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Osaka J. Math.
46 (2009), 563–597

THE STRUCTURE OF ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$ (THE NORMAL QUARTIC HYPERSURFACE CASE. II)

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(Received March 14, 2008)

Abstract

We obtain the affirmative answer for a special case of the linearization problem for algebraic embeddings of $\mathbb{C}^2$ into $\mathbb{C}^3$. Indeed, we determine all the compactifications (X, Y) of $\mathbb{C}^2$ such that X are normal quartic hypersurfaces in $\mathbb{P}^3$ without triple points and Y are hyperplane sections of X. Moreover, for each (X, Y), we construct a tame automorphism of $\mathbb{C}^3$ which transforms the hypersurface $X \setminus Y$ onto a coordinate hyperplane.

1. Introduction

A polynomial mapping $f\colon \mathbb{C}^n \to \mathbb{C}^m$ is called an *algebraic embedding* of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $m > n \geq 1$ if f is injective and if the image of f is a smooth algebraic subvariety of $\mathbb{C}^m$. Let $\mathrm{Aut}(\mathbb{C}^n)$ be the group of algebraic automorphisms of $\mathbb{C}^n$. Here we consider the following conjecture:

Conjecture. *Let $f\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ be an algebraic embedding. Then f is equivalent to a linear embedding up to* $\mathrm{Aut}(\mathbb{C}^n)$ *and* $\mathrm{Aut}(\mathbb{C}^{n+1})$, *equivalently to say, there exists an algebraic automorphism of $\mathbb{C}^{n+1}$ which transforms the image $f(\mathbb{C}^n)$ onto a coordinate hyperplane.*

For the case $n = 1$, Abhyankar-Moh [1] and Suzuki [14] showed that the conjecture is true. For the cases $n \geq 2$, the conjecture is still unsolved. In this paper, we will consider the case $n = 2$ only. Our approach is geometric and our main tool is a method of compactifications of $\mathbb{C}^2$. Let $f\colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be an algebraic embedding. We identify $\mathbb{C}^3$ with an affine part of the complex projective space $\mathbb{P}^3$ in the standard way. We denote by X_f the closure of the image of f in $\mathbb{P}^3$ and put $Y_f := X_f \setminus f(\mathbb{C}^2)$. By construction, we see that Y_f is a hyperplane section of X_f and that $X_f \setminus Y_f$ is biregular to $\mathbb{C}^2$, that is (X_f, Y_f) is a *compactification of* $\mathbb{C}^2$. We call Y_f the *boundary* of the compactification. Our main purpose is to write down explicitly a defining equation of the image of f up to affine transformations of $\mathbb{C}^3$ and to construct explicitly

2000 Mathematics Subject Classification. Primary 14R10; Secondary 32J05.

an algebraic automorphism of $\mathbb{C}^3$ linearizing the defining equation, when the image of f is of low degree. This explicit way is very important for us not only to obtain examples but also to find geometric invariants and inductive methods. In this direction, in Ohta [10], we showed that the conjecture is true when the degree of the image of f is less than or equal to three. For the case of degree three, we needed a so-called *Nagata automorphism* (cf. [10]) to linearize some embedding.

Next we consider the case of degree four. Then we have the following three possibilities: (1) X_f is normal and it has at least a triple point; (2) X_f is normal and it has no triple points; (3) X_f is non-normal. For the case (1), in Ohta [11], we showed that the conjecture is true, and we needed a generalization and an analogue of a Nagata automorphism to linearize some embeddings. In this paper, we will treat the case (2) only. The case (3) will be dealt with elsewhere. Thus it suffices to consider a compactification (X, Y) of $\mathbb{C}^2$ such that X is a normal quartic hypersurface in $\mathbb{P}^3$ without triple points and Y is a hyperplane section of X. First we will determine the defining equations of such compactifications (X, Y) by using the classification of minimal normal compactifications of $\mathbb{C}^2$ due to Morrow [9] and the structure theorem of minimally elliptic singularities due to Laufer [8]. Finally, for each (X, Y), we will construct a tame automorphism of $\mathbb{C}^3$ explicitly which linearizes the defining equation of $X \setminus Y$.

From now on to the end of this paper, we assume the following:

ASSUMPTION. Let X be a normal quartic hypersurface in $\mathbb{P}^3$ without triple points and Y a hyperplane section of X such that $X \setminus Y$ is biholomorphic to $\mathbb{C}^2$. Denote by H the hyperplane in $\mathbb{P}^3$ with $Y = X \cap H$.

We define some notations as follows. Let $Y = \bigcup_{i=1}^{t} Y_i$ be the irreducible decomposition of Y. We put $\mathcal{Y} := H|_X$. We note that $\operatorname{Supp}\mathcal{Y} = Y$ and $\mathcal{O}_H(X|_H) \cong \mathcal{O}_{\mathbb{P}^2}(4)$. We put $x := \operatorname{Sing} X = \{x_1, \dots, x_m\}$. Let $\pi\colon M \to X$ be the minimal resolution of X with exceptional set $E = \bigcup_{i=1}^{s} E_i := \pi^{-1}(x)$, where each E_i is irreducible. We denote by $\hat{C}$ the proper transform of a curve C in X by π. In §2, we shall see that X has a unique minimally elliptic double point, which is denoted by x_1, and that $Z^2 = -1, -2$ for the fundamental cycle Z of $\pi^{-1}(x_1)$. Then our main results are the following:

Theorem 1. *Let (X, Y) be a pair satisfying* Assumption. *Then the weighted dual graph of $\hat{Y} \cup E$ is one of* Fig. 1, *where the notations $\bullet$, $\circ$, $\triangle$ mean smooth rational curves with self-intersection numbers -1, -2, -3 respectively and all $\circ$, $\triangle$ are irreducible components of E.*

Theorem 2. *For each weighted dual graph of $\hat{Y} \cup E$ in* Theorem 1, *the defining equation of (X, Y) is one of the following up to* $\operatorname{Aut}(\mathbb{P}^3)$:

(XV) $X\colon\ (z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 + z_0z_3^3 + z_1^3z_3 = 0,\ \beta^2 - 4\alpha\gamma = 0,$

(XVI) $X\colon\ (z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 + z_0z_3^3 + z_1^3z_3 = 0,\ \beta^2 - 4\alpha\gamma \neq 0,$

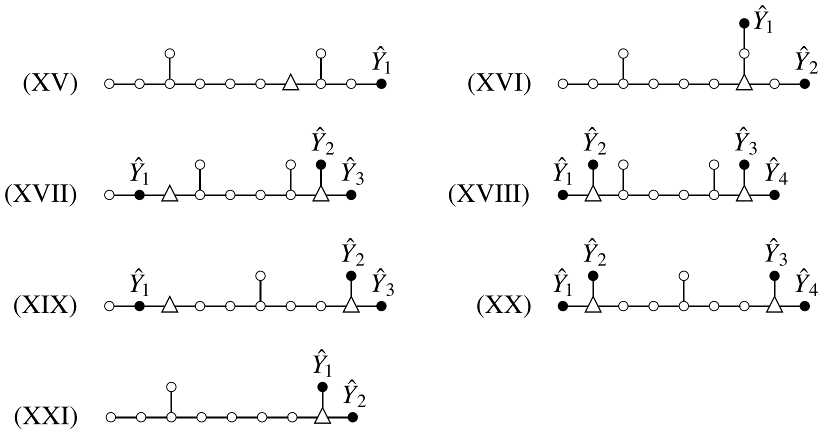


Fig. 1.

(XVII) X: $(z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - (z_0z_3 + z_1^2)^2 + z_1z_3^3 = 0$, $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} = 0$,
(XVIII) X: $(z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - (z_0z_3 + z_1^2)^2 + z_1z_3^3 = 0$, $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} \neq 0$,
(XIX) X: $(z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - z_1^4 + z_0z_3^3 + \delta z_1^2z_3^2 = 0$, $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} = 0$,
(XX) X: $(z_2z_3 + \alpha z_0^2 + \beta z_0z_1 + \gamma z_1^2)^2 - z_1^4 + z_0z_3^3 + \delta z_1^2z_3^2 = 0$, $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} \neq 0$,
(XXI) X: $z_2^2z_3^2 + (2z_0^3 + 3z_0z_1z_3)z_2 - z_1^3z_3 - (3/4)z_0^2z_1^2 + z_0z_3^3 + \delta(z_1z_3 + z_0^2)z_3^2 = 0$,
where $z = (z_0 : z_1 : z_2 : z_3)$ *is a homogeneous coordinate of* $\mathbb{P}^3$, $H = \{z_3 = 0\}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *and* $\alpha \neq 0$.

REMARK. In Theorems 1 and 2, we continue to number the types of (X, Y) from the previous paper [11] and we obtain some invariants as follows:
(XV) $Z^2 = -2$, $\mathcal{Y} = 4Y_1$ (Y_1: line), $x = \{x_1\}$.
(XVI) $Z^2 = -2$, $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: line), $x = \{x_1\}$.
(XVII) $Z^2 = -2$, $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i: line), $x = \{x_1, x_2\}$.
(XVIII) $Z^2 = -2$, $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: line), $x = \{x_1\}$.
(XIX) $Z^2 = -2$, $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i: line), $x = \{x_1, x_2\}$.
(XX) $Z^2 = -2$, $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: line), $x = \{x_1\}$.
(XXI) $Z^2 = -1$, $\mathcal{Y} = 2Y_1 + Y_2$ (Y_1: line, Y_2: conic), $x = \{x_1\}$.
For each type, $x_1 = (0 : 0 : 1 : 0)$ is the unique minimally elliptic double point and x_2

is a rational double point of type A_1, where

$$x_2 = \begin{cases} (\beta : -2\alpha : \beta : 0) & \text{for the type (XVII) and} \quad \beta^2 - 4\alpha(\gamma - 1) = 0. \\ (\beta : -2\alpha : -\beta : 0) & \text{for the type (XVII) and} \quad \beta^2 - 4\alpha(\gamma + 1) = 0. \\ (\beta : -2\alpha : 0 : 0) & \text{for the type (XIX).} \end{cases}$$

Moreover, every line in X through x_1 is an irreducible component of Y (see Lemma 3.2 (ii) and Lemma 4.1 (v)).

Here we recall some special subgroups of $\mathrm{Aut}(\mathbb{C}^3)$. Let $A(3, \mathbb{C})$ and $J(3, \mathbb{C})$ be the subgroups of all affine transformations and de Jonquières automorphisms respectively. Let us denote by $T(3, \mathbb{C})$ the subgroup generated by $A(3, \mathbb{C})$ and $J(3, \mathbb{C})$. An algebraic automorphism of $\mathbb{C}^3$ is said to be *tame* if it is an element of $T(3, \mathbb{C})$ (cf. [11]).

Theorem 3. *For each defining equation of* (X, Y) *in* Theorem 2, *there exists a tame automorphism of* $\mathbb{C}^3$ *which transforms* $X \setminus Y$ *onto a coordinate hyperplane.*

As a consequence of Theorems 2 and 3, we obtain the following:

Theorem 4. *Let* $f : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ *be an algebraic embedding. Assume that* X_f *is a normal quartic hypersurface in* $\mathbb{P}^3$ *without triple points. Then* f *is equivalent to a linear embedding up to* $\mathrm{Aut}(\mathbb{C}^2)$ *and* $T(3, \mathbb{C})$.

Indeed, if one has such an algebraic embedding f, then (X_f, Y_f) has one of the defining equations of the types (XV) through (XXI) up to $\mathrm{Aut}(\mathbb{P}^3)$ by Theorem 2 and there exists a tame automorphism of $\mathbb{C}^3$ transforming $f(\mathbb{C}^2) = X_f \setminus Y_f$ onto a coordinate hyperplane by Theorem 3. Thus we obtain Theorem 4.

NOTATION. $b_i(V) = \dim_{\mathbb{R}} H^i(V, \mathbb{R})$: i-th Betti number of V.
$\mathrm{Exc}\,\varphi$: exceptional set of birational morphism $\varphi : V \to W$.
$\mathrm{Pic}(V)$: Picard group of V.
K_V: canonical divisor of V.
ω_V: dualizing sheaf of V.
$\mathfrak{m}_{V,v}$: maximal ideal of $\mathcal{O}_{V,v}$.
$\mathrm{mult}_W V$: multiplicity of V at general point of W.
$D|_V$: restriction of Cartier divisor D to V.
$(D \cdot C)_{V,v}$: local intersection number of D and C at $v \in V$.
$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent.
(V, v): normal two-dimensional singularity.
$p_g(v)$: geometric genus of (V, v).
$p_g(v_1, \ldots, v_n) = \sum_{i=1}^{n} p_g(v_i)$.

$\mathbb{N} = \{1, 2, 3, \dots\}$: set of all positive integers.
$(-n)$-curve: smooth rational curve with self-intersection number $-n$.
$\overset{-n}{\circ}$: $(-n)$-curve.
$\odot$: 0-curve.
$\bullet$: (-1)-curve.
$\circ$: (-2)-curve.
$\triangle$: (-3)-curve.

2. Preliminaries

In this section, we shall describe the fundamental properties of a pair (X, Y) satisfying Assumption in §1. We use the same notation as that in §1. Let $Y = \bigcup_{i=1}^{t} Y_i$ be the irreducible decomposition of Y. We denote by $\deg Y_i$ the degree of Y_i as a plane curve of $H \cong \mathbb{P}^2$. We set $\mathcal{Y} := H|_X = \sum_{i=1}^{t} k_i Y_i$, where $k_i \in \mathbb{N}$ and $\sum_{i=1}^{t} k_i \deg Y_i = 4$. We put $x := \operatorname{Sing} X = \{x_1, \dots, x_m\}$. Let $\pi\colon M \to X$ be the minimal resolution of X with exceptional set $E = \bigcup_{j=1}^{s} E_j := \pi^{-1}(x)$, where each E_j is irreducible. We may assume that $\pi^{-1}(x_i) = \bigcup_{j=s_{i-1}+1}^{s_i} E_j$ for $1 \le i \le m$, where $0 =: s_0 \le s_1 \le \cdots \le s_m := s$. Let $Z^{(i)} = \sum_{j=s_{i-1}+1}^{s_i} a_j E_j$ be the fundamental cycle of $\pi^{-1}(x_i)$ with $a_j \in \mathbb{N}$. We denote by $\hat{C}$ the proper transform of a curve C in X by π. Let Γ be a general smooth hyperplane section of X with $\Gamma \cap x = \emptyset$. We have the relations $(\hat{\Gamma} \cdot \hat{Y}_i)_M = (\Gamma \cdot Y_i)_X = \deg Y_i$ and $\hat{\Gamma} \sim \sum_{i=1}^{t} k_i \hat{Y}_i + \sum_{j=1}^{s} b_j E_j$ with $b_j \in \mathbb{N}$. We note that $\omega_X = \mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and $x \subset Y$ and that $M \setminus (\hat{Y} \cup E)$ is biholomorphic to $\mathbb{C}^2$. By Kodaira [6] and Ramanujam [12], we see that $X \setminus Y$ and $M \setminus (\hat{Y} \cup E)$ are biregular to $\mathbb{C}^2$. In particular, X and M are rational surfaces. Then we have the next proposition.

Proposition 2.1 (Ohta [10]). *One obtains the following*:
(i) $H_0(X, \mathbb{Z}) \cong H_0(Y, \mathbb{Z}) = \mathbb{Z}$.
(ii) $H_1(X, \mathbb{Z}) \cong H_1(Y, \mathbb{Z}) = 0$.
(iii) $H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z}) = \bigoplus_{i=1}^{t} \mathbb{Z} \cdot Y_i$.
(iv) $H_3(X, \mathbb{Z}) \cong H_3(Y, \mathbb{Z}) = 0$.
(v) $H^1(X, \mathcal{O}_X) = 0$.
(vi) $p_g(x) = 1$.
(vii) *X is not a cone.*
(viii) $\gcd(\deg Y_1, \dots, \deg Y_t) = 1$.
(ix) $\operatorname{mult}_p X \le \sum_{i=1}^{t} k_i \operatorname{mult}_p Y_i$ $(\forall p \in Y = X \cap H)$.

REMARK. (1) By (i) and (ii), Y is a connected divisor without cycles. In particular, each Y_i is a rational curve without nodes. If Y contains at least two lines, then Y consists of lines which meet at only one point. Indeed, this follows since Y has no cycles and each Y_i is a plane curve.

(2) By (vi), (vii) and Assumption in §1, we may assume that x_1 is a minimally elliptic double point and $x \setminus \{x_1\}$ consists of at most rational double points. For simplic-

ity, we put $Z := Z^{(1)}$. By Artin [2] and Laufer [8], we see that $K_M \sim -Z$, $Z^2 = -1, -2$ and $Z^{(i)2} = -2$ for $2 \leq i \leq m$.

(3) Since $(M, \hat{Y} \cup E)$ also satisfies the assertions (i) through (v), $\hat{Y} \cup E$ is a connected divisor without cycles (cf. [10]). By Noether's formula, we obtain $b_2(\hat{Y}) + b_2(E) = b_2(M) = 10 - Z^2$. Thus $\hat{Y} \cup E$ consists of $10 - Z^2$ rational curves.

For the divisor $\mathcal{Y}$, we obtain the following classification. In the last part of this section, we will make this classification to be detailed.

Lemma 2.2. *There exist the following seven possibilities for the divisor* $\mathcal{Y}$:
(i) $\mathcal{Y} = 4Y_1$ (Y_1: *line*) *with* $x \subset Y_1$.
(ii) $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i: *line*) *with* $x \subset Y_1$.
(iii) $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: *line*) *with* $x \subset Y$.
(iv) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i: *line*) *with* $x \subset Y_1$ *and* $Y_1 \cap Y_2 \cap Y_3 = \{$*one point*$\}$.
(v) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: *line*) *with* $x = \{x_1\} = Y_1 \cap Y_2 \cap Y_3 \cap Y_4$.
(vi) $\mathcal{Y} = 2Y_1 + Y_2$ (Y_1: *line*, Y_2: *conic*) *with* $x \subset Y_1$ *and* $Y_1 \cap Y_2 = \{$*one point*$\}$.
(vii) $\mathcal{Y} = Y_1 + Y_2$ (Y_1: *line*, Y_2: *cuspidal cubic*) *with* $x \subset \operatorname{Sing} Y$ *and* $Y_1 \cap Y_2 = \{$*one point*$\}$.

