrm{reg}_x(T/\mathrm{in}(P)) &= \max_{i,j}\{a_{ij} - i\}\\ &\leq ti - i = (t-1)i\\ &\leq (t-1)\,\mathrm{pd}(T/\mathrm{in}(P)).\end{aligned}$$

Now (1.4) completes the proof. $\square$

One can see that now Theorem 1.1 is the special case of Proposition 2.2 with $t = 1$. However, this approach seems to be less effective. Our approach to generalize Theorem 1.1 is to change P with an isomorphic image $g(P)$ so that $\mathrm{in}(g(P))_{(k,\star)}$ only consists of terms with x-degree ≤ 1, for some k. To this end, we need a simple fact.

Let $<$ be any term order on $S = K[\underline{x}]$ and let $V \subseteq S$ be a K-vector space. Then with respect to the monomial order on S obtained by restricting $<$, by definition V is homogeneous if for any element f of V, $f = \sum_{i=0}^{n} f_i$, where f_i is an element of S of degree i, we have $f_i \in V$, $\forall i = 0, \ldots, n$. That is to say $V = \bigoplus_{i=0}^{\infty} V_i$, $V_i = V \cap S_i$. It yields that $\mathrm{in}(V) = \bigoplus_{i=0}^{\infty} \mathrm{in}(V_i)$ and so, $\mathrm{in}(V)_i = \mathrm{in}(V_i)$. Generalizing this idea to bigraded (or multigraded) situation is also well understood. Let F be a free S-module with a fixed basis and M a bigraded subvector space of it. Then

$$\mathrm{in}(M)_{(i,j)} = \mathrm{in}(M_{(i,j)}),$$

and so

$$\mathrm{in}(M)_{(k,\star)} := \bigoplus_{j \in \mathbb{Z}} \mathrm{in}(M)_{(k,j)} = \bigoplus_{j \in \mathbb{Z}} \mathrm{in}(M_{(k,j)}) = \mathrm{in}(M_{(k,\star)}). \tag{2.6}$$

See [7] Chapter 15.2 for more details. Furthermore since $\beta^S_{ij}(F/M) \leq \beta^S_{ij}(F/\mathrm{in}(M))$, it is easy to conclude with

$$\mathrm{reg}(F/M) \leq \mathrm{reg}(F/\mathrm{in}(M)). \tag{2.7}$$

Lemma 2.3. *Let P be the associated ideal of Rees ring $R(I)$ and let $T = R/P$. Then* $\mathrm{reg}([T/P]_{(k,\star)}) \leq \mathrm{reg}([T/\mathrm{in}(P)]_{(k,\star)})$.

Proof. Since P is a naturally bigraded ideal of T, and since easily $T_{(k,\star)}$ is a free S-module (see (2.5)), (2.6) implies that $\mathrm{in}(P)_{(k,\star)} = \mathrm{in}(P_{(k,\star)})$. Applying (2.7) for $F := T_{(k,\star)}$ and $M := P$ we obtain $\mathrm{reg}(T_{(k,\star)}/P_{(k,\star)}) \leq \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P_{(k,\star)}))$. Finally putting all together we get the required inequality.

$$\begin{aligned}\mathrm{reg}([T/P]_{(k,\star)}) = \mathrm{reg}(T_{(k,\star)}/P_{(k,\star)}) &\leq \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P_{(k,\star)}))\\ &= \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P)_{(k,\star)})\\ &= \mathrm{reg}([T/\mathrm{in}(P)]_{(k,\star)}). \qquad \square\end{aligned}$$

In the following the proof of Theorem 1.2 is given.

Proof. First of all notice that, since $g\colon K[\underline{x}, \underline{t}] \to K[\underline{x}, \underline{t}]$ is an invertible bi-homogenous transformation, we have the following bi-homogenous isomorphism:

$$\frac{K[\underline{x}, \underline{t}]}{P} \simeq \frac{K[\underline{x}, \underline{t}]}{g(P)},$$

and so we can simply take $g = id$ in the rest of proof. Write down the so-called Taylor resolution of T/G:

$$\cdots \to \begin{matrix} F_{2,0} \\ \oplus \\ F_{2,1} \\ \oplus \\ F_{2,2} \end{matrix} \to \begin{matrix} F_{1,0} \\ \oplus \\ F_{1,1} \end{matrix} \to T \to T/G \to 0, \tag{2.8}$$

