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SUBELLIPTIC HARMONIC MORPHISMS

Dedicated to Professor H. Urakawa on the occasion of his 60th birthday

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Abstract

We study subelliptic harmonic morphisms i.e. smooth maps $\phi\colon \Omega \to \tilde{\Omega}$ among domains $\Omega \subset \mathbb{R}^N$ and $\tilde{\Omega} \subset \mathbb{R}^M$, endowed with Hörmander systems of vector fields X and Y, that pull back local solutions to $H_Y v = 0$ into local solutions to $H_X u = 0$, where H_X and H_Y are Hörmander operators. We show that any subelliptic harmonic morphism is an open mapping. Using a subelliptic version of the Fuglede-Ishihara theorem (due to E. Barletta, [5]) we show that given a strictly pseudoconvex CR manifold M and a Riemannian manifold N for any heat equation morphism $\Psi\colon M \times (0, \infty) \to N \times (0, \infty)$ of the form $\Psi(x, t) = (\phi(x), h(t))$ the map $\phi\colon M \to N$ is a subelliptic harmonic morphism.

1. The Hörmander operator

Let $\Omega \subseteq \mathbb{R}^N$ be a domain and $X_a \in \mathcal{X}^\infty(\Omega)$, $1 \le a \le m$, a set of C^∞ vector fields on Ω. $X = (X_1, \dots, X_m)$ is a *Hörmander system* if the vector fields X_a together with their commutators up to a fixed length[1] r span the tangent space to $\mathbb{R}^N$ at each $x \in \Omega$. For instance, let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group (cf. e.g. [16], p. 11–14) with coordinates $(z^1, \dots, z^n, t)$. Let us consider the complex vector fields on $\mathbb{H}_n$ given by $L_{\overline{\alpha}} = \partial/\partial\overline{z}^\alpha - \sqrt{-1} z^\alpha\, \partial/\partial t$ (the *Lewy operators*). Then the following set of (left invariant) vector fields

$$X_\alpha = L_\alpha + L_{\overline{\alpha}}, \quad X_{\alpha+n} = \sqrt{-1}(L_\alpha - L_{\overline{\alpha}}), \quad 1 \le \alpha \le n, \tag{1}$$

is a Hörmander system $(r = 2)$ on $\mathbb{R}^{2n+1}$. Here $L_\alpha = \overline{L_{\overline{\alpha}}}$. If $X_a = b_a^i(x)\, \partial/\partial x^i$ we set $X_a^* f = -\partial(b_a^i f)/\partial x^i$ for any $f \in C_0^1(U)$ (the formal adjoint of X_a). The *Hörmander operator* is the second order differential operator $H = H_X$ given by

$$Hu \equiv -\sum_{a=1}^m X_a^* X_a u = \sum_{i,j=1}^N \frac{\partial}{\partial x^i}\left(a^{ij}(x) \frac{\partial u}{\partial x^j}\right)$$

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[1] A commutator of the form $[X_{a_1}, [X_{a_2}, [\dots, X_{a_r}] \cdots]$ has length r. By convention each X_a has length 1.

where $a^{ij}(x) = \sum_{a=1}^{m} b_a^i(x) b_a^j(x)$. When $m = N$ and $X_a = \partial/\partial x^a$ the Hörmander operator is the ordinary Laplacian on $\mathbb{R}^N$. In general $a^{ij}(x)$ is only positive semi-definite, so that H is a degenerate elliptic operator (and actually H satisfies the conditions (1)–(3) of M. Bony, [11], p. 278–279). Also, by a well known result of L. Hörmander, [23], H is hypoelliptic. The analogy to the theory of elliptic operators, and in particular to harmonic function theory, prompted the study of (local) properties of (weak) solutions to $Hu = 0$ (cf. e.g. M. Bony, [11], A. Sánchez-Calle, [34], A. Bonfiglioli and E. Lanconelli, [8]–[10], G. Citti, N. Garofallo and E. Lanconelli, [13], F. Uguzzoni and E. Lanconelli, [35]) and of solutions to certain nonlinear subelliptic systems of variational origin (with principal part H) such as the subelliptic harmonic map system (cf. J. Jost and C.-J. Xu, [27], Z.-R. Zhou, [39]).