Proof. By Proposition 2.1 (ii), (viii) and (ix), we obtain the assertions. □

For the fundamental cycles Z and $Z^{(i)}$, we shall prove some lemmas with strong effect to the structure of (X, Y).

Lemma 2.3 ([8], [10]). *One obtains the following*:
(1) *Assume that* $Z^2 = -2$. *Then*

$$\pi^* \mathfrak{m}_{X,x_1} \cong \mathcal{O}_M(-Z).$$

Moreover, the blowing-up morphism at x_1 *of* X *factors* π *and* $(\hat{C} \cdot Z) = \operatorname{mult}_{x_1} C$ *for any curve* C *in* X *through* x_1.
(2) *Assume that* $Z^2 = -1$. *Denote by* E_1 *a unique irreducible component* E_i *of* Z *with* $(E_i \cdot Z) = -1$ *and* $a_i = 1$. *Then there exists a unique point* p_0 *of* $E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$ *such that*

$$(\pi \circ \pi_0)^* \mathfrak{m}_{X,x_1} \cong \mathcal{O}_{M'}(-Z' - 2E'_0),$$

where $\pi_0 \colon M' \to M$ *is a blowing-up at* p_0 *with exceptional curve* E'_0 *and* Z' *is the proper transform of* Z *in* M'. *Moreover, the blowing-up morphism at* x_1 *of* X *factors* $\pi \circ \pi_0$ *and* $(\hat{C}' \cdot Z' + 2E'_0) = \operatorname{mult}_{x_1} C$ *for any curve* C *in* X *through* x_1, *where* $\hat{C}'$ *is the proper transform of* C *in* M'.

Proof. First we use Theorem 3.13 in [8] and the universal property of blowing-up (cf. Proposition II.7.14 in [5]). By applying the same argument as in the proof of Lemma 3 in [10], we obtain the assertions. □

REMARK. In (ii), we note that $\pi_0^* Z = Z' + E_0'$, $K_{M'} \sim -Z'$ and $Z'^2 = -2$ (cf. §4).

Lemma 2.4 ([2], [10]). *Assume that x contains at least two points. Then*

$$\pi^* \mathfrak{m}_{X,x_i} \cong \mathcal{O}_M(-Z^{(i)})$$

for $2 \le i \le m$. Moreover, the blowing-up morphism at x_i of X factors π and $(\hat{C} \cdot Z^{(i)}) = \operatorname{mult}_{x_i} C$ for any curve C in X through x_i.

Proof. First we use Theorem 4 in [2] and the universal property of blowing-up (cf. Proposition II.7.14 in [5]). By applying the same argument as in the proof of Lemma 3 in [10], we obtain the assertions. □

Lemma 2.5. *One obtains the following*:
(i) *Assume that C is a line or a conic in X. Then $\hat{C} \cong \mathbb{P}^1$ and $(\hat{C} \cdot Z^{(i)}) = 1$ when $x_i \in C$ $(1 \le i \le m)$. If $x_1 \in C$, then $\hat{C}$ is a (-1)-curve with $(\hat{C} \cdot Z) = 1$. If $x_1 \notin C$, then $\hat{C}$ is a (-2)-curve with $(\hat{C} \cdot Z) = 0$.*
(ii) *Assume that C is a plane cuspidal cubic in X and $\operatorname{Sing} C = \{x_1\}$. Then $\hat{C} \cong \mathbb{P}^1$ and $(\hat{C} \cdot Z^{(i)}) = 1$ when $x_i \in C$ $(2 \le i \le m)$. If $Z^2 = -2$, then $\hat{C}$ is a 0-curve with $(\hat{C} \cdot Z) = 2$. If $Z^2 = -1$, then $\hat{C}$ is a 0-curve with $(\hat{C} \cdot Z) = 2$ or a (-1)-curve with $(\hat{C} \cdot Z) = 1$.*

Proof. By Lemma 2.3, we note that the blowing-up morphism at x_1 of X factors π or $\pi \circ \pi_0$ if $Z^2 = -2$ or -1 respectively. By Lemmas 2.3, 2.4 and the adjunction formula, we obtain the assertions. □

REMARK. In (i) and (ii), we note that $\hat{C} \cup \pi^{-1}((x \setminus \{x_1\}) \cap C)$ is a simple normal crossing divisor of smooth rational curves. In (i), we see that $\hat{C}$ meets $\pi^{-1}(x_1)$ transversally at only one point if $x_1 \in C$.

Lemma 2.6 (Reid [13]). *One obtains the following*:
(i) *Z is a numerically 2-connected divisor of M with $\omega_Z \cong \mathcal{O}_Z$ and $p_a(Z) = h^0(\mathcal{O}_Z) = h^1(\mathcal{O}_Z) = 1$. Here an effective divisor D of smooth projective surface is said to be nummerically n-connected for $n \ge 0$ if it satisfies the condition $(D_1 \cdot D_2) \ge n$ for every effective decomposition $D = D_1 + D_2$ with $D_1, D_2 > 0$.*
(ii) *There exists an exact sequence*

$$0 \to \mathbb{C} \to \operatorname{Pic}(Z) \xrightarrow{\deg} \mathbb{Z}^{\oplus s_1} \to 0$$

of abelian groups, where the homomorphism deg *is given for $\mathcal{L} \in \operatorname{Pic}(Z)$ by*

$$\deg \mathcal{L} = \left(\deg_{E_1} \mathcal{L}|_{E_1}, \dots, \deg_{E_{s_1}} \mathcal{L}|_{E_{s_1}}\right).$$

(iii) *If $\mathcal{L}$ is a nef line bundle on Z with* $\deg_Z \mathcal{L} := \sum_{i=1}^{s_1} a_i \deg_{E_i} \mathcal{L}|_{E_i} = 1$, *then there exists a unique smooth point P of Z such that $\mathcal{L} \cong \mathcal{O}_Z(P)$.*
(iv) *If P and Q are smooth points of Z, then $P = Q$ if and only if $\mathcal{O}_Z(n(P-Q)) \cong \mathcal{O}_Z$ for some integer $n \geq 1$.*

Proof. (i) By the adjunction formula, Lemma 3.11 and Theorem 4.21 in [13], we obtain the assertions.

(ii) Note that $H^1(Z, \mathbb{Z}) = 0$ and $H^2(Z, \mathbb{Z}) \cong \mathbb{Z}^{\oplus s_1}$ since $\hat{Y} \cup E$ has no cycles and $\operatorname{Supp} Z$ consists of s_1 rational curves. Note that $H^1(\mathcal{O}_Z) \cong \mathbb{C}$ and $H^2(\mathcal{O}_Z) = 0$ by (i) and $\dim_{\mathbb{C}} Z = 1$. By applying the exponential cohomology sequence of sheaves on Z, we obtain the assertion.

(iii) By Lemma 4.23 in [13], we obtain the assertion.

(iv) By noting (ii) and (iii), we obtain the assertion. □

We shall prove some useful lemmas for smooth compactifications of $\mathbb{C}^2$. It is well-known that the weighted dual graph of a boundary of minimal normal compactification of $\mathbb{C}^2$ is a linear tree of smooth rational curves by Ramanujam [12] and these graphs are classified by Morrow [9] (cf. Proposition 2 in [10]). Here a smooth compactification (S, C) of $\mathbb{C}^2$ is said to be *minimally normal* if it satisfies the following two conditions: (1) C is a simple normal crossing divisor; (2) any (-1)-curve in C meets at least three other irreducible components of C.

Lemma 2.7. *There exists no boundary C of smooth compactification of $\mathbb{C}^2$ satisfying the following conditions*:
(i) *C contains a smooth rational curve C_0 with $C_0^2 \geq -1$.*
(ii) *$\overline{C \setminus C_0}$ consists of at least three connected components, which are denoted by C_1, $C_2, \ldots, C_n$ with $n \geq 3$.*
(iii) *C_i meets C_0 transversally at only one point for any $1 \leq i \leq n$.*
(iv) *C_i is a simple normal crossing divisor of smooth rational curves whose self-intersection numbers are less than or equal to -2 for any $i = 1, 2$.*

Proof. Assume that there exists such a smooth compactification (S, C) of $\mathbb{C}^2$. By applying some blowing-ups on $(C_3 \cup \cdots \cup C_n) \setminus C_0$, we obtain a smooth compactification (S', C') of $\mathbb{C}^2$ with simple normal crossing boundary, where C' is the total transform of C in S'. Let C_i' be the total transform of C_i in S' for $0 \leq i \leq n$. Then C' satisfies the following conditions:
(1) C_0' is a smooth rational curve in C' with $(C_0')^2 \geq -1$;
(2) $\overline{C' \setminus C_0'}$ consists of the n connected components $C_1', C_2', \ldots, C_n'$ with $n \geq 3$;
(3) C_i' meets C_0' transversally at only one point for any $1 \leq i \leq n$;
(4) C_i' is a simple normal crossing divisor of smooth rational curves whose self-intersection numbers are less than or equal to -2 for any $i = 1, 2$;
(5) C_i' is a simple normal crossing divisor of smooth rational curves for any $3 \leq i \leq n$.

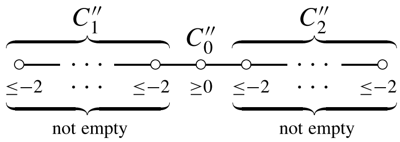


Fig. 2.

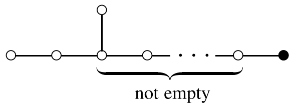


Fig. 3.

By [12], we obtain a linear tree in Fig. 2 as a boundary of minimal normal compactification of $\mathbb{C}^2$ by applying some blowing-downs in C', where C_0'', C_1'' and C_2'' are the proper transforms of C_0', C_1' and C_2' respectively. However this dual graph is not found in Morrow's classification. This is a contradiction. □

Lemma 2.8. *Assume that C is a boundary of smooth compactification of $\mathbb{C}^2$ satisfying the following conditions*:

(i) *C contains a (-1)-curve C_0.*

(ii) *$\overline{C \setminus C_0}$ consists of exactly two connected components C_1 and C_2.*

(iii) *C_i meets C_0 transversally at only one point for any $i = 1, 2$.*

(iv) *C_1 is a simple normal crossing divisor of (-2)-curves.*

Then the weighted dual graph of $C_0 \cup C_1$ is a linear tree $\bullet\!\!-\!\!\circ\!\!-\!\!\circ\!\!- \cdots -\!\!\circ$.

Proof. Assume that the weighted dual graph of $C_0 \cup C_1$ is not such a linear tree. Then there exists an irreducible component $C_{1,1}$ of C_1 such that $\overline{(C_0 \cup C_1) \setminus C_{1,1}}$ consists of at least three connected components and such that the weighted dual graph of the connected component of $\overline{(C_0 \cup C_1) \setminus C_{1,1}}$ containing C_0 is a linear tree. By contracting the connected component of $\overline{(C_0 \cup C_1) \setminus C_{1,1}}$ containing C_0, we obtain a boundary of smooth compactification of $\mathbb{C}^2$ satisfying the conditions in Lemma 2.7. This is a contradiction. □

Lemma 2.9. *Assume that C is a simple normal crossing boundary of smooth compactification of $\mathbb{C}^2$ which is a union of only one (-1)-curve and some (-2)-curves. Then the weighted dual graph of C is a linear tree* $\circ\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ$ *or a tree as in* Fig. 3.

Proof. Let C_0 be the unique (-1)-curve in C and $\overline{C \setminus C_0} = \bigcup_{i=1}^{n} C_i$ the decomposition into connected components with $n \geq 1$. By Lemma 2.7, we obtain $n = 1, 2$. Then we consider the following cases (1) and (2).

(1) Assume that $n = 2$. By Lemma 2.8, the weighted dual graphs of $C_0 \cup C_1$ and $C_0 \cup C_2$ are linear trees $\bullet\!\!-\!\!\circ\!\!-\!\!\circ\!- \cdots -\!\circ$. By using Morrow's classification after the contraction of $C_0 \cup C_1$ or $C_0 \cup C_2$, we see that the weighted dual graph of C is a linear tree $\circ\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ$.

(2) Assume that $n = 1$. Note that the weighted dual graph of C is not a linear tree by the assumption. Then there exists an irreducible component $C_{1,1}$ of C_1 such that $\overline{C \setminus C_{1,1}}$ consists of at least three connected components and such that the weighted dual graph of the connected component of $\overline{C \setminus C_{1,1}}$ containing C_0 is a linear tree. By contracting the connected component of $\overline{C \setminus C_{1,1}}$ containing C_0, we obtain a simple normal crossing boundary D of smooth compactification of $\mathbb{C}^2$ which is a union of only one (-1)-curve D_0 and some (-2)-curves and such that $\overline{D \setminus D_0}$ consists of at least two connected components. By Lemma 2.7, $\overline{D \setminus D_0}$ consists of exactly two connected components. By the same argument as that in (1), the weighted dual graph of D is a linear tree $\circ\!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\circ$. Hence the weighted dual graph of C is obtained as in Fig. 3. $\square$

From now on to the end of this section, we shall show that some cases do not occur for the classification of the divisor $\mathcal{Y}$ in Lemma 2.2. In the proofs of the following lemmas, we mainly use Lemmas 2.5, 2.7 and 2.8. Especially, we always note Remark of Lemma 2.5.

Lemma 2.10. *It does not occur the case where*

$$\mathcal{Y} = 2Y_1 + 2Y_2 \quad (Y_i\colon \text{ line}) \text{ with } \quad x_1 \notin Y_1 \cap Y_2.$$

Proof. Assume that this case occurs. We may assume that $x_1 \in Y_1 \setminus Y_2$. By Lemma 2.5, $\hat{Y}_1$ is a (-1)-curve and $\hat{Y}_2$ is a (-2)-curve in M. Note the linear equivalence

$$\hat{\Gamma} \sim 2\hat{Y}_1 + 2\hat{Y}_2 + \sum_i b_i E_i$$

with $b_i \in \mathbb{N}$. Then we consider the following cases (1) and (2).

(1) Assume that $Y_1 \cap Y_2$ is a smooth point of X. Note that $(\hat{Y}_1 \cdot \hat{Y}_2) = 1$ by computing the intersection number of the above linear equivalence and $\hat{Y}_1$. By Lemma 2.7, $x \cap (Y_1 \setminus Y_2)$ consists of only one point x_1. Since $\left(\sum_i b_i E_i \cdot \hat{Y}_2\right) = 3 > 0$, $x \cap (Y_2 \setminus Y_1)$ contains at least one point. By Lemma 2.8, $x \cap (Y_2 \setminus Y_1)$ consists of exactly one point, which is denoted by x_2, and the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)$ is a linear tree as in Fig. 4 (1), where $E_{s_1+1}, \ldots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ $(s_2 - s_1 \geq 1)$. By computing the intersection number of the above linear equivalence and $\hat{Y}_2 + Z^{(2)}$, we obtain $b_{s_2} = -1$. This is a contradiction.

(2) Assume that $Y_1 \cap Y_2$ is a singular point of X, which is denoted by x_2. By Lemma 2.4, the blowing-up morphism at x_2 of X factors π. Thus we obtain $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$.

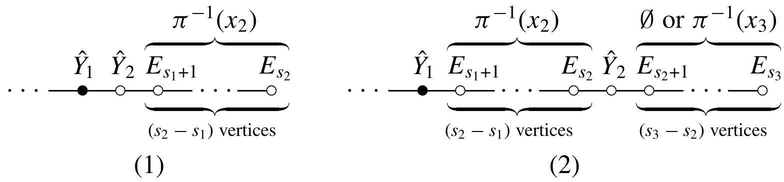


Fig. 4.

By Lemma 2.7, $x \cap (Y_1 \setminus Y_2)$ consists of only one point x_1. By Lemma 2.8, $x \cap (Y_2 \setminus Y_1)$ consists of no points or exactly one point, which is denoted by x_3, and the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x \cap Y_2)$ is a linear tree as in Fig. 4 (2), where $E_{s_1+1}, \dots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ and $E_{s_2+1}, \dots, E_{s_3}$ are the irreducible components of $\pi^{-1}(x_3)$ ($s_2 - s_1 \geq 1$, $s_3 - s_2 \geq 0$). If $x = \{x_1, x_2\}$, then we have $b_{s_1+1} = -1$ by computing the intersection number of the above linear equivalence and $\hat{Y}_2 + Z^{(2)}$. This is a contradiction. If $x = \{x_1, x_2, x_3\}$, then we have $b_{s_1+1} + b_{s_3} = 1$ by computing the intersection number of the above linear equivalence and $\hat{Y}_2 + Z^{(2)} + Z^{(3)}$. This is a contradiction. □

Lemma 2.11. *It does not occur the case where*

$$\mathcal{Y} = 2Y_1 + Y_2 + Y_3 \quad (Y_i\text{: } \textit{line}) \textit{ with } \quad x_1 \in Y_1 \setminus (Y_2 \cup Y_3).$$

Proof. Assume that this case occurs. Note that $x \subset Y_1$ and that $\hat{Y}_1$, $\hat{Y}_2$, $\hat{Y}_3$ are a (-1)-curve and two (-2)-curves in M respectively by Lemma 2.5. If $Y_1 \cap Y_2 \cap Y_3$ is a smooth point of X, then we have $(\hat{Y}_1 \cdot \hat{Y}_2) = (\hat{Y}_2 \cdot \hat{Y}_3) = (\hat{Y}_3 \cdot \hat{Y}_1) = 1$ by computing the intersection number of each $\hat{Y}_i$ and $\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + \hat{Y}_3 + \sum_i b_i E_i$ ($b_i \in \mathbb{N}$). By applying the blowing-up on $\hat{Y}_1 \cap \hat{Y}_2 \cap \hat{Y}_3$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus $Y_1 \cap Y_2 \cap Y_3$ is a singular point of X, which is denoted by x_2. By Lemma 2.4, the blowing-up morphism at x_2 of X factors π. Thus we have $(\hat{Y}_i \cdot \hat{Y}_j) = 0$ for $i \neq j$. Hence the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \hat{Y}_3 \cup \pi^{-1}(x_2)$ is not a linear tree and $x = \{x_1, x_2\}$ by Lemma 2.7. On the other hand, the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \hat{Y}_3 \cup \pi^{-1}(x_2)$ is a linear tree by Lemma 2.8. This is a contradiction. □

Lemma 2.12. *It does not occur the case where*

$$\mathcal{Y} = 2Y_1 + Y_2 \quad (Y_1\text{: } \textit{line},\ Y_2\text{: } \textit{conic}) \textit{ with } \quad x_1 \in Y_1 \setminus Y_2.$$

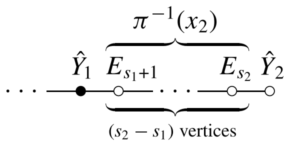


Fig. 5.