where $F_{i,j} = \bigoplus_{a\in\mathbb{Z}} T(-a, -j)^{\beta_{i,(a,j)}(T/G)}$. Note that $\beta_{i,(a,j)}(T/G)$, is an integer number which depends on i, a, and j. Since $(k, \star)$ is an exact functor, the following complex of $K[\underline{x}]$-modules is exact:

$$\cdots \to \begin{matrix} (F_{2,0})_{(k,\star)} \\ \oplus \\ (F_{2,1})_{(k,\star)} \\ \oplus \\ (F_{2,2})_{(k,\star)} \end{matrix} \to \begin{matrix} (F_{1,0})_{(k,\star)} \\ \oplus \\ (F_{1,1})_{(k,\star)} \end{matrix} \to T_{(k,\star)} \to [T/G]_{(k,\star)} \to 0. \tag{2.9}$$

Using formula (2.3) we obtain $T(-a,-b)_{(k,\star)} = S(-b)^{N_{a,k}}$, so for $F_{i,j}$ we get

$$(F_{i,j})_{(k,\star)} = \bigoplus_{a\in\mathbb{Z}} S(-j)^{N_{a,k}\beta_{i,(a,j)}(T/G)}. \tag{2.10}$$

It follows that (2.9) is a (possibly non-minimal) graded free $K[\underline{x}]$-resolution of $[T/G]_{(k,\star)}$. Since $\deg_x(G) \le 1$, from (2.9) and (2.10) we conclude that

$$\operatorname{reg}([T/G]_{(k,\star)}) = 0 \quad \text{for all} \quad k. \tag{2.11}$$

Now we have

$$\begin{aligned} dk \le \operatorname{reg}(Q^k) \le \operatorname{reg}([T/P]_{(k,\star)}) + dk &\le \operatorname{reg}([T/\operatorname{in}(P)]_{(k,\star)}) + dk \\ &= \operatorname{reg}([T/G]_{(k,\star)}) + dk \quad \text{for all} \quad k \ge k_0 \\ &= 0 + dk = dk, \end{aligned} \tag{2.12}$$

where the second (in)equality in (2.12) follows from (2.2), the third inequality is due to Lemma 2.3, and the forth comes from the easy argument $[T/\operatorname{in}(P)]_{(k,\star)} = T_{(k,\star)}/\operatorname{in}(P)_{(k,\star)} = T_{(k,\star)}/G_{(k,\star)} = [T/G]_{(k,\star)}$.

Finally (2.12) implies that $\operatorname{reg}(Q^k) = kd$ for all $k \ge k_0$ as desired. □

3. Examples and applications

In this section we provide some applications of Theorem 1.2. But before that we examine our condition on the decomposition of $\operatorname{in}(P)$ in a closer view. In the following a reformulation of our results is provided.

With the assumptions and notation introduced in Theorem 1.2 assume that $B = (m_1,\dots,m_p)$ and $\operatorname{bideg}(m_i) = (t_i, \ge 2)$. By $(t_i, \ge 2)$ we mean that the $\deg_x(m_i) \ge 2$. It is harmless to assume that $t_1 \le \cdots \le t_p$. If for all $i = 1,\dots,p$ and all $\alpha \in \mathbb{N}^m$ with $|\alpha| = t_p + 1 - t_i$ we have $\underline{t}^\alpha m_i \subseteq G$ then $I_{(k,\star)} = G_{(k,\star)}$ for all $k > t_p + 1$.

Using this strategy and as an application for our main result we give an answer to the Question A proposed in the Introduction.

EXAMPLE 3.1. Let $S = \mathbb{Q}[x_1,\dots,x_6]$ and let J be the ideal of (1.3). Let $T = \mathbb{Q}[x_1,\dots,x_6,t_1,\dots,t_{10}]$ with order $\underline{x} > \underline{t}$ (and DegRevLex). We also use J for the ideal of T generated by the same generators as of J in S. Let P be the defining ideal of the Rees ring of J, so $R(J) = T/P$. One can check that P has 15 elements of bidegree $(1,1)$, 10 elements of bidegree $(3,0)$, and 15 elements of bidegree $(4,0)$. Take G and B as in Theorem 1.2. We have checked that $|G| = 60$, $B = \operatorname{Ideal}(t_6x_4x_5, t_4x_3x_5, t_4t_6x_5^2)$, and so $\max\{\deg_t(h) \mid h \in B\} = 2$. But $(\underline{t})^2(t_6x_4x_5) \nsubseteq G$, $(\underline{t})^2(t_4x_3x_5) \nsubseteq G$, $(\underline{t})(t_4t_6x_5) \nsubseteq G$. So in DegRevLex (also Lex) order and $\underline{x} > \underline{t}$, we were unable to admit the conditions of Theorem 1.2. We have observed that the same story happens for ordering $\underline{t} > \underline{x}$