On the same line of thought E. Barletta, [4], started the study of *subelliptic harmonic morphisms* i.e. localizable[2] maps $\phi\colon \Omega \to N$, where N is a Riemannian manifold, such that for any local harmonic function $v\colon V \to \mathbb{R}$ (with $V \subseteq N$ open) one has i) $v \circ \phi \in L^1_{\mathrm{loc}}(U)$ for any open subset $U \subset \Omega$ such that $\phi(U) \subset V$, and ii) $H(v \circ \phi) = 0$ in distributional sense. Any subelliptic harmonic morphism is easily seen to be a C^∞ map, as a consequence of the existence of harmonic local coordinates on the target Riemannian manifold N. By a result of E. Barletta (cf. *op. cit.*) if $\dim(N) = \nu > m$ then there are no nonconstant subelliptic harmonic morphisms $\phi\colon \Omega \to N$. Moreover, if $\nu \leq m$ then every subelliptic harmonic morphism is a subelliptic harmonic map (in the sense of J. Jost and C.-J. Xu, cf. *op. cit.*). The elliptic counterpart of this result is the well known Fuglede-Ishihara theorem (cf. B. Fuglede, [19], T. Ishihara, [25]).

The present work is devoted to further exploring the geometry of subelliptic harmonic morphisms and their variants. One of the main results is

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ and $\tilde{\Omega} \subset \mathbb{R}^M$ be two domains and $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ two Hörmander systems of vector fields on $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. Let $\phi\colon \Omega \to \tilde{\Omega}$ be a smooth map pulling back the local harmonics of the Hörmander operator associated to Y to local harmonics of the Hörmander operator associated to X. Then ϕ is an open map, that is ϕ maps open subsets of Ω into open subsets of $\tilde{\Omega}$.*

Theorem 1 extends (from elliptic to subelliptic theory) a result of B. Fuglede, [20]. The ingredients in the proof of Theorem 1 are the existence of fundamental solutions to the Hörmander operator H (due to M. Bony, [11]), the estimates on the fundamental solution to H (due to A. Sánchez-Calle, [34]) and a version of the Harnack inequality for degenerate elliptic operators (due again to M. Bony, cf. *op. cit.*). Our second main result is

[2]That is for any $x_0 \in \Omega$ there is an open neighborhood $x_0 \in U \subseteq \Omega$ and a local coordinate system $(V, y^1, \ldots, y^\nu)$ on N such that $\phi(U) \subseteq V$.

Theorem 2. *Let M be a strictly pseudoconvex CR manifold and θ a contact form such that the Levi form G_θ is positive definite. Let N be a Riemannian manifold and $\Psi : M \times (0, +\infty) \to N \times (0, +\infty)$ a smooth map of the form $\Psi(x, t) = (\phi(x), h(t))$ for any $x \in M$, $t > 0$. Then Ψ is a heat equation morphism if and only if $\phi : M \to N$ is a subelliptic harmonic morphism of constant θ-dilation λ and $h(t) = \lambda^2 t + C$ for some $C \in \mathbb{R}$.*

The paper is organized as follows. In Section 2 we discuss complex valued subelliptic harmonic morphisms from the lowest dimensional Heisenberg group $\mathbb{H}_1$. Various generalizations of subelliptic harmonic morphisms, both in the context of Hörmander systems of vector fields and within CR geometry, are considered in Section 3 (where Theorem 1 is proved) and Section 4 where we present the subelliptic version of a result of E. Loubeau, [30] (cf. Theorem 2).

2. $\mathbb{C}$-valued subelliptic harmonic morphisms from $\mathbb{H}_1$

Let ϕ: $U \to \mathbb{C}$ be a C^2 function, with $U \subseteq \mathbb{H}_1$ open. Adopting the so called Jacobi trick (cf. C.G.J. Jacobi, [26]) we seek solutions to $H\phi = 0$ of the form $v \circ \phi$ where $v : V \to \mathbb{C}$ is a holomorphic function (with $V \subseteq \mathbb{C}$ open). The Hörmander operator (the *sublaplacian*) on $\mathbb{H}_1$ is given by $H = X_1^2 + X_2^2$ where X_a are given by (1) i.e.

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}.$$

As $v_{\overline{z}} = 0$ in V

$$X(v \circ \phi) = X(\phi)v_z \circ \phi, \quad X^2(v \circ \phi) = X^2(\phi)v_z \circ \phi + X(\phi)^2 v_{zz} \circ \phi,$$

for any $X \in T(\mathbb{H}_1)$ hence

$$H(v \circ \phi) = (H\phi)v_z \circ \phi + \sum_a X_a(\phi)^2 v_{zz} \circ \phi.$$