Proof. Assume that this case occurs. Note that $x \subset Y_1$ and that $\hat{Y}_1$ and $\hat{Y}_2$ are a (-1)-curve and a (-2)-curve in M respectively by Lemma 2.5. Note the linear equivalence

$$\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + \sum_i b_i E_i$$

with $b_i \in \mathbb{N}$. If $Y_1 \cap Y_2$ is a smooth point of X, then we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 2$ by computing the intersection number of the above linear equivalence and $\hat{Y}_2$. By applying the blowing-ups twice on $\hat{Y}_1 \cap \hat{Y}_2$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus $Y_1 \cap Y_2$ is a singular point of X, which is denoted by x_2. By computing the intersection number of the above linear equivalence and $\hat{Y}_1$, we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1$. If $(\hat{Y}_1 \cdot \hat{Y}_2) = 1$, then each pair of $\hat{Y}_1$, $\hat{Y}_2$ and $\pi^{-1}(x_2)$ meets transversally at only one point since $(\hat{Y}_1 \cdot Z^{(2)}) = (\hat{Y}_2 \cdot Z^{(2)}) = 1$. By applying the blowing-up on $\hat{Y}_1 \cap \hat{Y}_2 \cap \pi^{-1}(x_2)$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$. Hence $x = \{x_1, x_2\}$ by Lemma 2.7 and the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)$ is a linear tree as in Fig. 5 by Lemma 2.8, where $E_{s_1+1}, \cdots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ $(s_2 - s_1 \geq 1)$. By computing the intersection number of the above linear equivalence and $\hat{Y}_2 + Z^{(2)}$, we obtain $b_{s_1+1} = -1$. This is a contradiction. $\square$

Lemma 2.13. *It does not occur the case where*

$$\mathcal{Y} = Y_1 + Y_2 \quad (Y_1\text{: } \textit{line},\ Y_2\text{: } \textit{cuspidal cubic}) \textit{ with } \quad Y_1 \cap Y_2 = \{x_1\} \neq \operatorname{Sing} Y_2.$$

Proof. Assume that this case occurs. In $H \cong \mathbb{P}^2$, Y_1 and Y_2 meet tangentially to the third order at x_1 which is a smooth point of Y_2. By Lemmas 2.3 and 2.5, $\hat{Y}_1$ is a (-1)-curve in M and $(\hat{Y}_1 \cdot Z) = (\hat{Y}_2 \cdot Z) = 1$. Note that $(\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1$ by computing the intersection number of $\hat{Y}_1$ and $\hat{\Gamma} \sim \hat{Y}_1 + \hat{Y}_2 + \sum_i b_i E_i$ $(b_i \in \mathbb{N})$. If $\operatorname{Sing} Y_2$ is a smooth point of X, then we see that $\hat{Y}_2 \cong Y_2$ and $\hat{Y}_2^2 = 1$ by the adjunction formula. By applying the blowing-ups three times on $\operatorname{Sing} \hat{Y}_2$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus $\operatorname{Sing} Y_2$ is a singular point of X, which is denoted by x_2, and in particular $x = \{x_1, x_2\}$. By Lemma 2.4, the blowing-up morphism at x_2 of X factors π. Thus $\hat{Y}_2$ is a smooth curve and in particular a (-1)-curve in M since $(\hat{Y}_2 \cdot Z) = 1$. Note that $(\hat{Y}_2 \cdot Z^{(2)}) = 2$ by Lemma 2.4 and $Z^{(2)} = \sum_{i=s_1+1}^{s_2} a_i E_i$ $(a_i \in \mathbb{N})$. Then we consider the following cases (1), (2) and (3).

(1) Assume that there exists an irreducible component E_{i_1} of $\pi^{-1}(x_2)$ such that $(\hat{Y}_2 \cdot E_{i_1}) = 2$ and $a_{i_1} = 1$. Note that $(\hat{Y}_2 \cdot Z^{(2)} - E_{i_1}) = 0$. By applying the blowing-ups twice on $\hat{Y}_2 \cap E_{i_1}$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction.

(2) Assume that there exist two irreducible components E_{i_1} and E_{i_2} of $\pi^{-1}(x_2)$ such $(\hat{Y}_2 \cdot E_{i_1}) = (\hat{Y}_2 \cdot E_{i_2}) = 1$ and $a_{i_1} = a_{i_2} = 1$. Note that $(\hat{Y}_2 \cdot Z^{(2)} - E_{i_1} - E_{i_2}) = 0$. By applying the blowing-up on $\hat{Y}_2 \cap E_{i_1} \cap E_{i_2}$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction.

(3) Assume that there exists an irreducible component E_{i_1} of $\pi^{-1}(x_2)$ such that $(\hat{Y}_2 \cdot E_{i_1}) = 1$ and $a_{i_1} = 2$. Note that $(\hat{Y}_2 \cdot Z^{(2)} - 2E_{i_1}) = 0$. Note that $\hat{Y}_2 \cup \pi^{-1}(x_2)$ is of simple normal crossing and that x_2 is a rational double point not of type A. If $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$, then x_2 is a rational double point of type A by Lemma 2.8. This is a contradiction. Thus we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 1$. Since $(\hat{Y}_1 \cdot \hat{Y}_2) = (\hat{Y}_1 \cdot Z) = (\hat{Y}_2 \cdot Z) = 1$, each pair of $\hat{Y}_1$, $\hat{Y}_2$ and $\pi^{-1}(x_1)$ meets transversally at only one point. By applying the blowing-up on $\hat{Y}_1 \cap \hat{Y}_2 \cap \pi^{-1}(x_1)$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. □

Lemma 2.14. *It does not occur the case where*

$$\mathcal{Y} = Y_1 + Y_2 \quad (Y_1\text{: } \textit{line},\ Y_2\text{: } \textit{cuspidal cubic}) \textit{ with } \quad Y_1 \cap Y_2 \neq \{x_1\} = \operatorname{Sing} Y_2.$$

Proof. Assume that this case occurs. Note that Y_1 and Y_2 meet in $H \cong \mathbb{P}^2$ tangentially to the third order at a smooth point of Y_2 and that $x \setminus \{x_1\}$ is contained in $Y_1 \cap Y_2$ and $\{x_1\} = \operatorname{Sing} Y_2$. By Lemma 2.5, $\hat{Y}_1$ is a (-2)-curve in M with $(\hat{Y}_1 \cdot Z) = 0$ and $\hat{Y}_2$ is a (-1)-curve with $(\hat{Y}_2 \cdot Z) = 1$ or a 0-curve with $(\hat{Y}_2 \cdot Z) = 2$. Note the linear equivalence

$$\hat{\Gamma} \sim \hat{Y}_1 + \hat{Y}_2 + \sum_i b_i E_i$$

with $b_i \in \mathbb{N}$. If $Y_1 \cap Y_2$ is a smooth point of X, then we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 3$ by computing the intersection number of the above linear equivalence and $\hat{Y}_1$. By applying the blowing-ups three times on $\hat{Y}_1 \cap \hat{Y}_2$, we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7. This is a contradiction. Thus $Y_1 \cap Y_2$ is a singular point of X, which is denoted by x_2, and in particular $x = \{x_1, x_2\}$. Note that $(\hat{Y}_1 \cdot Z^{(2)}) = (\hat{Y}_2 \cdot Z^{(2)}) = 1$ by Lemma 2.4. By computing the intersection number of the above linear equivalence and $\hat{Y}_1$, we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0, 1, 2$. If $(\hat{Y}_1 \cdot \hat{Y}_2) = 1, 2$, then there exists a unique irreducible component E_{i_1} of $\pi^{-1}(x_2)$ such that $\hat{Y}_1$, $\hat{Y}_2$ and E_{i_1} meet at only one point since $(\hat{Y}_1 \cdot Z^{(2)}) = (\hat{Y}_2 \cdot Z^{(2)}) = 1$. By applying the blowing-ups $(\hat{Y}_1 \cdot \hat{Y}_2)$ times on $\hat{Y}_1 \cap \hat{Y}_2 \cap E_{i_1}$, we obtain a smooth compactification of $\mathbb{C}^2$ with the

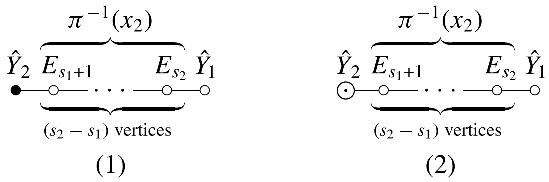


Fig. 6.

conditions in Lemma 2.7. This is a contradiction. Hence we have $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$. Then we consider the following cases (1) and (2).

(1) Assume that $\hat{Y}_2$ is a (-1)-curve in M with $(\hat{Y}_2 \cdot Z) = 1$. By Lemma 2.8, the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)$ is a linear tree as in Fig. 6 (1), where $E_{s_1+1}, \dots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ $(s_2 - s_1 \geq 1)$. By computing the intersection number of the above linear equivalence and $\hat{Y}_1 + Z^{(2)}$, we have $b_{s_1+1} = -1$. This is a contradiction.

(2) Assume that $\hat{Y}_2$ is a 0-curve in M with $(\hat{Y}_2 \cdot Z) = 2$. Since $(\hat{Y}_2 \cdot Z) = 2$, we have $\mathrm{mult}_p Z = 1, 2$, where $\{p\} := \hat{Y}_2 \cap \mathrm{Supp}\, Z$. If $\mathrm{mult}_p Z = 1$, then we obtain a smooth compactification of $\mathbb{C}^2$ with the conditions in Lemma 2.7 by applying the blowing-ups twice on p. This is a contradiction. If $\mathrm{mult}_p Z = 2$, then after applying the blowing-up on p, by Lemma 2.8, we have the weighted dual graph of $\hat{Y}_1 \cup \hat{Y}_2 \cup \pi^{-1}(x_2)$ as in Fig. 6 (2), where $E_{s_1+1}, \dots, E_{s_2}$ are the irreducible components of $\pi^{-1}(x_2)$ $(s_2 - s_1 \geq 1)$. By computing the intersection number of the above linear equivalence and $\hat{Y}_1 + Z^{(2)}$, we obtain $b_{s_1+1} = -1$. This is a contradiction. $\square$

As a consequence, we obtain the following refined classification of the divisor $\mathcal{Y}$.

Proposition 2.15. *There exist the following seven possibilities for the divisor* $\mathcal{Y}$:
(i) $\mathcal{Y} = 4Y_1$ (Y_1: *line*) *with* $x \subset Y_1$.
(ii) $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i: *line*) *with* $x \subset Y_1$.
(iii) $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: *line*) *with* $x \subset Y$ *and* $Y_1 \cap Y_2 = \{x_1\}$.
(iv) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i: *line*) *with* $x \subset Y_1$ *and* $Y_1 \cap Y_2 \cap Y_3 = \{x_1\}$.
(v) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: *line*) *with* $x = \{x_1\} = Y_1 \cap Y_2 \cap Y_3 \cap Y_4$.
(vi) $\mathcal{Y} = 2Y_1 + Y_2$ (Y_1: *line*, Y_2: *conic*) *with* $x \subset Y_1$ *and* $Y_1 \cap Y_2 = \{x_1\}$.
(vii) $\mathcal{Y} = Y_1 + Y_2$ (Y_1: *line*, Y_2: *cuspidal cubic*) *with* $x = \{x_1\} = Y_1 \cap Y_2 = \mathrm{Sing}\, Y_2$.
In particular, for each case, Y *contains at least one line through* x_1.

3. Proof of Theorem 1 for $Z^2 = -2$

In this section, we shall prove Theorem 1 for the case $Z^2 = -2$. Let (X, Y) be a pair satisfying Assumption in §1 and $Z^2 = -2$. We use the same notation as that in §1 and §2. We mainly consider a projection from x_1 and a blowing-up at x_1 to investigate the pair (X, Y). First we note that $\hat{Y} \cup E$ is a connected divisor without cycles which consists of twelve rational curves. For each irreducible component E_i of Z, we may assume that $(E_i \cdot Z) < 0$ for $1 \le i \le s_{0,1}$ and $(E_i \cdot Z) = 0$ for $s_{0,1} < i \le s_1$, where $s_{0,1}$ is an integer with $1 \le s_{0,1} \le s_1$. We put $Z_1 := \sum_{i=1}^{s_{0,1}} a_i E_i$ and $Z_2 := \sum_{i=s_{0,1}+1}^{s_1} a_i E_i$, where $Z_2 = 0$ is allowed. Thus we obtain an effective decomposition $Z = Z_1 + Z_2$. Since $Z^2 = -2$, we note that $s_{0,1} = 1, 2$. Let $\sigma\colon \overline{\mathbb{P}^3} \to \mathbb{P}^3$ be the blowing-up at x_1 with exceptional divisor Δ, which is isomorphic to $\mathbb{P}^2$. Let $\overline{T}$ be the proper transform of a closed algebraic subset T of $\mathbb{P}^3$ by σ. We have that $\sigma|_{\overline{\mathbb{P}^3}\setminus\Delta}\colon \overline{\mathbb{P}^3} \setminus \Delta \cong \mathbb{P}^3 \setminus \{x_1\}$ and $\mathcal{O}_{\overline{\mathbb{P}^3}}(\Delta)|_\Delta \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. We set $\overline{E} := \overline{X} \cap \Delta$. We have that $\sigma|_{\overline{X}\setminus\overline{E}}\colon \overline{X} \setminus \overline{E} \cong X \setminus \{x_1\}$ and that $(\overline{X}, \overline{Y} \cup \overline{E})$ is a compactification of $\mathbb{C}^2$ with dualizing sheaf $\omega_{\overline{X}} \cong \mathcal{O}_{\overline{X}}$. Let $\nu\colon \overline{X}^\nu \to \overline{X}$ be the normalization of $\overline{X}$. Let $\overline{C}^\nu$ be the proper transform of a curve C in X by $\sigma|_{\overline{X}} \circ \nu$. We have that $\nu|_{\overline{X}^\nu\setminus\nu^{-1}(\overline{E})}\colon \overline{X}^\nu \setminus \nu^{-1}(\overline{E}) \cong \overline{X} \setminus \overline{E}$ and that $(\overline{X}^\nu, \overline{Y}^\nu \cup \nu^{-1}(\overline{E}))$ is a compactification of $\mathbb{C}^2$. Let $\psi\colon \mathbb{P}^3 \cdots\to \mathbb{P}^2$ be the projection from x_1 and $\overline{\psi}\colon \overline{\mathbb{P}^3} \to \mathbb{P}^2$ the resolution of indeterminacy of ψ. We have that $\overline{\psi}|_\Delta\colon \Delta \to \mathbb{P}^2$ is an isomorphism and $\overline{\psi}|_{\overline{X}}\colon \overline{X} \to \mathbb{P}^2$ is a generically finite morphism of degree two. We note that $\overline{\Gamma} \sim \overline{H}|_{\overline{X}} + \Delta|_{\overline{X}}$, $\overline{\Gamma}^\nu \sim \nu^*(\overline{H}|_{\overline{X}}) + \nu^*(\Delta|_{\overline{X}})$ and $\hat{\Gamma} \sim \sum_{i=1}^t k_i \hat{Y}_i + \sum_{j=1}^s b_j E_j$ with $b_j \in \mathbb{N}$. Then we have some fundamental lemmas.