either DegRevLex or Lex. One could try to take g “generic”, as in (3.1).

$$
\begin{aligned}
&g := g_1 \times g_2,\\
&g_1 := x_i \mapsto \mathrm{Random}(\mathrm{Sum}(x_1, \dots, x_6)),\\
&g_2 := t_j \mapsto \mathrm{Random}(\mathrm{Sum}(t_1, \dots, t_{10})),
\end{aligned}
\tag{3.1}
$$

for all $i = 1, \dots, 6$ and all $j = 1, \dots, 10$, where by $\mathrm{Random}(\mathrm{Sum}(x_1, \dots, x_6))$ we mean a linear combination of $x_1, \dots, x_6$ with random coefficients and the same interpretation for $t_1, \dots, t_{10}$. But we realized that a properly chosen sparse random upper triangular g does the job as well. We continue in DegRevLex order and $\underline{t} > \underline{x}$.

We have implemented some functions (in CoCoA) to look for a desired upper triangular bi-change of coordinates. For example, the following g works fine for J, indeed there exists many of such g:

$$g := g_1 \times g_2 \in \mathrm{GL}_6(\mathbb{Q}) \times \mathrm{GL}_{10}(\mathbb{Q}),$$

where $g_1 \colon \mathbb{Q}[\underline{x}] \to \mathbb{Q}[\underline{x}]$ is given by

$$
\begin{aligned}
x_4 &\mapsto x_1 + x_4,\\
x_6 &\mapsto x_3 + x_6,
\end{aligned}
$$

and sends x_i for $i \neq 4, 6$ to itself and let g_2 to be the identity map over $\mathbb{Q}[\underline{t}]$. One can compute that $|G| = 98$, $B = (t_7x_3^2, t_4t_6x_5^2)$. It is easy to verify that

$$
I_{(k,\star)} = G_{(k,\star)}, \quad \text{for} \quad k > 2 \iff \begin{cases} (t_7x_3^2)(t_1, \dots, t_{10})^2 \subseteq G, \\ (t_4t_6x_5^2)(t_1, \dots, t_{10}) \subseteq G, \end{cases}
\tag{3.2}
$$

and since in the right side of (3.2) both containments are valid we conclude with $\mathrm{reg}(J^k) = 3k$ for all $k > 2$.

Taking several ideas from Example 3.1 now we are able to quickly find an answer to Question A for J_1. In the following we show that $\mathrm{reg}(J_1^k) = 3k$, for all $k > 2$.

EXAMPLE 3.2. Let $S = \mathbb{Q}[x_1, \dots, x_6]$ and let J_1 be the ideal of (1.5). Let $T = \mathbb{Q}[t_1, \dots, t_{10}, x_1, \dots, x_6]$ in DegRevLex order, and let P_1 be the defining ideal of the Rees ring of J_1, so $R(J_1) = T/P_1$. One can observe that P has 15 elements of bidegree (1, 1), 10 elements of bidegree (3, 0), and 12 elements of bidegree (4, 0). Take g to be the following simple upper triangular bi-transformation:

$$g := g_1 \times g_2 \in \mathrm{GL}_6(\mathbb{Q}) \times \mathrm{GL}_{10}(\mathbb{Q}),$$

where $g_1 : \mathbb{Q}[\underline{x}] \to \mathbb{Q}[\underline{x}]$ shall be given by

$$x_4 \mapsto x_2 + x_4,$$
$$x_6 \mapsto x_1 + x_6,$$

and sending the rest to themselves and take $g_2 : \mathbb{Q}[\underline{t}] \to \mathbb{Q}[\underline{t}]$ to be