Consequently we obtain the following

Proposition 1. *Let $\phi : U \subseteq \mathbb{H}_1 \to \mathbb{C}$ be a harmonic of the sublaplacian on $\mathbb{H}_1$. Then $H(v \circ \phi) = 0$ on $\phi^{-1}(V)$ for each holomorphic function $v : V \subseteq \mathbb{C} \to \mathbb{C}$ if and only if*

$$X_1(\phi)^2 + X_2(\phi)^2 = 0 \tag{2}$$

everywhere in U. Moreover if ϕ satisfies (2) *then so does $v \circ \phi$ for any holomorphic function v.*

Confined to the case of complex valued maps from the lowest dimensional Heisenberg group we may adopt the following temporary definition. A C^0 map ϕ: $U \subseteq \mathbb{H}_1 \to \mathbb{C}$ is a *subelliptic harmonic morphism* if for every harmonic function $h\colon V \subseteq \mathbb{C} \to \mathbb{R}$ with $\phi^{-1}(V) \neq \emptyset$ the function $h \circ \phi$ is a harmonic of H on $\phi^{-1}(V)$. Any subelliptic harmonic morphism $\phi\colon U \to \mathbb{C}$ is C^∞. Indeed, let $\phi = \phi_1 + \sqrt{-1}\phi_2$ be the real and imaginary parts of ϕ. As Re, Im: $\mathbb{C} \to \mathbb{R}$ are harmonic functions it follows that $H\phi_i = 0$ in distributional sense. Yet H is hypoelliptic hence $\phi_i \in C^\infty(U)$. As a consequence of Proposition 1

Corollary 1. *Let $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ be a continuous map. Then ϕ is a subelliptic harmonic morphism if and only if $H\phi = 0$ and $X_1(\phi)^2 + X_2(\phi)^2 = 0$.*

For instance let $f(z)$ be an entire function. Then (by Corollary 1) $\phi(z, t) = f(z)$, $(z, t) \in \mathbb{H}_1$, is a subelliptic harmonic morphism.

Proof of Corollary 1. We start by proving sufficiency. Let $h\colon V \subseteq \mathbb{C} \to \mathbb{R}$ be a harmonic function with $\phi^{-1}(V) \neq \emptyset$. We may assume that V is connected (otherwise the same proof applies to any subdomain of V) hence there is a holomorphic function $v\colon V \to \mathbb{C}$ such that $\mathrm{Re}(v) = h$. By Proposition 1 the identity (2) implies $H(v \circ \phi) = 0$, hence $H(h \circ \phi) = 0$ (as H is a real operator).

Viceversa, let $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ be a subelliptic harmonic morphism. The very definition (applied twice, for $h = \mathrm{Re}$ and $h = \mathrm{Im}$) implies $H\phi = 0$. Let $v\colon V \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then $H(v \circ \phi) = 0$ (as the real and imaginary parts of v are harmonic) and (2) follows from Proposition 1. $\square$

Note that the identity

$$H(\phi^2) = 2\left\{\phi H\phi + \sum_a (X_a\phi)^2\right\}$$

yields the following

Corollary 2. *A C^0 map $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ is a subelliptic harmonic morphism if and only if both ϕ and ϕ^2 are harmonics of the sublaplacian on $\mathbb{H}_1$.*

We recall (cf. e.g. [16], p. 12) that $\theta = dt + i(z\,d\bar{z} - \bar{z}\,dz)$ is a contact form on $\mathbb{H}_1$. Let us consider the *Levi form*

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(\mathbb{H}_1).$$

Here $H(\mathbb{H}_1)$ is the span of $\{X_1, X_2\}$ (the *Levi distribution*) and $J\colon H(\mathbb{H}_1) \to H(\mathbb{H}_1)$ its natural complex structure i.e. $JX_1 = X_2$ and $JX_2 = -X_1$. We set $\|X\|_\theta = G_\theta(X, X)^{1/2}$. Also the *horizontal gradient* of a function $u \in C^1(\mathbb{H}_1)$ is given by $\nabla^H u = \sum_a (X_a u) X_a$.

If $\phi = \phi_1 + \sqrt{-1}\phi_2$ then $\sum_a (X_a\phi)^2 = 0$ is equivalent to

$$\sum_a (X_a\phi_1)^2 = \sum_a (X_a\phi_2)^2, \quad \sum_a (X_a\phi_1)(X_a\phi_2) = 0,$$

or

$$\|\nabla^H\phi_1\|_\theta = \|\nabla^H\phi_2\|_\theta, \quad G_\theta(\nabla^H\phi_1, \nabla^H\phi_2) = 0. \tag{3}$$