Lemma 3.1. *One obtains the following*:
(i) $\overline{X}$ *is non-normal. Moreover,* $\overline{X}|_\Delta = 2\overline{E} = 2$*line and* $\Delta \cap \operatorname{Sing}\overline{X} = \overline{E}$.
(ii) $\operatorname{Sing}\overline{X}^\nu$ *consists of at most rational double points.*
(iii) *There exists a birational morphism* $\overline{\pi}\colon M \to \overline{X}$ *satisfying* $\sigma|_{\overline{X}} \circ \overline{\pi} = \pi$. *Then* $\overline{\pi}^*(\Delta|_{\overline{X}}) = Z$ *and* $\overline{\pi}^*(\overline{\psi}|_{\overline{X}})^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_M(\hat{\Gamma} - Z)$. *Moreover,* $\deg(\overline{\pi}|_{E_i}) = -(E_i \cdot Z)$ *for each irreducible component* E_i *of* Z_1 *and* $\overline{\pi}(\operatorname{Supp} Z_2)$ *is a finite set. In particular,* $\overline{\pi}|_{M\setminus E}\colon M \setminus E \cong \overline{X} \setminus (\overline{E} \cup (\sigma|_{\overline{X}})^{-1}(x \setminus \{x_1\}))$.
(iv) *There exists a birational morphism* $\overline{\pi}^\nu\colon M \to \overline{X}^\nu$ *satisfying* $\nu \circ \overline{\pi}^\nu = \overline{\pi}$. *Then* $\deg(\overline{\pi}^\nu|_{E_i}) = 1$ *and* $\deg(\nu|_{\overline{E_i}^\nu}) = -(E_i \cdot Z)$ *with* $\overline{E_i}^\nu := \overline{\pi}^\nu(E_i)$ *for each irreducible component* E_i *of* Z_1, *and* $\overline{\pi}^\nu(\operatorname{Supp} Z_2)$ *is a finite set. Moreover,* $\overline{\pi}^\nu$ *is a minimal resolution of* $\overline{X}^\nu$ *with* $\operatorname{Exc}\overline{\pi}^\nu = \operatorname{Supp} Z_2 \cup \pi^{-1}(x \setminus \{x_1\})$. *Here one puts the push-forward Weil divisor* $\overline{Z}^\nu := (\overline{\pi}^\nu)_*(Z)$ *of* $\overline{X}^\nu$.
(v) *Let* $\overline{X}^\nu \xrightarrow{g} V \xrightarrow{h} \mathbb{P}^2$ *be the Stein factorization of* $\overline{\psi}|_{\overline{X}} \circ \nu$. *Then* V *is normal,* g *is a birational morphism and* h *is a finite morphism of degree two. In particular,* $g|_{\overline{X}^\nu\setminus\operatorname{Exc} g}\colon \overline{X}^\nu \setminus \operatorname{Exc} g \cong V \setminus g(\operatorname{Exc} g)$ *and* $\operatorname{Exc} g$ *is the proper transform of the union of all lines in* X *through* x_1 *by* $\sigma|_{\overline{X}} \circ \nu$. *Thus one obtains the commutative diagram as in* Fig. 7.
(vi) *Assume that* l *is a line in* X *through* x_1. *Then* $\bar{l}^\nu \cap \operatorname{Sing}\overline{X}^\nu$ *consists of at most one rational double point of type* A *and the weighted dual graph of* $(\overline{\pi}^\nu)^{-1}(\bar{l}^\nu) = \hat{l} \cup (\overline{\pi}^\nu)^{-1}(\bar{l}^\nu \cap \operatorname{Sing}\overline{X}^\nu)$ *is a linear tree* $\bullet$ *or* $\bullet\!\!-\!\!\circ\!\!-\!\!\circ\!\!- \cdots -\!\!\circ$. *In particular,* $g(\bar{l}^\nu)$ *is a smooth point of* V *and* $x \cap l = \{x_1\}$, $\{x_1, A_n\}$ *for some* $n \ge 1$.
(vii) $\operatorname{Sing} V$ *consists of at most rational double points.*

$$
\begin{tikzcd}
 & \overline{X}^{\nu} \arrow[r, "g"] \arrow[d, "\nu"'] & V \arrow[dd, "h"] \\
M \arrow[ur, "\overline{\pi}^{\nu}"] \arrow[r, "\overline{\pi}"] \arrow[dr, "\pi"'] & \overline{X} \arrow[d, "\sigma|_{\overline{X}}"'] \arrow[dr, "\overline{\psi}|_{\overline{X}}"] & \\
 & X \arrow[r, dashed, "\psi|_X"'] & \mathbb{P}^2
\end{tikzcd}
$$

Fig. 7.

Proof. (i), (ii) The assertions are the general properties of the minimally elliptic singularity (X, x_1) with $Z^2 = -2$. Indeed, we can check the assertions by applying a blowing-up of the local analytic defining equation of (X, x_1) in Theorem 4.57 (2) of [7] (cf. [8]).

(iii) There exists a birational morphism $\overline{\pi}\colon M \to \overline{X}$ satisfying $\sigma|_{\overline{X}} \circ \overline{\pi} = \pi$ by Lemma 2.3 (i). In particular, we have $(\overline{\pi})^{-1}(\overline{E}) = \operatorname{Supp} Z$. By the isomorphisms

$$\overline{\pi}^*(\mathcal{O}_{\overline{\mathbb{P}^3}}(-\Delta)|_{\overline{X}}) \cong \overline{\pi}^*(\sigma|_{\overline{X}})^*\mathfrak{m}_{X,x_1} \cong \pi^*\mathfrak{m}_{X,x_1} \cong \mathcal{O}_M(-Z),$$

we obtain $\overline{\pi}^*(\Delta|_{\overline{X}}) \sim Z$. Since $\overline{\pi}^*(\Delta|_{\overline{X}})$ is an effective divisor of M whose support equals to $\operatorname{Supp} Z$ and the intersection matrix of $\operatorname{Supp} Z$ is negative definite, we obtain $\overline{\pi}^*(\Delta|_{\overline{X}}) = Z$. In particular, we have

$$\overline{\pi}^*(\overline{\psi}|_{\overline{X}})^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \overline{\pi}^*\mathcal{O}_{\overline{X}}(\overline{H}|_{\overline{X}}) \cong \overline{\pi}^*\mathcal{O}_{\overline{X}}(\overline{\Gamma} - \Delta|_{\overline{X}}) \cong \mathcal{O}_M(\hat{\Gamma} - Z).$$

Let E_i be any irreducible component of Z. Since $\pi|_{E_i}$ is identified with $\overline{\psi}|_{\overline{X}} \circ \overline{\pi}|_{E_i}$, we obtain $\deg(\overline{\pi}|_{E_i}) = -(E_i \cdot Z)$. By noting that $(\sigma|_{\overline{X}})^{-1}(x) = \overline{E} \cup (\sigma|_{\overline{X}})^{-1}(x \setminus \{x_1\})$, we have $\overline{\pi}|_{M \setminus E}\colon M \setminus E \cong \overline{X} \setminus (\overline{E} \cup (\sigma|_{\overline{X}})^{-1}(x \setminus \{x_1\}))$.

(iv) There exists a birational morphism $\overline{\pi}^{\nu}\colon M \to \overline{X}^{\nu}$ satisfying $\nu \circ \overline{\pi}^{\nu} = \overline{\pi}$ by the lifting property of normalization (cf. Proposition 8.4.3 in [4]). By noting (iii) and that ν is a finite morphism, we obtain the assertions.

(v) Note the general properties of Stein factorization (cf. Corollary III.11.5 in [5]).

(vi) First we note (ii) and that $\hat{l}$ is a (-1)-curve in M by Lemma 2.5 (i). By applying Lemma 4 in [10] for the morphisms $\overline{\pi}^{\nu}\colon (M, (\overline{\pi}^{\nu})^{-1}(\bar{l}^{\nu})) \to (\overline{X}^{\nu}, \bar{l}^{\nu})$ and $g\colon (\overline{X}^{\nu}, \bar{l}^{\nu}) \to (V, g(\bar{l}^{\nu}))$, we obtain the assertions.

(vii) By using (ii), (v) and (vi), we obtain the assertion. □

Lemma 3.2. *One obtains the following*:

(i) $\overline{H} \cap \Delta = \overline{E}$. *In particular, Y is a union of lines through x_1.*

(ii) *Every line in X through x_1 is contained in Y.*

(iii) $\hat{Y}_i$ *is a (-1)-curve in M with* $(\hat{\Gamma} \cdot \hat{Y}_i) = (\hat{Y}_i \cdot Z) = 1$ $(1 \leq i \leq t)$.

(iv) $\overline{Y_i} \cap \overline{Y_j} = \overline{Y_i}^{\nu} \cap \overline{Y_j}^{\nu} = \hat{Y}_i \cap \hat{Y}_j = \emptyset$ $(i \neq j)$.
(v) $x \setminus \{x_1\}$ *consists of at most rational double points of type* A.
(vi) $\operatorname{Exc} g = \overline{Y}^{\nu}$ *and* $g|_{\overline{X}^{\nu} \setminus \overline{Y}^{\nu}} : \overline{X}^{\nu} \setminus \overline{Y}^{\nu} \cong V \setminus g(\overline{Y}^{\nu})$.
(vii) $(\operatorname{Sing} \overline{X}^{\nu}) \setminus \overline{Y}^{\nu} = g^{-1}(\operatorname{Sing} V)$.
(viii) $(\overline{X}^{\nu}, \overline{Y}^{\nu} \cup \operatorname{Supp} \overline{Z}^{\nu})$ *and* $(V, g(\operatorname{Supp} \overline{Z}^{\nu}))$ *are compactifications of* $\mathbb{C}^2$.

Proof. (i) Assume that $\overline{H} \cap \Delta \neq \overline{E}$. Let l be any line in H such that $x_1 \in l \not\subset X$ and $\bar{l} \cap \overline{E} = \emptyset$. Since $(\overline{X} \cdot \bar{l})_{\overline{\mathbb{P}^3}} = 2$ and $(\bar{l} \cap \Delta) \cap \overline{X} = \emptyset$, we have

$$\sum_{p \in l \setminus \{x_1\}} \left(\sum_{i=1}^{t} k_i Y_i \cdot l \right)_{H,p} = \sum_{p \in l \setminus \{x_1\}} (X \cdot l)_{\mathbb{P}^3, p} = 2.$$

On the other hand, by Proposition 2.15, we have $\sum_{p \in l \setminus \{x_1\}} \left(\sum_{i=1}^{t} k_i Y_i \cdot l \right)_{H,p} = 0, 1$ for a general line l in H through x_1. This is a contradiction. Thus we obtain $\overline{H} \cap \Delta = \overline{E}$. Since $\overline{E} \subset \operatorname{Sing} \overline{X}$ and $(\overline{X} \cdot \bar{l})_{\overline{\mathbb{P}^3}} = 2$ for a general line l in H through x_1, we see that Y is a union of lines through x_1.

(ii) Let l be any line in X through x_1. Then we have $\bar{l} \cap \Delta \subset \overline{X} \cap \Delta = \overline{E} = \overline{H} \cap \Delta$. Hence we obtain $l \subset H$ and thus $l \subset X \cap H = Y$.

(iii) Note that each Y_i is a line in X through x_1 by (i).

(iv) Since each $\overline{Y_i}$ is the proper transform of a line through x_1 by the blowing-up at x_1, we see that $\overline{Y_i} \cap \overline{Y_j} = \emptyset$ $(i \neq j)$. By noting this, we obtain the assertions.

(v) By using (i), (ii) and Lemma 3.1 (vi), we obtain the assertion.

(vi) By using (i), (ii) and Lemma 3.1 (v), we obtain the assertions.

(vii) Note (vi) and that $g(\overline{Y}^{\nu})$ consists of smooth points of V.

(viii) Note (vi) and that $\nu^{-1}(\overline{E}) = \operatorname{Supp} \overline{Z}^{\nu}$. □

Lemma 3.3. *One obtains the following*:
(i) $\overline{Z}^{\nu}$ *is the Cartier divisor* $\nu^*(\Delta|_{\overline{X}})$ *of* $\overline{X}^{\nu}$. *In particular,* $(\overline{\pi}^{\nu})^*(\overline{Z}^{\nu}) = Z$.
(ii) $g_*(\overline{Z}^{\nu}) = h^*(\overline{\psi}(\overline{E}))$, $g^* g_*(\overline{Z}^{\nu}) = \overline{Z}^{\nu} + \sum_{i=1}^{t} k_i \overline{Y_i}^{\nu} = \nu^*(\overline{H}|_{\overline{X}})$.
(iii) $k_i \overline{Y_i}^{\nu}$ *is a Cartier divisor of* $\overline{X}^{\nu}$ $(1 \leq i \leq t)$.
(iv) $\overline{\Gamma}^{\nu} \sim \sum_{i=1}^{t} k_i \overline{Y_i}^{\nu} + 2\overline{Z}^{\nu}$, $\hat{\Gamma} \sim \sum_{i=1}^{t} (\overline{\pi}^{\nu})^*(k_i \overline{Y_i}^{\nu}) + 2Z$.
(v) $g_*(\overline{\Gamma}^{\nu})$ *is a smooth Cartier divisor of* V *with* $g_*(\overline{\Gamma}^{\nu}) \cap \operatorname{Sing} V = \emptyset$.
(vi) $g_*(\overline{\Gamma}^{\nu}) \sim 2 g_*(\overline{Z}^{\nu})$, $g^* g_*(\overline{\Gamma}^{\nu}) = \overline{\Gamma}^{\nu} + \sum_{i=1}^{t} k_i \overline{Y_i}^{\nu}$.
(vii) $(g_*(\overline{\Gamma}^{\nu}) \cdot g_*(\overline{\Gamma}^{\nu}))_V = 8$, $(g_*(\overline{\Gamma}^{\nu}) \cdot g_*(\overline{Z}^{\nu}))_V = 4$, $(g_*(\overline{Z}^{\nu}) \cdot g_*(\overline{Z}^{\nu}))_V = 2$.
(viii) $(g_*(\overline{\Gamma}^{\nu}) \cdot g_*(\overline{Z}^{\nu}))_{V, g(\overline{Y_i}^{\nu})} = (\overline{\psi}(\overline{\Gamma}) \cdot \overline{\psi}(\overline{E}))_{\mathbb{P}^2, \overline{\psi}(\overline{Y_i})} = k_i$ $(1 \leq i \leq t)$.
(ix) $K_{\overline{X}^{\nu}} \sim -\overline{Z}^{\nu}$, $K_V \sim -g_*(\overline{Z}^{\nu}) = -h^*(\overline{\psi}(\overline{E}))$.

Proof. (i) By Lemma 3.1 (iii) and (iv), we have $(\overline{\pi}^{\nu})^* \nu^*(\Delta|_{\overline{X}}) = Z$. By pushing this forward, we obtain $\nu^*(\Delta|_{\overline{X}}) = (\overline{\pi}^{\nu})_*(Z) = \overline{Z}^{\nu}$.

(ii) By noting that $\nu^*(\Delta|_{\overline{X}}) = \overline{Z}^\nu$ and $\overline{\psi}|_{\overline{E}}\colon \overline{E} \cong \overline{\psi}(\overline{E})$, we have $g_*(\overline{Z}^\nu) = h^*(\overline{\psi}(\overline{E}))$. Thus we obtain $g^*g_*(\overline{Z}^\nu) = \nu^*(\overline{\psi}|_{\overline{X}})^*(\overline{\psi}(\overline{E})) = \nu^*(\overline{H}|_{\overline{X}}) = \overline{Z}^\nu + \sum_{i=1}^t k_i \overline{Y_i}^\nu$.

(iii) By (i) and (ii), we see that $\sum_{i=1}^t k_i \overline{Y_i}^\nu$ is a Cartier divisor of $\overline{X}^\nu$. By Lemma 3.2 (iv), we obtain the assertion.

(iv) Note (i), (ii), (iii) and that $\overline{\Gamma}^\nu \sim \nu^*(\overline{H}|_{\overline{X}}) + \nu^*(\Delta|_{\overline{X}})$.

(v) First we note that $\overline{\Gamma}^\nu$ is a smooth Cartier divisor of $\overline{X}^\nu$ with $\overline{\Gamma}^\nu \cap \operatorname{Sing} \overline{X}^\nu = \emptyset$ which intersects each $\overline{Y_i}^\nu$ transversally at only one point by $(\Gamma \cdot Y_i)_X = 1$. Since each $g(\overline{Y_i}^\nu)$ is a smooth point of V, we obtain the assertion.

(vi) By pushing the first relation of (iv) forward, we obtain the first assertion. By noting (i), (ii), (iii) and (iv), we have $(\pi^\nu)^*(g^*g_*(\overline{\Gamma}^\nu)) \sim \hat{\Gamma} + \sum_{i=1}^t (\pi^\nu)^*(k_i \overline{Y_i}^\nu)$. Since $\operatorname{Supp}(\pi^\nu)^*(g^*g_*(\overline{\Gamma}^\nu)) = \hat{\Gamma} \cup (\pi^\nu)^{-1}(\overline{Y}^\nu)$ and the intersection matrix of $(\pi^\nu)^{-1}(\overline{Y}^\nu) = (g \circ \pi^\nu)^{-1}(g(\overline{Y}^\nu))$ is negative definite, we have $(\pi^\nu)^*(g^*g_*(\overline{\Gamma}^\nu)) = \hat{\Gamma} + \sum_{i=1}^t (\pi^\nu)^*(k_i \overline{Y_i}^\nu)$. By pushing this forward, we obtain the second assertion.

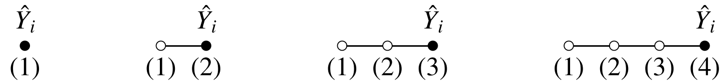
(vii) Note (vi) and that $\deg h = 2$ and $\mathcal{O}_V(g_*(\overline{Z}^\nu)) \cong h^*\mathcal{O}_{\mathbb{P}^2}(1)$.

(viii) By noting that $h|_{g_*(\overline{\Gamma}^\nu)}\colon g_*(\overline{\Gamma}^\nu) \cong \overline{\psi}(\overline{\Gamma})$, we obtain the first equality of the assertion. By the same argument as in the proof of Proposition 2.2 (vi) in [11], we obtain the second equality of the assertion.

(ix) Since both of $\operatorname{Sing} \overline{X}^\nu$ and $\operatorname{Sing} V$ consist of at most rational double points, we see that $K_{\overline{X}^\nu}$ and K_V are Cartier divisors. Since π^ν is the minimal resolution of $\overline{X}^\nu$, we have $(\pi^\nu)^*K_{\overline{X}^\nu} \sim K_M \sim -Z$. By pushing this forward, we also have $K_{\overline{X}^\nu} \sim -\overline{Z}^\nu$. By noting Lemma 3.2 (vi), we obtain $K_V \sim -g_*(\overline{Z}^\nu)$. □

REMARK. The branch locus B of h is a reduced plane quartic curve. Indeed, this is showed as follows. First we note that $\operatorname{Pic}(V)$ is torsion-free by Lemma 3.1 (vii), Lemma 3.2 (viii), and Proposition 1 in [10]. Thus we obtain the injectivity of $h^*\colon \operatorname{Pic}(\mathbb{P}^2) \cong \mathbb{Z} \to \operatorname{Pic}(V)$. Let R be the ramification divisor of h. Since $\deg h = 2$, we have $K_V \sim h^*K_{\mathbb{P}^2} + R$ and $h^*B = 2R$. By noting (ix), we obtain $h^*(B - 4L) \sim 0$ and hence $B - 4L \sim 0$, where L is a line in $\mathbb{P}^2$. In the following, we omit the investigation of a detailed structure of B since there is no necessity in our arguments.