$$t_8 \mapsto t_7 + t_8,$$

and for $i \neq 8$, $t_i \mapsto t_i$. Computations by CoCoA shows that $|G| = 144$, $B = (t_{10}x_2x_3, t_2t_4x_5^2)$. Since $I := \text{in}(g(P)) = G + B$, we have

$$I_{(k,\star)} = G_{(k,\star)}, \quad \text{for} \quad k > 2 \iff \begin{cases} (t_{10}x_2x_3)(t_1, \ldots, t_{10})^2 \subseteq G, \\ (t_2t_4x_5^2)(t_1, \ldots, t_{10}) \subseteq G, \end{cases} \tag{3.3}$$

and since it is easy to check that the right side of (3.3) is holding, we obtain that $\text{reg}(J_1^k) = 3k$ for all $k > 2$.

We conclude with the following two corollaries which indicate that ideals J, (1.3), and J_1, (1.5), are very tightly related.

Corollary 3.3. *When the characteristic of the base field is zero, all the powers of J, and J_1, but the second power have linear resolution.*

Since the least exponent k_0 for J^k, and also for J_1^k in order to have linear resolution for all $k > k_0$ is 2, the following question seems to be interesting to discover:

QUESTION B. Does there exist an ideal Q with generators of the same degree d over some polynomial ring $S = K[x_1, \ldots, x_r]$, for which $\text{reg}(Q^k) = kd$, $\forall k \neq 3$ or $\forall k \neq 2, 3$?

As we mentioned in Introduction, it is easy to check that T/P and T/P_1 have the same multigraded Hilbert series, where P, and P_1 are the defining ideals of Rees rings of J and J_1 correspondingly. The immediate result is as follows:

Corollary 3.4. $\text{HS}(S/J^k) = \text{HS}(S/J_1^k) \forall k$, *and so* $\beta_{i,j}(J^k) = \beta_{i,j}(J_1^k)\ \forall i, j, \forall k$.

ACKNOWLEDGMENT. The results of this paper were obtained during the visit of the author to Dipartimento di Matematica, Universitá di Genova, Italia. The author would like to express his deep gratitude to Professor Aldo Conca for the kind invitation and for his warm hospitality whose guidance and support were crucial for the successful completion of this project. During the stay in Genova, the author was supported by a grant within the frame of the Italian network on “Commutative, Combinatorial, Computational Algebra” PRIN 2006-07, directed by Professor Valla. It is a pleasure for the author to warmly thank him and also the kind staff in DIMA as well. Finally, the role of the software package CoCoA in computations of concrete examples as we worked on this project is acknowledged.

References

[1] A. Aramova, K. Crona and E. De Negri: *Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions*, J. Pure Appl. Algebra **150** (2000), 215–235.

[2] W. Bruns and J. Herzog: Cohen-Macaulay Rings, Cambridge Univ. Press, Cambridge, 1993.

[3] CoCoATeam, *CoCoA*: *a system for doing Computations in Commutative Algebra*, available at `http://cocoa.dima.unige.it`.

[4] A. Conca: *Regularity jumps for powers of ideals*; in Commutative Algebra, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall/CRC, Boca Raton, FL, 2006,

[5] A. Conca and J. Herzog: *Castelnuovo-Mumford regularity of products of ideals*, Collect. Math. **54** (2003), 137–152.

[6] S.D. Cutkosky, J. Herzog and N.V. Trung: *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math. **118** (1999), 243–261.

[7] D. Eisenbud: Commutative Algebra, With a View Toward Algebraic Geometry, Graduate Texts in Mathematics **150**, Springer, New York, 1995.

[8] D. Eisenbud: The Geometry of Syzygies, A Second Course in Commutative Algebra and Algebraic Geometry, University of California, Berkeley, 2002.

[9] D. Eisenbud and S. Goto: *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

[10] J. Herzog, T. Hibi and X. Zheng: *Monomial ideals whose powers have a linear resolution*, Math. Scand. **95** (2004), 23–32.

[11] V. Kodiyalam: *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128** (2000), 407–411.

[12] D. Mumford: Lectures on Curves on an Algebraic Surface, Princeton Univ. Press, Princeton, N.J., 1966.

[13] T. Römer: *Homological properties of bigraded algebras*, Illinois J. Math. **45** (2001), 1361–1376.

Faculty of Mathematical Sciences and Computer
Tarbiat Moallem University
Tehran
Iran
e-mail: borna@tmu.ac.ir