Lemma 3.4. *One obtains the following*:
(i) *The weighted dual graph of* $(\pi^\nu)^{-1}(\overline{Y_i}^\nu)$ *is given as in* Fig. 8(a), (b), (c), (d) *for* $k_i = 1, 2, 3, 4$ *respectively, where the integers adjacent to vertices are coefficients in the divisor* $(\pi^\nu)^*(k_i \overline{Y_i}^\nu)$.
(ii) $b_2((\pi^\nu)^{-1}(\overline{Y}^\nu)) = 4$. *In particular,* $8 \leq b_2(\pi^{-1}(x_1)) \leq 11$.
(iii) $(\operatorname{Sing} \overline{X}^\nu) \setminus \overline{Y}^\nu \neq \emptyset$. *In particular,* $\operatorname{Sing} \overline{X}^\nu \neq \emptyset$ *and* $\operatorname{Sing} V \neq \emptyset$.
(iv) *If* $x \cap Y_i$ *contains more than one point for some* i, *then* $2 \leq k_i \leq 4$ *and* $(x \cap Y_i) \setminus \{x_1\}$ *consists of only one rational double point of type* $A_{k_i - 1}$.
(v) $\hat{Y} \cup E$ *is a simple normal crossing divisor of twelve smooth rational curves whose weighted dual graph is not a linear tree. In particular, each irreducible component* E_i *of* E *is a smooth rational curve with* $E_i^2 = (E_i \cdot Z) - 2$.



(a) (b) (c) (d)

Fig. 8.

Proof. (i) By Lemma 3.3 (vi), we have $(\overline{\pi}^{\nu})^*(g^*g_*(\overline{\Gamma}^{\nu})) = \hat{\Gamma} + \sum_{i=1}^{t}(\overline{\pi}^{\nu})^*(k_i\overline{Y_i}^{\nu})$. By noting that $(g \circ \overline{\pi}^{\nu})|_{M\setminus(g\circ\overline{\pi}^{\nu})^{-1}(\mathrm{Sing}\,V)} : M \setminus (g \circ \overline{\pi}^{\nu})^{-1}(\mathrm{Sing}\,V) \to V \setminus \mathrm{Sing}\,V$ is a composite of blowing-ups whose exceptional set is $(\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) = (g \circ \overline{\pi}^{\nu})^{-1}(g(\overline{Y}^{\nu}))$ and that $(\overline{\pi}^{\nu})^*(g^*g_*(\overline{\Gamma}^{\nu}))$ is the total transform of $g_*(\overline{\Gamma}^{\nu})$ by $g \circ \overline{\pi}^{\nu}$, we obtain the assertion.

(ii) By noting (i) and that $\hat{Y} \cup E = (\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) \cup \pi^{-1}(x_1)$, we obtain the assertions.

(iii) Assume that $\mathrm{Sing}\,\overline{X}^{\nu} \subset \overline{Y}^{\nu}$. Then we have $\hat{Y} \cup E = (\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) \cup \mathrm{Supp}\,Z_1$. Thus we obtain $b_2(\hat{Y} \cup E) \le 4 + 2 = 6$. This is a contradiction.

(iv) By noting (i) and that $\overline{X}^{\nu} \setminus \nu^{-1}(\overline{E}) \cong X \setminus \{x_1\}$, we obtain the assertion.

(v) By using $b_2(\pi^{-1}(x_1)) \ge 4$ and Proposition 3.5 in [8], we have that $\pi^{-1}(x_1)$ is a simple normal crossing divisor of smooth rational curves. By using $p_g(x_1) \ne 0$ and Satz 2.10 in Brieskorn [3], we see that the weighted dual graph of $\pi^{-1}(x_1)$ is not a linear tree. By Lemma 3.1 (vi) and $(\hat{Y}_i \cdot Z) = 1$ $(1 \le i \le t)$, we obtain the assertions. □

Since $Z^2 = -2$, there exist the following three possibilities for the divisor $Z_1 = \sum_{i=1}^{s_{0,1}} a_i E_i$. From now on, we shall consider these cases separately:
(1) $Z_1 = E_1$ with $(E_1 \cdot Z) = -2$.
(2) $Z_1 = 2E_1$ with $(E_1 \cdot Z) = -1$.
(3) $Z_1 = E_1 + E_2$ with $(E_1 \cdot Z) = (E_2 \cdot Z) = -1$.

3.1. The case $Z_1 = E_1$ with $(E_1 \cdot Z) = -2$.

Proposition 3.5. *This case does not occur.*

Proof. Assume that this case occurs. By Lemma 3.4 (v), we have that E_1 is a (-4)-curve in M. By Lemma 3.3 (iv) and Lemma 3.4 (i), (v), we see that $(\hat{Y}_i \cdot E_1) = 1$ and $((\overline{\pi}^{\nu})^*(k_i\overline{Y_i}^{\nu}) - k_i\hat{Y}_i \cdot E_1) = 0$ $(1 \le i \le t)$. In particular, $\overline{Y}^{\nu} \cap \overline{E_1}^{\nu}$ consists of smooth points of $\overline{X}^{\nu}$. By contracting the curve $(\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu})$, we obtain a boundary of a minimal normal compactification of $\mathbb{C}^2$ which is a union of seven (-2)-curves and one 0-curve. However its weighted dual graph cannot be found in Morrow's classification. This is a contradiction. □

3.2. The case $Z_1 = 2E_1$ with $(E_1 \cdot Z) = -1$. By Lemma 3.4 (v), we have that E_1 is a (-3)-curve in M. By Lemma 3.1 (iii) and the isomorphism $\overline{\psi}|_{\overline{E}}$, we obtain $E_1 \cong \overline{E_1}^{\nu} \cong g(\overline{E_1}^{\nu}) \cong \overline{\psi}(\overline{E}) \cong \overline{E} \cong \mathbb{P}^1$. By Lemma 3.1 (iv) and Lemma 3.3 (ii), we also obtain $\overline{E_1}^{\nu} = \overline{\pi}^{\nu}(\mathrm{Supp}\,Z)$ and $g(\overline{E_1}^{\nu}) = h^{-1}(\overline{\psi}(\overline{E}))$.

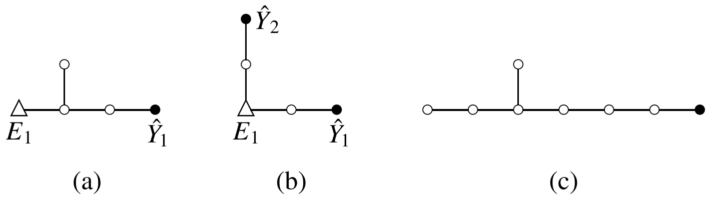


Fig. 9.

Proposition 3.6. *There exist the following two cases*:

(i) $\mathcal{Y} = 4Y_1$ (Y_1: *line*) *with* $x = \{x_1\}$.

(ii) $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: *line*) *with* $x = \{x_1\} = Y_1 \cap Y_2$.

Moreover, for the cases (i) *and* (ii), *the weighted dual graphs of* $\hat{Y} \cup E$ *are of types* (XV) *and* (XVI) *in* Theorem 1 *respectively.*

Proof. First we note that $\hat{Y} \cup E$ is of simple normal crossing. Since $a_1 = 2$ and $(\hat{Y}_i \cdot Z) = 1$, we obtain $(\hat{Y}_i \cdot E_1) = 0$ $(1 \le i \le t)$. In particular, $\overline{Y}^\nu \cap \overline{E_1}^\nu$ consists of singular points of $\overline{X}^\nu$. By Lemma 3.4 (i), we have that $x = \{x_1\}$ and $2 \le k_i \le 4$ $(1 \le i \le t)$. Thus we obtain $\mathcal{Y} = 4Y_1$ or $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: line). By Lemma 3.3 (iv), we obtain $\left(\sum_{i=1}^{t} (\overline{\pi}^\nu)^*(k_i \overline{Y}_i^\nu) \cdot E_1\right) = 2$. By noting this, we see that the weighted dual graphs of $(\overline{\pi}^\nu)^{-1}(\overline{Y}^\nu) \cup E_1$ are given as in Fig. 9 (a) and (b) for the cases $\mathcal{Y} = 4Y_1$ and $\mathcal{Y} = 2Y_1 + 2Y_2$ respectively. By contracting the curve $(\overline{\pi}^\nu)^{-1}(\overline{Y}^\nu)$, we have a boundary of a smooth compactification of $\mathbb{C}^2$ which is a simple normal crossing divisor of seven (-2)-curves and one (-1)-curve. By Lemma 2.9, its weighted dual graph is given as in Fig. 9 (c). Since the (-1)-curve in Fig. 9 (c) is the proper transform of E_1, we obtain the assertions. □

3.3. The case $Z_1 = E_1 + E_2$ with $(E_1 \cdot Z) = (E_2 \cdot Z) = -1$. By Lemma 3.4 (v), we have that E_1 and E_2 are (-3)-curves in M. By Lemma 3.1 (iv), (v) and Lemma 3.3 (ii), we also have that $\overline{E_1}^\nu \neq \overline{E_2}^\nu$, $g(\overline{E_1}^\nu) \neq g(\overline{E_2}^\nu)$, $\overline{E_1}^\nu \cup \overline{E_2}^\nu = \overline{\pi}^\nu(\mathrm{Supp}\, Z)$ and $g(\overline{E_1}^\nu) \cup g(\overline{E_2}^\nu) = h^{-1}(\overline{\psi}(\overline{E}))$. Since Supp Z is connected, both of $\overline{E_1}^\nu \cup \overline{E_2}^\nu$ and $g(\overline{E_1}^\nu) \cup g(\overline{E_2}^\nu)$ are also connected.

Lemma 3.7. *One obtains the following*:

(i) $E_i \cong \overline{E_i}^\nu \cong g(\overline{E_i}^\nu) \cong \overline{\psi}(\overline{E}) \cong \overline{E} \cong \mathbb{P}^1$ $(i = 1, 2)$.

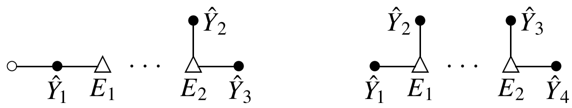
(ii) *Both of* $\overline{E_1}^\nu \cap \overline{E_2}^\nu$ *and* $g(\overline{E_1}^\nu) \cap g(\overline{E_2}^\nu)$ *consist of only one point.*

(iii) Sing $V = g(\overline{E_1}^\nu) \cap g(\overline{E_2}^\nu)$, $(\mathrm{Sing}\, \overline{X}^\nu) \setminus \overline{Y}^\nu = \overline{E_1}^\nu \cap \overline{E_2}^\nu$.

(iv) $(\hat{Y}_i \cdot E_1 + E_2) = 1$, $((\overline{\pi}^\nu)^*(k_i \overline{Y}_i^\nu) - k_i \hat{Y}_i \cdot E_1 + E_2) = 0$ $(1 \le i \le t)$.

(v) $(\mathrm{Sing}\, \overline{X}^\nu) \cap (\overline{E_1}^\nu \cup \overline{E_2}^\nu) = \overline{E_1}^\nu \cap \overline{E_2}^\nu$.

(vi) $(\overline{\pi}^\nu)^{-1}(\overline{E_1}^\nu \cap \overline{E_2}^\nu) = \mathrm{Supp}(Z - E_1 - E_2)$, $b_2(\mathrm{Supp}(Z - E_1 - E_2)) = 6$.



(a) (b)

Fig. 10.

(vii) *There exist the following two cases*:

(a) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i: *line*), $x = x \cap Y_1 = \{x_1, A_1\}$, $\{x_1\} = Y_1 \cap Y_2 \cap Y_3$.

(b) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: *line*), $x = \{x_1\} = Y_1 \cap Y_2 \cap Y_3 \cap Y_4$.

Moreover, for the cases (a) *and* (b), *the weighted dual graphs of* $(\overline{\pi}^\nu)^{-1}(\overline{Y}^\nu) \cup E_1 \cup E_2$ *are given as in* Fig. 10 (a) *and* (b) *respectively.*

(viii) $E_1 \cap E_2 = \emptyset$. *Moreover, there exists no irreducible component of* $\mathrm{Supp}(Z - E_1 - E_2)$ *intersecting both of* E_1 *and* E_2.

Proof. (i) By Lemma 3.1 (iii) and the isomorphism $\overline{\psi}|_{\overline{E}}$, we obtain the assertion.

(ii) First we note that $(\overline{X}^\nu, \overline{Y}^\nu \cup \overline{E_1}^\nu \cup \overline{E_2}^\nu)$ and $(V, g(\overline{E_1}^\nu) \cup g(\overline{E_2}^\nu))$ are compactifications of $\mathbb{C}^2$ and that $\overline{X}^\nu$ and V are normal. By Proposition 1 (ii) in [10], we obtain the assertion.

(iii) By Lemma 3.4 (iii), we have that $\mathrm{Sing}\, V \neq \emptyset$. Since $h^*(\overline{\psi}(\overline{E})) = g(\overline{E_1}^\nu) + g(\overline{E_2}^\nu)$ and $\sum_{q \in h^{-1}(p)} \mathrm{mult}_q V \leq \deg h = 2$ for any point p of $\mathbb{P}^2$, we obtain the first assertion. By Lemma 3.2 (vii), we also obtain the second assertion.

(iv) By noting (iii), Lemma 3.3 (iv) and Lemma 3.4 (i), (v), we obtain the assertions.

(v) By using (iii) and (iv), we obtain the assertion.

(vi) By using (v) and $b_2((\overline{\pi}^\nu)^{-1}(\overline{Y}^\nu)) = 4$, we obtain the assertions.

(vii) By Lemma 3.3 (iv), we obtain $\left(\sum_{i=1}^t (\overline{\pi}^\nu)^*(k_i \overline{Y_i}^\nu) \cdot E_j\right) = 2$ ($j = 1, 2$). By noting this and (iv), we have that $\mathcal{Y} = 2Y_1 + 2Y_2$, $2Y_1 + Y_2 + Y_3$ or $Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: line). Now we assume that $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i: line). Then we may assume that $\hat{Y}_i$ and $E_i \setminus \mathrm{Sing}(\mathrm{Supp}\, Z)$ meet transversally at only one point, which is denoted by p_i, for each $i = 1, 2$. Let L be a line in $\mathbb{P}^2$ such that $(h \circ g \circ \overline{\pi}^\nu)(p_1) \in L \neq \overline{\psi}(\overline{E})$. Since $h^*(\overline{\psi}(\overline{E})) = g(\overline{E_1}^\nu) + g(\overline{E_2}^\nu)$, we see that the divisor $(\overline{\pi}^\nu)^* g^* h^* L$ intersects Z transversally at only two points p_1 and q_2, where q_2 is a point of $E_2 \setminus \mathrm{Sing}(\mathrm{Supp}\, Z)$. By noting this and Lemma 3.1 (iii), we obtain

$$\mathcal{O}_Z(Z) \cong \mathcal{O}_Z(-(\hat{\Gamma} - Z)) \cong \mathcal{O}_Z(-(\overline{\pi}^\nu)^* g^* h^* L) \cong \mathcal{O}_Z(-p_1 - q_2).$$

By Lemma 3.3 (iv), we obtain $\mathcal{O}_Z(2Z) \cong \mathcal{O}_Z(-2p_1 - 2p_2)$. Thus we obtain $\mathcal{O}_Z(2(q_2 - p_2)) \cong \mathcal{O}_Z$. By Lemma 2.6 (iv), we see that $q_2 = p_2$. Hence we obtain

$$(h \circ g \circ \overline{\pi}^\nu)(p_1) = (h \circ g \circ \overline{\pi}^\nu)(q_2) = (h \circ g \circ \overline{\pi}^\nu)(p_2).$$

By noting that $(h \circ g \circ \overline{\pi}^{\nu})(p_i) = \overline{\psi}(\overline{Y_i})$ $(i = 1, 2)$, we have $\overline{\psi}(\overline{Y_1}) = \overline{\psi}(\overline{Y_2})$. On the other hand, we have $\overline{\psi}(\overline{Y_1}) \neq \overline{\psi}(\overline{Y_2})$ since Y_1 and Y_2 are distinct two lines through x_1. This is a contradiction. Thus we obtain the assertions.

(viii) Since $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ consists of only one point and $\operatorname{Supp} Z$ is of simple normal crossing, we have $E_1 \cap E_2 = \emptyset$. Next we assume that there exists an irreducible component of $\operatorname{Supp}(Z - E_1 - E_2)$ intersecting both of E_1 and E_2. By contracting the curve $(\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) \cup E_1 \cup E_2$, we obtain a boundary of a minimal normal compactification of $\mathbb{C}^2$ which is a union of five (-2)-curves and one 0-curve. However its weighted dual graph cannot be found in Morrow's classification. This is a contradiction. Thus we obtain the assertions. □

Now we have that $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is a rational double point of $\overline{X}^{\nu}$ of type A_6, D_6 or E_6 by Lemma 3.7 (vi). Let W be the fundamental cycle of $(\overline{\pi}^{\nu})^{-1}(\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu})$. Then we note that $(\overline{\pi}^{\nu})^{-1}(\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}) = \operatorname{Supp}(Z - E_1 - E_2) = \operatorname{Supp} W = \bigcup_{i=3}^{8} E_i$ and $Z = E_1 + E_2 + \sum_{i=3}^{8} a_i E_i$ with $a_3, \dots, a_8 \in \mathbb{N}$. By Lemmas 2 and 3 in [10], we also note that $(E_1 \cdot W) = (E_2 \cdot W) = 1$. Since $(Z \cdot E_i) = 0$ $(3 \leq i \leq 8)$, we obtain $(Z \cdot W) = 0$.

Proposition 3.8. *Assume that $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is of type A_6. Then the weighted dual graph of $\hat{Y} \cup E$ is of type* (XVII) *or* (XVIII) *in* Theorem 1, *for the case $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ or $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i: line) respectively.*

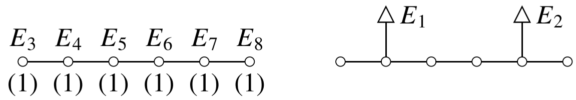
Proof. Assume that $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is of type A_6. Then the weighted dual graph of $\operatorname{Supp} W$ is given as in Fig. 11 (a), where the integers adjacent to vertices are coefficients in W. We note that $(E_3 \cdot W) = (E_8 \cdot W) = -1$ and $(E_i \cdot W) = 0$ $(4 \leq i \leq 7)$. By computing the intersection number $(Z \cdot W)$, we have

$$0 = (Z \cdot W) = (E_1 \cdot W) + (E_2 \cdot W) + \sum_{i=3}^{8} a_i (E_i \cdot W) = 2 - a_3 - a_8.$$

Thus we obtain $a_3 = a_8 = 1$. Moreover, by computing the intersection numbers $(Z \cdot E_1)$ and $(Z \cdot E_2)$, we obtain $\left(\sum_{i=3}^{8} a_i E_i \cdot E_1\right) = \left(\sum_{i=3}^{8} a_i E_i \cdot E_2\right) = 2$. By noting that $\operatorname{Supp} Z$ is of simple normal crossing, we see that $E_1 \cap (E_3 \cup E_8) = E_2 \cap (E_3 \cup E_8) = \emptyset$. Here we note Lemma 3.7 (viii). By contracting the curve $(\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) \cup E_1 \cup E_2$ and suitable curves in $\hat{Y} \cup E$, we have a boundary of a minimal normal compactification of $\mathbb{C}^2$. Since its weighted dual graph must be found in Morrow's classification, the weighted dual graph of $\operatorname{Supp} Z$ is uniquely determined as in Fig. 11 (b). By Lemma 3.7 (vii), we obtain the assertion. □

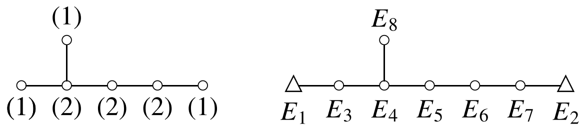
Proposition 3.9. *It does not occur the case where $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is of type D_6.*

Proof. Assume that $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is of type D_6. Then the weighted dual graph of $\operatorname{Supp} W$ is given as in Fig. 12 (a), where the integers adjacent to vertices are co-



(a) (b)

Fig. 11.



(a) (b)

Fig. 12.

efficients in W. We note that $(E_1 \cdot W) = (E_2 \cdot W) = 1$. By contracting the curve $(\overline{\pi}^{\nu})^{-1}(\overline{Y}^{\nu}) \cup E_1 \cup E_2$ and suitable curves in $\hat{Y} \cup E$, we have a boundary of a minimal normal compactification of $\mathbb{C}^2$. Since its weighted dual graph must be found in Morrow's classification, the weighted dual graph of $\operatorname{Supp} Z$ is uniquely determined as in Fig. 12 (b). We note that $(E_6 \cdot W) = -1$ and $(E_i \cdot W) = 0$ $(i = 3, 4, 5, 7, 8)$. By computing the intersection numbers $(Z \cdot W)$ and $(Z \cdot E_7)$, we have $0 = (Z \cdot W) = 2 - a_6$ and $0 = (Z \cdot E_7) = a_6 - 2a_7 + 1$. Thus we obtain $a_7 = 3/2 \notin \mathbb{N}$. This is a contradiction. $\square$

Proposition 3.10. *Assume that* $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ *is of type* E_6. *Then the weighted dual graph of* $\hat{Y} \cup E$ *is of type* (XIX) *or* (XX) *in* Theorem 1, *for the case* $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ *or* $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ $(Y_i$: *line*) *respectively.*

Proof. Assume that $\overline{E_1}^{\nu} \cap \overline{E_2}^{\nu}$ is of type E_6. Then the weighted dual graph of $\operatorname{Supp} W$ is given as in Fig. 13 (a), where the integers adjacent to vertices are coefficients in W. By Lemma 3.7 (viii) and $(E_1 \cdot W) = (E_2 \cdot W) = 1$, the weighted dual graph of $\operatorname{Supp} Z$ is uniquely determined as in Fig. 13 (b). By Lemma 3.7 (vii), we obtain the assertion. $\square$

Thus we complete the proof of Theorem 1 for the case $Z^2 = -2$.

4. Proof of Theorem 1 for $Z^2 = -1$

In this section, we shall prove Theorem 1 for the case $Z^2 = -1$. Let (X, Y) be a pair satisfying Assumption in §1 and $Z^2 = -1$. We use the same notation as that in §1 and §2. We mainly consider a projection from x_1 and a blowing-up at x_1 to investigate the pair (X, Y). First we note that $\hat{Y} \cup E$ is a connected divisor without

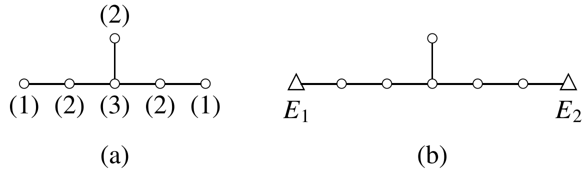


Fig. 13.

cycles which consists of eleven rational curves. Since $Z^2 = -1$, we may assume that E_1 is a unique irreducible component of $Z = \sum_{i=1}^{s_1} a_i E_i$ such that $(E_1 \cdot Z) = -1$ and $a_1 = 1$. Then we have that $(E_i \cdot Z) = 0$ for any irreducible component E_i of $Z - E_1$. By Lemma 2.3 (ii), there exists a unique point p_0 of $E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$ such that $(\pi \circ \pi_0)^* \mathfrak{m}_{X,x_1} \cong \mathcal{O}_{M'}(-Z' - 2E_0')$, where $\pi_0 \colon M' \to M$ is a blowing-up at p_0 with exceptional curve E_0' and Z' is the proper transform of Z by π_0. We put $\pi' := \pi \circ \pi_0$. Let $\hat{C}'$ be the proper transform of a curve C in X by π'. Let E_i' and E' be the proper transforms of E_i and E by π_0 respectively for $i \geq 1$. We note that $\pi_0^* Z = Z' + E_0'$, $\operatorname{Supp} \pi_0^* Z = (\pi')^{-1}(x_1)$, $\operatorname{Exc} \pi' = Z' \cup E_0'$, $K_{M'} \sim -Z'$ and $Z'^2 = -2$. Let $\sigma \colon \overline{\mathbb{P}^3} \to \mathbb{P}^3$ be the blowing-up at x_1 with exceptional divisor Δ, which is isomorphic to $\mathbb{P}^2$. Let $\overline{T}$ be the proper transform of a closed algebraic subset T of $\mathbb{P}^3$ by σ. We have that $\sigma|_{\overline{\mathbb{P}^3} \setminus \Delta} \colon \overline{\mathbb{P}^3} \setminus \Delta \cong \mathbb{P}^3 \setminus \{x_1\}$ and $\mathcal{O}_{\overline{\mathbb{P}^3}}(\Delta)|_\Delta \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. We set $\overline{E} := \overline{X} \cap \Delta$. We have that $\sigma|_{\overline{X} \setminus \overline{E}} \colon \overline{X} \setminus \overline{E} \cong X \setminus \{x_1\}$ and that $(\overline{X}, \overline{Y} \cup \overline{E})$ is a compactification of $\mathbb{C}^2$ with dualizing sheaf $\omega_{\overline{X}} \cong \mathcal{O}_{\overline{X}}$. Let $\psi \colon \mathbb{P}^3 \cdots \to \mathbb{P}^2$ be the projection from x_1 and $\overline{\psi} \colon \overline{\mathbb{P}^3} \to \mathbb{P}^2$ the resolution of indeterminacy of ψ. We have that $\overline{\psi}|_\Delta \colon \Delta \to \mathbb{P}^2$ is an isomorphism and $\overline{\psi}|_{\overline{X}} \colon \overline{X} \to \mathbb{P}^2$ is a generically finite morphism of degree two. We note that $\overline{\Gamma} \sim \overline{H}|_{\overline{X}} + \Delta|_{\overline{X}}$ and $\hat{\Gamma} \sim \sum_{i=1}^t k_i \hat{Y}_i + \sum_{j=1}^s b_j E_j$ with $b_j \in \mathbb{N}$. Then we have some fundamental lemmas.

Lemma 4.1. *One obtains the following*:

(i) $\overline{X}$ *is normal. Moreover,* $\overline{X}|_\Delta = 2\overline{E} = 2$*line and* $\overline{E} \cap \operatorname{Sing} \overline{X}$ *consists of only one point, which is denoted by* $\overline{x_1}$.

(ii) $\operatorname{Sing} \overline{X}$ *consists of exactly one minimally elliptic singular point* $\overline{x_1}$ *and at most rational double points*.

(iii) *There exists a birational morphism* $\overline{\pi}' \colon M' \to \overline{X}$ *satisfying* $(\sigma|_{\overline{X}}) \circ \overline{\pi}' = \pi'$. *Then* $(\overline{\pi}')^*(\Delta|_{\overline{X}}) = Z' + 2E_0'$, $(\overline{\pi}')^*(\overline{\psi}|_{\overline{X}})^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{M'}(\hat{\Gamma}' - Z' - 2E_0')$, $\overline{\pi}'|_{E_0'} \colon E_0' \cong \overline{E}$ *and* $\overline{\pi}'(\operatorname{Supp} Z') = \{\overline{x_1}\}$. *Moreover,* $\overline{\pi}'$ *is a minimal resolution of* $\overline{X}$ *with* $\operatorname{Exc} \overline{\pi}' = (\overline{\pi}')^{-1}(\overline{x_1}) \cup (\pi')^{-1}(x \setminus \{x_1\})$.

(iv) Z' *is the fundamental cycle of* $(\overline{\pi}')^{-1}(\overline{x_1})$ *with* $Z'^2 = -2$. *Moreover,* $\overline{x_1}$ *is a minimally elliptic double point of* $\overline{X}$ *and* $(\overline{\pi}')^* \mathfrak{m}_{\overline{X}, \overline{x_1}} \cong \mathcal{O}_{M'}(-Z')$.

$$\begin{tikzcd}
M' \arrow[r, "\overline{\pi}'"] \arrow[d, "\pi_0"'] \arrow[dr, "\pi'"] & \overline{X} \arrow[r, "g"] \arrow[d, "\sigma|_{\overline{X}}"'] \arrow[dr, "\overline{\psi}|_{\overline{X}}"] & V \arrow[d, "h"] \\
M \arrow[r, "\pi"'] & X \arrow[r, dashed, "\psi|_X"'] & \mathbb{P}^2
\end{tikzcd}$$

Fig. 14.

(v) *There exists exactly one line* l_1 *in* X *through* x_1. *Then* $x_1 \in l_1 \subset Y$ *and* $\overline{x_1} \in \overline{l_1} \subset \overline{Y}$. *Moreover,* $\hat{l}_1'$ *is a* (-1)*-curve in* M' *with* $(\hat{l}_1' \cdot Z') = 1$ *and* $(\hat{l}_1' \cdot E_0') = 0$ *and* $\hat{l}_1$ *is a* (-1)*-curve in* M *with* $(\hat{l}_1 \cdot Z) = 1$ *and* $p_0 \notin \hat{l}_1$.

(vi) *Let* $\overline{X} \xrightarrow{g} V \xrightarrow{h} \mathbb{P}^2$ *be the Stein factorization of* $\overline{\psi}|_{\overline{X}}$. *Then* V *is normal,* g *is a birational morphism and h is a finite morphism of degree two. In particular,* $g|_{\overline{X}\setminus\overline{l_1}}\colon \overline{X} \setminus \overline{l_1} \cong V \setminus g(\overline{l_1})$ *with* $\operatorname{Exc} g = \overline{l_1} = g^{-1}(g(\overline{x_1}))$ *and* $(V, g(\overline{Y} \cup \overline{E}))$ *is a compactification of* $\mathbb{C}^2$. *Thus one obtains the commutative diagram as in* Fig. 14.

(vii) V *is a projective normal Gorenstein surface with dualizing sheaf* $\omega_V \cong \mathcal{O}_V$. *Moreover,* $\operatorname{Sing} V$ *consists of exactly one minimally elliptic double point* $g(\overline{x_1})$ *and at most rational double points.*

Proof. (i) The assertions are the general properties of the minimally elliptic singularity (X, x_1) with $Z^2 = -1$. Indeed, we can check the assertions by applying a blowing-up of the local analytic defining equation of (X, x_1) in Theorem 4.57 (3) of [7] (cf. [8]).

(ii) By using $K_{\overline{X}} \sim 0$ and Proposition 1 (vi) in [10], we obtain $p_g(\operatorname{Sing} \overline{X}) = 1$. Since $\sigma|_{\overline{X}\setminus\overline{E}} : \overline{X} \setminus \overline{E} \cong X \setminus \{x_1\}$ and $\overline{E} \cap \operatorname{Sing} \overline{X} = \{\overline{x_1}\}$, we obtain the assertion.

(iii) There exists a birational morphism $\overline{\pi}'\colon M' \to \overline{X}$ satisfying $(\sigma|_{\overline{X}}) \circ \overline{\pi}' = \pi'$ by Lemma 2.3 (ii). In particular, we obtain $(\overline{\pi}')^{-1}(\overline{E}) = \operatorname{Supp}(Z' + 2E_0')$. By the isomorphisms

$$(\overline{\pi}')^*(\mathcal{O}_{\overline{\mathbb{P}^3}}(-\Delta)|_{\overline{X}}) \cong (\overline{\pi}')^*(\sigma|_{\overline{X}})^*\mathfrak{m}_{X,x_1} \cong \mathcal{O}_{M'}(-Z' - 2E_0'),$$

we obtain $(\overline{\pi}')^*(\Delta|_{\overline{X}}) \sim Z' + 2E_0'$. Since $(\overline{\pi}')^*(\Delta|_{\overline{X}})$ is an effective divisor of M' whose support equals to $\operatorname{Supp}(Z' + 2E_0')$ and the intersection matrix of $\operatorname{Supp}(Z' + 2E_0')$ is negative definite, we obtain $(\overline{\pi}')^*(\Delta|_{\overline{X}}) = Z' + 2E_0'$. In particular, we have

$$(\overline{\pi}')^*(\overline{\psi}|_{\overline{X}})^*\mathcal{O}_{\mathbb{P}^2}(1) \cong (\overline{\pi}')^*\mathcal{O}_{\overline{X}}(\overline{H}|_{\overline{X}}) \cong (\overline{\pi}')^*\mathcal{O}_{\overline{X}}(\overline{\Gamma} - \Delta|_{\overline{X}}) \cong \mathcal{O}_{M'}(\hat{\Gamma}' - Z' - 2E_0').$$

Let E_i' be any irreducible component of $(\pi')^{-1}(x_1)$ for $i \geq 0$. Since $\overline{\pi}'|_{E_i'}$ is identified with $(\overline{\psi}|_{\overline{X}}) \circ (\overline{\pi}'|_{E_i'})$, we obtain $\deg(\overline{\pi}'|_{E_i'}) = -(Z' + 2E_0' \cdot E_i')_{M'}$. Thus we see that $\overline{\pi}'|_{E_0'}\colon E_0' \cong \overline{E}$ and $\overline{\pi}'(\operatorname{Supp} Z') = \{\overline{x_1}\}$. Here we note that $\operatorname{Exc} \overline{\pi}' = (\overline{\pi}')^{-1}(\overline{x_1}) \cup (\pi')^{-1}(x \setminus \{x_1\})$. Since $\operatorname{Supp} Z = \pi^{-1}(x_1)$ and $p_0 \in E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$, there exist no (-1)-curves in $\operatorname{Supp} Z' = (\overline{\pi}')^{-1}(\overline{x_1})$. Thus $\overline{\pi}'$ is a minimal resolution of $\overline{X}$.

(iv) Note that $\overline{\pi}'$ gives a minimal resolution of the minimally elliptic singularity $(\overline{X}, \overline{x_1})$ and $K_{M'} \sim -Z'$. By using Theorem 3.4 in [8], we have that Z' is the fundamental cycle of $(\overline{\pi}')^{-1}(\overline{x_1})$. By using $Z'^2 = -2$ and Theorem 3.13 in [8], we see that $\mathrm{mult}_{\overline{x_1}} \overline{X} = 2$ and $(\overline{\pi}')^* \mathfrak{m}_{\overline{X}, \overline{x_1}} \cong \mathcal{O}_{M'}(-Z')$.

(v) There exists at least a line l in X through x_1 by Proposition 2.15. By Lemma 2.3 (ii), we obtain $(\hat{l}' \cdot Z' + 2E'_0) = 1$. Hence $\hat{l}'$ is a (-1)-curve in M' with $(\hat{l}' \cdot Z') = 1$ and $(\hat{l}' \cdot E'_0) = 0$. In particular, $\hat{l}$ is a (-1)-curve in M with $(\hat{l} \cdot Z) = 1$ and $p_0 \notin \hat{l}$. Since $(\hat{l}' \cdot Z') = 1$, we see that $\bar{l}$ passes through $\overline{x_1}$ necessarily. From this, the existence of l is unique. Thus we put $l_1 := l$. By Proposition 2.15 again, we obtain $l_1 \subset Y$ and in particular $\overline{l_1} \subset \overline{Y}$.

(vi) Note the general properties of Stein factorization (cf. Corollary III.11.5 in [5]).

(vii) Since $\deg g = 2$, $\mathrm{Sing}\, V$ consists of at most double points. Note that any double point is a hypersurface singularity and in particular a Gorenstein singularity. Thus V is Gorenstein. Since $g|_{\overline{X} \setminus \overline{l_1}} \colon \overline{X} \setminus \overline{l_1} \cong V \setminus g(\overline{l_1})$ and $K_{\overline{X}} \sim 0$, we obtain $K_V \sim 0$. By applying Proposition 1 (vi) in [10], we obtain $p_g(\mathrm{Sing}\, V) = 1$. Hence we obtain the assertions. □

REMARK. The branch locus B of h is a reduced plane sextic curve. Indeed, this is showed as follows. First we note that $\mathrm{Pic}(V)$ is torsion-free by Lemma 4.1, and Proposition 1 in [10]. Thus we obtain the injectivity of $h^* \colon \mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z} \to \mathrm{Pic}(V)$. Let R be the ramification divisor of h. Since $\deg h = 2$, we have $K_V \sim h^* K_{\mathbb{P}^2} + R$ and $h^* B = 2R$. By noting that $K_V \sim 0$, we obtain $h^*(B - 6L) \sim 0$ and hence $B - 6L \sim 0$, where L is a line in $\mathbb{P}^2$. In the following, we omit the investigation of a detailed structure of B since there is no necessity in our arguments.

Lemma 4.2. *One obtains the following*:
(i) $x = \{x_1\}$, $\mathrm{Sing}\, \overline{X} = \{\overline{x_1}\}$, $1 \leq b_2(Y) \leq 2$, $b_2(E) = 11 - b_2(Y)$. *In particular,* $\mathrm{Sing}\, V = \{g(\overline{x_1})\}$.
(ii) *E is a simple normal crossing divisor of one (-3)-curve E_1 and some (-2)-curves whose weighted dual graph is not a linear tree.*
(iii) *$\hat{l}_1$ is a (-1)-curve in M with $(\hat{l}_1 \cdot E_1) = 1$, $(\hat{l}_1 \cdot Z - E_1) = 0$ and $p_0 \notin \hat{l}_1$.*
(iv) *$\hat{l}_1 \cap \mathrm{Supp}\, Z$ is a smooth point of Z, which is denoted by p_1. In particular, $p_0 \neq p_1 \in E_1 \setminus \mathrm{Sing}(\mathrm{Supp}\, Z)$.*
(v) $\mathcal{O}_Z(Z) \cong \mathcal{O}_Z(-p_0)$.

Proof. (i) First we show that $(\overline{l_1} \cap \mathrm{Sing}\, \overline{X}) \setminus \{\overline{x_1}\} = \emptyset$. Assume that $(\overline{l_1} \cap \mathrm{Sing}\, \overline{X}) \setminus \{\overline{x_1}\} \neq \emptyset$. By Lemma 2.5 (i) and Lemma 4.1 (ii), there exists an irreducible component E'_i of $(\overline{\pi}')^{-1}((\overline{l_1} \cap \mathrm{Sing}\, \overline{X}) \setminus \{\overline{x_1}\})$ which is a (-2)-curve in M' with $(\hat{l}'_1 \cdot E'_i) = 1$. By using $Z'^2 = -2$ and Lemma 4.1 (v), we obtain $(Z' + E'_i + 2\hat{l}'_1)^2 = 0$ directly. On the other hand, the intersection matrix of $(g \circ \overline{\pi}')^{-1}(g(\overline{l_1})) = \hat{l}'_1 \cup \mathrm{Supp}\, Z' \cup (\overline{\pi}')^{-1}((\overline{l_1} \cap \mathrm{Sing}\, \overline{X}) \setminus \{\overline{x_1}\})$ is negative definite. This is a contradiction. Thus we have that $\overline{l_1} \cap \mathrm{Sing}\, \overline{X} = \{\overline{x_1}\}$ and in particular $x \cap l = \{x_1\}$. By noting Proposition 2.15 and Lemma 4.1 (v), we obtain $1 \leq b_2(Y) \leq 2$ and $x = \{x_1\}$. Immediately, we also obtain the other assertions.

(ii) By using $b_2(E) \geq 4$ and Proposition 3.5 in [8], we have that E is a simple normal crossing divisor of smooth rational curves. By using $p_g(x_1) \neq 0$ and Satz 2.10 in [3], we see that the weighted dual graph of E is not a linear tree. By the adjunction formula, we obtain the assertion.

(iii) Note that $\hat{l}_1'$ is a (-1)-curve in M' with $(\hat{l}_1' \cdot Z') = 1$ and $(\hat{l}_1' \cdot E_0') = 0$ by Lemma 4.1 (v). Since $(\hat{l}_1' \cdot Z') = 1$, there exists a unique irreducible component E_i' of Z' such that $(\hat{l}_1' \cdot E_i') = 1$ and $(\hat{l}_1' \cdot Z' - E_i') = 0$. Since the intersection matrix of $(g \circ \overline{\pi}')^{-1}(g(\overline{l_1})) = \hat{l}_1' \cup \operatorname{Supp} Z'$ is negative definite, we obtain $3(E_i')^2 + 6 = (Z' + E_i' + 2\hat{l}_1')^2 < 0$ and thus $(E_i')^2 \leq -3$. Since $p_0 \in E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$, we have $E_i' = E_1'$. By noting that $(\hat{l}_1' \cdot E_0') = 0$, we obtain the assertion.

(iv) By noting (iii), we obtain the assertions.

(v) Let L be a line in $\mathbb{P}^2$ such that $(h \circ g)(\overline{x_1}) = \overline{\psi}(\overline{x_1}) \notin L$. Then we have $\operatorname{Supp}(h \circ g \circ \overline{\pi}')^* L \cap \operatorname{Supp} Z' = \emptyset$. By the projection formula and $(h \circ g \circ \overline{\pi}')|_{E_0'} \colon E_0' \cong \overline{\psi}(\overline{E})$, we also have $((h \circ g \circ \overline{\pi}')^* L \cdot E_0')_{M'} = (L \cdot \overline{\psi}(\overline{E}))_{\mathbb{P}^2} = 1$. Hence $(\pi_0)_*(h \circ g \circ \overline{\pi}')^* L$ intersects Z transversally at only one point p_0, which is a smooth point of Z. By Lemma 4.1 (iii), we obtain $\hat{\Gamma} - Z \sim (\pi_0)_*(h \circ g \circ \overline{\pi}')^* L$. By restricting this relation to Z, we obtain the assertion. $\square$

Lemma 4.3. *There exists only the case where*

$$\mathcal{Y} = 2Y_1 + Y_2 \ (Y_1\colon \text{ line},\ Y_2\colon \text{ conic}) \text{ with } x = Y_1 \cap Y_2 = \{x_1\}.$$

In this case, one has that $Y_1 = l_1$, $\overline{\psi}(\overline{H}) = \overline{\psi}(\overline{E})$ *and* $\overline{H}|_{\overline{X}} = 2\overline{Y_1} + \overline{Y_2} + \overline{E}$. *Moreover, one has that* $g(\overline{Y_2}) \neq g(\overline{E})$, $g(\overline{Y_2}) + g(\overline{E}) = h^*(\overline{\psi}(\overline{E}))$ *and* $\overline{Y_2} \cong g(\overline{Y_2}) \cong \overline{\psi}(\overline{E}) \cong g(\overline{E}) \cong \overline{E} \cong \mathbb{P}^1$.

Proof. By Proposition 2.15, Lemma 4.1 (v) and Lemma 4.2 (i), there exist the following four possibilities:
(1) $\mathcal{Y} = 4Y_1$ (Y_1: line) with $x = x \cap Y_1 = \{x_1\}$.
(2) $\mathcal{Y} = 3Y_1 + Y_2$ (Y_1, Y_2: line) with $x = x \cap (Y_1 \setminus Y_2) = \{x_1\}$.
(3) $\mathcal{Y} = 2Y_1 + Y_2$ (Y_1: line, Y_2: conic) with $x = Y_1 \cap Y_2 = \{x_1\}$.
(4) $\mathcal{Y} = Y_1 + Y_2$ (Y_1: line, Y_2: cuspidal cubic) with $x = Y_1 \cap Y_2 = \operatorname{Sing} Y_2 = \{x_1\}$.
For each case, we note that $Y_1 = l_1$, $\overline{x_1} \in \overline{Y_1}$ and $\overline{Y_i} \cong \mathbb{P}^1$ for any i. By noting the position of x_1 in Y and that $\overline{\psi}|_{\overline{X}}$ is a generically finite morphism of degree two, we obtain $\overline{\psi}(\overline{H}) = \overline{\psi}(\overline{E})$. First we consider the case (3). In this case, we obtain $\overline{H}|_{\overline{X}} = 2\overline{Y_1} + \overline{Y_2} + \overline{E}$ since $\overline{\psi}(\overline{H}) = \overline{\psi}(\overline{E})$. By noting Lemma 4.1 (vi) and that $\overline{\psi}|_{\overline{E}} \colon \overline{E} \cong \overline{\psi}(\overline{E})$, we have that $g(\overline{Y_2}) \neq g(\overline{E})$, $g(\overline{Y_2}) + g(\overline{E}) = h^*(\overline{\psi}(\overline{E}))$ and $\overline{Y_2} \cong g(\overline{Y_2}) \cong \overline{\psi}(\overline{E}) \cong g(\overline{E}) \cong \overline{E} \cong \mathbb{P}^1$. Next we show that the cases (1), (2) and (4) do not occur.

(1) Assume that the case (1) occurs. Since $\overline{\psi}(\overline{H}) = \overline{\psi}(\overline{E})$, we obtain $\overline{H}|_{\overline{X}} = 4\overline{Y}_1 + 2\overline{E}$. By Lemma 4.1 (iii), there exists an effective divisor D of M such that $\operatorname{Supp} D =$

$\operatorname{Supp} Z$ and $\hat{\Gamma} - Z \sim 4\hat{Y}_1 + D$. From this, we obtain $(D \cdot E_i) = (3Z \cdot E_i)$ for each irreducible component E_i of Z. Since the intersection matrix of $\operatorname{Supp} Z = \pi^{-1}(x_1)$ is negative definite, we have $D = 3Z$ and thus $\hat{\Gamma} \sim 4\hat{Y}_1 + 4Z$. In particular, we have $\mathcal{O}_Z(4Z) \cong \mathcal{O}_Z(-4p_1)$. By Lemma 4.2 (v), we also have $\mathcal{O}_Z(4(p_0 - p_1)) \cong \mathcal{O}_Z$. By Lemma 2.6 (iv), we obtain $p_0 = p_1$. This is a contradiction.

(2) Assume that the case (2) occurs. Since $\overline{\psi}(\overline{H}) = \overline{\psi}(\overline{E})$, we obtain $\overline{H}|_{\overline{X}} = 3\overline{Y_1} + \overline{Y_2} + \overline{E}$. By Lemma 4.1 (iii), there exists an effective divisor D of M such that $\operatorname{Supp} D = \operatorname{Supp} Z$ and $\hat{\Gamma} - Z \sim 3\hat{Y}_1 + \hat{Y}_2 + D$. From this, we obtain $(D \cdot E_i) = (2Z \cdot E_i)$ for each irreducible component E_i of Z. Since the intersection matrix of $\operatorname{Supp} Z = \pi^{-1}(x_1)$ is negative definite, we have $D = 2Z$ and thus $\hat{\Gamma} \sim 3\hat{Y}_1 + \hat{Y}_2 + 3Z$. In particular, we have $\mathcal{O}_Z(3Z) \cong \mathcal{O}_Z(-3p_1)$. By Lemma 4.2 (v), we have $\mathcal{O}_Z(3(p_0 - p_1)) \cong \mathcal{O}_Z$. By Lemma 2.6 (iv), we obtain $p_0 = p_1$. This is a contradiction.

(4) Assume that the case (4) occurs. Note that $\overline{Y_1}$ and $\overline{Y_2}$ meet at only one point $\overline{x_1}$ transversally. In particular, $\overline{Y_2}$ is smooth. By using $(\overline{\pi}')^* \mathfrak{m}_{\overline{X},\overline{x_1}} \cong \mathcal{O}_{M'}(-Z')$ and Lemma 3 in [10], we obtain $(\hat{Y}_2' \cdot Z') = 1$. On the other hand, we obtain $(\hat{Y}_2' \cdot Z') = 0, 2$ by Lemma 2.3 (ii). This is a contradiction. □

Lemma 4.4. *One obtains the following*:

(i) $\hat{Y}_2$ *is a* (-1)*-curve in* M *with* $(\hat{Y}_2 \cdot Z) = 1$, $(\hat{Y}_1 \cdot \hat{Y}_2) = 0$ *and* $p_0 \notin \hat{Y}_2$.

(ii) $\hat{Y}_2 \cap \operatorname{Supp} Z$ *is a smooth point of* Z, *which is denoted by* p_2, *and* $p_2 \neq p_0, p_1$.

(iii) $\hat{Y} \cup E$ *is a simple normal crossing divisor of two* (-1)*-curves* $\hat{Y}_1$, $\hat{Y}_2$, *one* (-3)*-curve* E_1 *and eight* (-2)*-curves whose weighted dual graph is not a linear tree.*

(iv) $\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + 3Z$. *In particular,* $\mathcal{O}_Z(3Z) \cong \mathcal{O}_Z(-2p_1 - p_2)$.

(v) $\mathcal{O}_Z(3p_0 - 2p_1 - p_2) \cong \mathcal{O}_Z$. *In particular,* $p_2 \in E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$.

Proof. (i) Note that each pair of $\overline{Y_1}$, $\overline{Y_2}$ and $\overline{E}$ meet transversally at only one point $\overline{x_1}$ and that the blowing-up morphism at $\overline{x_1}$ of $\overline{X}$ factors $\overline{\pi}'$ by Lemma 4.1 (iv) and Proposition II.7.14 in [5]. From these, we have $(\hat{Y}_1' \cdot \hat{Y}_2') = (\hat{Y}_1' \cdot E_0') = (\hat{Y}_2' \cdot E_0') = 0$. Thus we obtain the assertion.

(ii) By using (i), we obtain the assertions.

(iii) By (i), (ii) and Lemma 4.2 (ii), (iii), we obtain the assertion.

(iv) Note that $\overline{H}|_{\overline{X}} = 2\overline{Y_1} + \overline{Y_2} + \overline{E}$. By Lemma 4.1 (iii), there exists an effective divisor D of M such that $\operatorname{Supp} D = \operatorname{Supp} Z$ and $\hat{\Gamma} - Z \sim 2\hat{Y}_1 + \hat{Y}_2 + D$. From this, we have $(D \cdot E_i) = (2Z \cdot E_i)$ for each irreducible component E_i of Z. Since the intersection matrix of $\operatorname{Supp} Z = \pi^{-1}(x_1)$ is negative definite, we obtain $D = 2Z$ and thus $\hat{\Gamma} \sim 2\hat{Y}_1 + \hat{Y}_2 + 3Z$. In particular, we obtain $\mathcal{O}_Z(3Z) \cong \mathcal{O}_Z(-2p_1 - p_2)$.

(v) By (iv) and Lemma 4.2 (v), we have $\mathcal{O}_Z(3p_0 - 2p_1 - p_2) \cong \mathcal{O}_Z$ and in particular $\deg \mathcal{O}_Z(3p_0 - 2p_1 - p_2) = \deg \mathcal{O}_Z = (0, \dots, 0)$. By noting (ii) and that $p_0, p_1 \in E_1$, we obtain $p_2 \in E_1 \setminus \operatorname{Sing}(\operatorname{Supp} Z)$. □

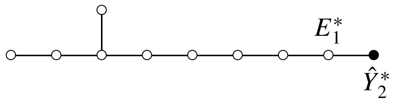


Fig. 15.

Proposition 4.5. *There exists only the case where*

$$\mathcal{Y} = 2Y_1 + Y_2 \ (Y_1\colon \textit{ line},\ Y_2\colon \textit{ conic}) \textit{ with } x = Y_1 \cap Y_2 = \{x_1\}.$$

Moreover, the weighted dual graph of $\hat{Y} \cup E$ *is of type* (XXI) *in* Theorem 1.

Proof. We have already obtained the first assertion. Now we prove the second assertion. Let $\hat{Y}_2^*$, E_i^* and E^* be the proper transforms of $\hat{Y}_2$, E_i and E by the contraction morphism of $\hat{Y}_1$ respectively. By Lemma 4.2 (iv) and Lemma 4.4 (ii), (iii), $\hat{Y}_2^* \cup E^*$ is a boundary of a smooth compactification of $\mathbb{C}^2$ which is a simple normal crossing divisor of one (-1)-curve $\hat{Y}_2^*$ and nine (-2)-curves. By Lemma 2.9 and $p_2 \in E_1 \setminus \mathrm{Sing}(\mathrm{Supp}\, Z)$, the weighted dual graph of $\hat{Y}_2^* \cup E^*$ is given as in Fig. 15. Since $p_1 \in E_1 \setminus \mathrm{Sing}(\mathrm{Supp}\, Z)$ and $p_1 \neq p_2$, we obtain the assertion. □

Thus we complete the proof of Theorem 1 for the case $Z^2 = -1$.

5. Proof of Theorems 2 and 3

In this section, we shall prove Theorems 2 and 3. We use the same notation as that in the previous sections. For each weighted dual graph of $\hat{Y} \cup E$ of type (XV) through (XXI) in Theorem 1, we know the shape of the divisor $\hat{\Gamma} \cup (\hat{Y} \cup E)$ by noting that $(\hat{\Gamma} \cdot \hat{Y}_i)_M = (\Gamma \cdot Y_i)_X = \deg Y_i$ and $(\hat{\Gamma} \cdot E)_M = 0$. By contracting suitable (-1)-curves in $\hat{Y} \cup E$ repeatedly, we can obtain $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ as a compactification of $\mathbb{C}^2$. Let $\tau = \tau_1 \circ \cdots \circ \tau_N\colon M_N := M \to \cdots \to M_0$ be the composite of blowing-downs to $M_0 := \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, where $10 \le N \le 11$. Conversely, we obtain M by applying blowing-ups of M_0 on $\tau(\hat{Y} \cup E)$ repeatedly. We denote by P_{i-1} the center of the blowing-up τ_i and by F_i the proper transform of $\mathrm{Exc}\, \tau_i = \tau_i^{-1}(P_{i-1})$ in M for $1 \le i \le N$. The birational map $\phi := \pi \circ \tau^{-1}\colon M_0 \cdots \to X$, which has points of indeterminacy at $\tau(\mathrm{Exc}\, \tau)$, gives an isomorphism $M_0 \setminus \tau(\hat{Y} \cup E) \cong X \setminus Y$. The commutative diagram in Fig. 16 gives a resolution of indeterminacy of ϕ. The image $G := \tau_* \hat{\Gamma}$ is an irreducible curve on M_0 with $\mathrm{Sing}\, G = \tau(\mathrm{Exc}\, \tau)$. Since π is determined by the linear system $|\hat{\Gamma}|$ on M, the map ϕ is determined by the linear system $\tau_*|\hat{\Gamma}| = |G - m_0 P_0 - m_1 P_1 - \cdots - m_{N-1} P_{N-1}|$ on M_0 with $m_i \ge 1$. By chasing the process of the resolution of indeterminacy of ϕ, we can determine a basis of the four-dimensional $\mathbb{C}$-vector space associated with $\tau_*|\hat{\Gamma}|$. Thus we write down the map ϕ and the pair (X, Y) as the image of ϕ concretely. Finally we construct a tame automorphism of $\mathbb{C}^3$ explicitly which linearizes the hypersurface

$$
\begin{tikzcd}
M \arrow[d, "\tau"'] \arrow[dr, "\pi"] & \\
M_0 \arrow[r, dashed, "\phi"'] & X
\end{tikzcd}
$$

Fig. 16.

$X \setminus Y$ of $\mathbb{P}^3 \setminus H = \mathbb{C}^3$. Let $w = (w_0 : w_1 : w_2)$ and $z = (z_0 : z_1 : z_2 : z_3)$ be homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively. Let $(x, y) = ((x_0 : x_1), (y_0 : y_1))$ be a bihomogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$.

5.1. The types (XV) and (XVI). For each type, there exists a composite $\tau = \tau_1 \circ \cdots \circ \tau_{11} : M_{11} = M \to \cdots \to M_0 = \mathbb{P}^2$ of blowing-downs to $\mathbb{P}^2$ such that $\operatorname{Exc}\tau$ is contained in $\hat{Y} \cup E$. Let L be the image $\tau(\hat{Y} \cup E)$, which is a line in $\mathbb{P}^2$, and $\overline{L}$ the proper transform of L in M. We note that $\tau(\operatorname{Exc}\tau) = \{P_0\}$ and $\hat{Y} \cup E = \overline{L} \cup \left(\bigcup_{i=1}^{11} F_i\right)$, whose weighted dual graph is given as in Fig. 17(XV) or (XVI). By the shape of $\hat{\Gamma} \cup (\hat{Y} \cup E)$ and $\hat{\Gamma}^2 = 4$, we see that G is a plane sextic curve with $\operatorname{Sing} G = G \cap L = \{P_0\}$ and that ϕ is determined by the linear system $|6L - 2P_0 - 2P_1 - 2P_2 - 2P_3 - 2P_4 - 2P_5 - 2P_6 - P_7 - P_8 - P_9 - P_{10}|$, whose base locus consists of only one point P_0. We may assume that $L = \{w_2 = 0\}$ and $P_0 = (0 : 1 : 0)$. Then we have the next proposition by computing directly.

Proposition 5.1. *One obtains the following*:
(i) *The map ϕ is given, up to automorphisms of $\mathbb{P}^2$ and $\mathbb{P}^3$, as follows*:

$$
\phi : \begin{cases} z_0 = w_0 w_2^5 \\ z_1 = f_1(w) w_2^3 \\ z_2 = w_1 w_2^5 + \{f_1(w) + \lambda_1 w_0 w_2^2\}\{f_1(w) + \lambda_2 w_0 w_2^2\} \\ z_3 = w_2^6 \end{cases}
$$

with $f_1(w) = f_1(w_0, w_1, w_2) = w_0^3 + w_1^2 w_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, where $\lambda_1 = \lambda_2$ for the type (XV) *and $\lambda_1 \neq \lambda_2$ for the type* (XVI).
(ii) *The pair (X, Y) is given, up to automorphisms of $\mathbb{P}^3$, as follows*:

$$
\begin{cases} X : (z_2 z_3 + \alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 + z_0 z_3^3 + z_1^3 z_3 = 0 \\ Y : z_3 = (\alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 = 0 \end{cases}
$$

with $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha \neq 0$, where $\beta^2 - 4\alpha\gamma = 0$ for the type (XV) *and $\beta^2 - 4\alpha\gamma \neq 0$ for the type* (XVI).

(iii) *For each type, there exists a tame automorphism of* $\mathbb{C}^3$ *which transforms the hypersurface* $X \setminus Y$ *onto a coordinate hyperplane.*

5.2. The types (XVII) and (XVIII). For each type, there exists a composite $\tau = \tau_1 \circ \cdots \circ \tau_{10} \colon M_{10} = M \to \cdots \to M_0 = \mathbb{P}^1 \times \mathbb{P}^1$ of blowing-downs to $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\operatorname{Exc} \tau$ is contained in $\hat{Y} \cup E$. Let $C_1 \cup C_2$ be the image $\tau(\hat{Y} \cup E)$, which is a union of fibers of the two standard projections of $\mathbb{P}^1 \times \mathbb{P}^1$, and $\overline{C_1} \cup \overline{C_2}$ the proper transform of $C_1 \cup C_2$ in M. We note that $\tau(\operatorname{Exc} \tau) = \{P_0, P_5\}$ and $\hat{Y} \cup E = \overline{C_1} \cup \overline{C_2} \cup \left(\bigcup_{i=1}^{10} F_i\right)$, whose weighted dual graph is given as in Fig. 17 (XVII) or (XVIII). By the shape of $\hat{\Gamma} \cup (\hat{Y} \cup E)$ and $\hat{\Gamma}^2 = 4$, we see that G is an irreducible curve of bidegree (4, 4) with $\operatorname{Sing} G = G \cap (C_1 \cup C_2) = \{P_0, P_5\}$ and that ϕ is determined by the linear system $|(4C_1 + 4C_2) - (2P_0 + 2P_1 + 2P_2 + P_3 + P_4) - (2P_5 + 2P_6 + 2P_7 + P_8 + P_9)|$, whose base locus consists of two points P_0 and P_5. We may assume that $C_1 = \{y_1 = 0\}$, $C_2 = \{x_1 = 0\}$, $P_0 = ((0:1), (1:0))$ and $P_5 = ((1:0), (0:1))$. Then we have the next proposition by computing directly.

Proposition 5.2. *One obtains the following*:
(i) *The map* ϕ *is given, up to automorphisms of* $\mathbb{P}^1 \times \mathbb{P}^1$ *and* $\mathbb{P}^3$, *as follows*:

$$\phi\colon \begin{cases} z_0 = x_0 x_1^3 y_0 y_1^3 \\ z_1 = f_2(x, y) x_1^2 y_1^2 \\ z_2 = \lambda_3 x_1^4 y_0 y_1^3 + \{f_2(x, y) + \lambda_1 x_0 x_1 y_0 y_1\}\{f_2(x, y) + \lambda_2 x_0 x_1 y_0 y_1\} \\ z_3 = x_1^4 y_1^4 \end{cases}$$

with $f_2(x, y) = f_2(x_0, x_1, y_0, y_1) = x_0^2 y_0^2 + x_0 x_1 y_1^2 + x_1^2 y_0 y_1$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ *and* $\lambda_3 \neq 0$, *where* $(\lambda_1 - \lambda_2)^2\{(\lambda_1 - \lambda_2)^2 + 4\lambda_3\} = 0$ *for the type* (XVII) *and* $(\lambda_1 - \lambda_2)^2\{(\lambda_1 - \lambda_2)^2 + 4\lambda_3\} \neq 0$ *for the type* (XVIII).
(ii) *The pair* (X, Y) *is given, up to automorphisms of* $\mathbb{P}^3$, *as follows*:

$$\begin{cases} X\colon (z_2 z_3 + \alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 - (z_0 z_3 + z_1^2)^2 + z_1 z_3^3 = 0 \\ Y\colon z_3 = (\alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 - z_1^4 = 0 \end{cases}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$ *and* $\alpha \neq 0$, *where* $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} = 0$ *for the type* (XVII) *and* $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} \neq 0$ *for the type* (XVIII).
(iii) *For each type, there exists a tame automorphism of* $\mathbb{C}^3$ *which transforms the hypersurface* $X \setminus Y$ *onto a coordinate hyperplane.*

5.3. The types (XIX) and (XX). For each type, there exists a composite $\tau = \tau_1 \circ \cdots \circ \tau_{11} \colon M_{11} = M \to \cdots \to M_0 = \mathbb{P}^2$ of blowing-downs to $\mathbb{P}^2$ such that $\operatorname{Exc} \tau$ is contained in $\hat{Y} \cup E$. Let L be the image $\tau(\hat{Y} \cup E)$, which is a line in $\mathbb{P}^2$, and $\overline{L}$ the proper transform of L in M. We note that $\tau(\operatorname{Exc} \tau) = \{P_0, P_7\}$ and $\hat{Y} \cup E =$

$\overline{L} \cup \left(\bigcup_{i=1}^{11} F_i\right)$, whose weighted dual graph is given as in Fig. 17 (XIX) or (XX). By the shape of $\hat{\Gamma} \cup (\hat{Y} \cup E)$ and $\hat{\Gamma}^2 = 4$, we see that G is a plane sextic curve with $\operatorname{Sing} G = G \cap L = \{P_0, P_7\}$ and that ϕ is determined by the linear system $|6L - (2P_0 + 2P_1 + 2P_2 + 2P_3 + 2P_4 + P_5 + P_6) - (2P_7 + 2P_8 + P_9 + P_{10})|$, whose base locus consists of two points P_0 and P_7. We may assume that $L = \{w_2 = 0\}$, $P_0 = (0 : 1 : 0)$ and $P_7 = (1 : 0 : 0)$. Then we have the next proposition by computing directly.

Proposition 5.3. *One obtains the following*:
(i) *The map ϕ is given, up to automorphisms of $\mathbb{P}^2$ and $\mathbb{P}^3$, as follows*:

$$\phi\colon \begin{cases} z_0 = w_0 w_2^5 \\ z_1 = f_3(w) w_2^3 \\ z_2 = w_0^2 w_2^4 + w_1 w_2^5 + \{f_3(w) + \lambda_1 w_0 w_2^2\}\{f_3(w) + \lambda_2 w_0 w_2^2\} \\ z_3 = w_2^6 \end{cases}$$

with $f_3(w) = f_3(w_0, w_1, w_2) = w_0^2 w_1 + w_1^2 w_2 + \lambda_3 w_1 w_2^2$ *and* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, *where* $(\lambda_1 - \lambda_2)^2\{(\lambda_1 - \lambda_2)^2 - 4\} = 0$ *for the type* (XIX) *and* $(\lambda_1 - \lambda_2)^2\{(\lambda_1 - \lambda_2)^2 - 4\} \neq 0$ *for the type* (XX).
(ii) *The pair (X, Y) is given, up to automorphisms of $\mathbb{P}^3$, as follows*:

$$\begin{cases} X\colon (z_2 z_3 + \alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 - z_1^4 + z_0 z_3^3 + \delta z_1^2 z_3^2 = 0 \\ Y\colon z_3 = (\alpha z_0^2 + \beta z_0 z_1 + \gamma z_1^2)^2 - z_1^4 = 0 \end{cases}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *and* $\alpha \neq 0$, *where* $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} = 0$ *for the type* (XIX) *and* $\{\beta^2 - 4\alpha(\gamma - 1)\}\{\beta^2 - 4\alpha(\gamma + 1)\} \neq 0$ *for the type* (XX).
(iii) *For each type, there exists a tame automorphism of $\mathbb{C}^3$ which transforms the hypersurface $X \setminus Y$ onto a coordinate hyperplane.*

5.4. The type (XXI). For this type, there exists a composite $\tau = \tau_1 \circ \cdots \circ \tau_{10}\colon M_{10} = M \to \cdots \to M_0 = \mathbb{P}^2$ of blowing-downs to $\mathbb{P}^2$ such that $\operatorname{Exc}\tau$ is contained in $\hat{Y} \cup E$. Let L be the image $\tau(\hat{Y} \cup E)$, which is a line in $\mathbb{P}^2$, and $\overline{L}$ the proper transform of L in M. We note that $\tau(\operatorname{Exc}\tau) = \{P_0\}$, $F_9 = \hat{Y}_2$, $F_{10} = \hat{Y}_1$ and $\hat{Y} \cup E = \overline{L} \cup \left(\bigcup_{i=1}^{11} F_i\right)$, whose weighted dual graph is given as in Fig. 17 (XXI). By the shape of $\hat{\Gamma} \cup (\hat{Y} \cup E)$ and $\hat{\Gamma}^2 = 4$, we see that G is a plane curve of degree nine with $\operatorname{Sing} G = G \cap L = \{P_0\}$ and that ϕ is determined by the linear system $|9L - 3P_0 - 3P_1 - 3P_2 - 3P_3 - 3P_4 - 3P_5 - 3P_6 - 3P_7 - 2P_8 - P_9|$, whose base locus consists of only one point P_0. We may assume that $L = \{w_2 = 0\}$ and $P_0 = (0 : 1 : 0)$. Then we have the next proposition by computing directly.

Proposition 5.4. *One obtains the following*:

(i) *The map ϕ is given, up to automorphisms of $\mathbb{P}^2$ and $\mathbb{P}^3$, as follows*:

$$\phi: \begin{cases} z_0 = f_4(w)w_2^6 \\ z_1 = w_0 w_2^8 + f_4(w)^2 w_2^3 \\ z_2 = w_1 w_2^8 - f_4(w)^3 + \dfrac{3}{2} f_4(w)\{w_0 w_2^5 + f_4(w)^2\} \\ z_3 = w_2^9 \end{cases}$$

with $f_4(w) = f_4(w_0, w_1, w_2) = w_0^3 + w_1^2 w_2 + \lambda w_0 w_2^2$ *and* $\lambda \in \mathbb{C}$.

(ii) *The pair (X, Y) is given, up to automorphisms of $\mathbb{P}^3$, as follows*:

$$\begin{cases} X: z_2^2 z_3^2 + (2z_0^3 + 3z_0 z_1 z_3)z_2 - z_1^3 z_3 - \dfrac{3}{4} z_0^2 z_1^2 + z_0 z_3^3 + \delta(z_1 z_3 + z_0^2) z_3^2 = 0 \\ Y: z_3 = z_0^2 \left(z_0 z_2 - \dfrac{3}{8} z_1^2 \right) = 0 \end{cases}$$

with $\delta \in \mathbb{C}$.

(iii) *There exists a tame automorphism of $\mathbb{C}^3$ which transforms the hypersurface $X \setminus Y$ onto a coordinate hyperplane.*

REMARK. In (ii), the hypersurface $X \setminus Y$ is expressed as follows:

$$\begin{aligned} 0 &= z_2^2 + (2z_0^3 + 3z_0 z_1)z_2 - z_1^3 - \frac{3}{4} z_0^2 z_1^2 + z_0 + \delta(z_1 + z_0^2) \\ &= \left(z_2 + z_0^3 + \frac{3}{2} z_0 z_1 \right)^2 - (z_1 + z_0^2)^3 + z_0 + \delta(z_1 + z_0^2) \end{aligned}$$

where (z_0, z_1, z_2) is a coordinate of $\mathbb{C}^3 = \mathbb{P}^3 \setminus H$.

Thus we complete the proof of Theorems 2 and 3 for the types (XV) through (XXI) in Theorem 1.

ACKNOWLEDGMENTS. The author would like to express his hearty gratitude to Professor Mikio Furushima for his invaluable advice and encouragement during the preparation of this paper. Furthermore, the author is grateful to the referee for his/her suggestions and corrections.

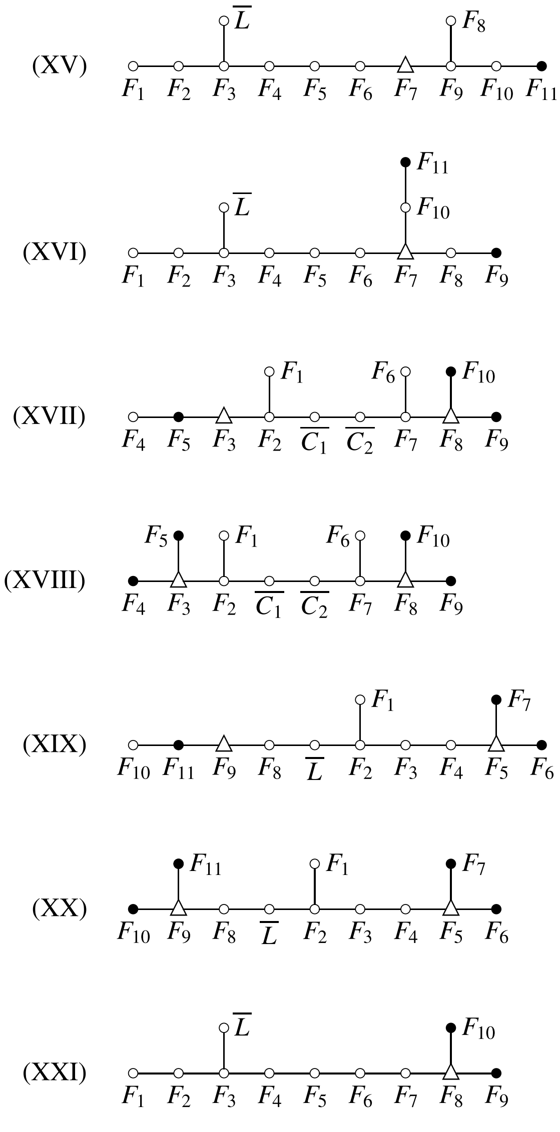


Fig. 17.

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