A C^∞ map $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ satisfying (3) is said to be *semiconformal*. Note that the identities (3) are *CR-invariant* i.e. invariant under a transformation $\hat{\theta} = f\theta$, for any C^∞ function $f: \mathbb{H}_1 \to \mathbb{R} \setminus \{0\}$. Given a semiconformal map $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ we set $\lambda = \|\nabla^H\phi_1\|_\theta = \|\nabla^H\phi_2\|_\theta$ (the *θ-dilation*) and note that $\lambda \in C^0(U)$ while λ^2 is smooth. We adopt the following definitions. A point $x_0 \in U$ is *critical* (respectively *regular*) if $\lambda(x_0) = 0$ (respectively $\lambda(x_0) \neq 0$). The notions of critical and regular point of ϕ are CR-invariant notions. We shall establish the following

Proposition 2. *Let $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ be a semiconformal map. For any regular point $x_0 \in U$ of ϕ there is an open neighborhood $x_0 \in \Omega \subseteq U$ such that $\phi\colon \Omega \to \mathbb{C}$ is a submersion.*

Proof. The proof is rather elementary. Note first that the Jacobi matrix of ϕ may be written as

$$\begin{pmatrix} X_1\phi_1 - 2yT\phi_1 & X_1\phi_2 - 2yT\phi_2 \\ X_2\phi_1 + 2xT\phi_1 & X_2\phi_2 + 2xT\phi_2 \\ T\phi_1 & T\phi_2 \end{pmatrix} \tag{4}$$

where $T = \partial/\partial t$. Let D_a ($a = 1, 2$) be the determinant consisting of the a-th and third rows in (4). Let $v = (D_1, D_2)$. We distinguish two cases as I) $v(x_0) \neq 0$, and then $\mathrm{rank}(d_{x_0}\phi) = 2$, or II) $v(x_0) = 0$. In the second case the determinant consisting of the first two rows in (4) is $(X_1\phi_1)(X_2\phi_2) - (X_1\phi_2)(X_2\phi_1) \neq 0$ at x_0 as (by semiconformality) $(X_1\phi_a, X_2\phi_a)_{x_0}$, $a = 1, 2$, are orthogonal vectors in $\mathbb{R}^2$. □

Note that $\|\nabla^H\phi\|_\theta = \lambda\sqrt{2}$ hence for any semiconformal map $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ point $x \in U$ is regular if and only if $\|\nabla^H\phi\|_\theta(x) \neq 0$.

The problem whether one may produce subelliptic harmonic morphisms $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ by solving implicit equations, analogous to [26] or [2], p. 6, is open. It should be noted that both the result[3] claimed in Proposition 1.2.1 in [2], p. 6–7, and its tentative proof[4] are actually wrong[5]. Nevertheless examples of implicit equations whose solutions are subelliptic harmonic morphisms do exist.

Proposition 3. *Let $V \subseteq \mathbb{C}$ be an open set and $\xi : V \to \mathbb{C}^2$ a null holomorphic map i.e. if $\xi = (\xi_1, \xi_2)$ then each ξ_a is a holomorphic function and*

$$\sum_a \xi_a(\zeta)^2 = 0, \quad \sum_a |\xi_a(\zeta)|^2 \neq 0,$$

for any $\zeta \in V$. Let

$$G(x, \zeta) = \xi_1(\zeta)x_1 + \xi_2(\zeta)x_2, \quad x = (x_1 + \sqrt{-1}x_2, t) \in \mathbb{H}_1, \quad \zeta \in V.$$

Then any smooth solution $\phi : U \subseteq \mathbb{H}_1 \to V \subseteq \mathbb{C}$ to $G(x, \phi(x)) = 0$, $x \in U$, is a submersive subelliptic harmonic morphism.

Proof. A calculation shows that $\sum_a |X_a G|^2 = \sum_a |\xi_a(\zeta)|^2 \neq 0$ and $H_x G(x, \zeta) = 0$ and $\sum_a (X_a G)^2 = \sum_a \xi_a(\zeta)^2 = 0$, for any $(x, \zeta) \in \mathbb{H}_1 \times V$, hence Proposition 3 follows from Proposition 2 and the following

Lemma 1. *Let $G\colon A \to \mathbb{C}$ be a smooth function with $A \subseteq \mathbb{H}_1 \times \mathbb{C}$ open such that $G(x, \zeta)$ is holomorphic in ζ. Let us assume that $(X_1 G, X_2 G)_{(x,\zeta)} \neq 0$ and $(HG)(x, \zeta) = 0$ and $\sum_a (X_a G)(x, \zeta)^2 = 0$ for any $(x, \zeta) \in A$. Then any smooth solution $\phi\colon U \subseteq \mathbb{H}_1 \to \mathbb{C}$ to $G(x, \phi(x)) = 0$, $x \in U$, is a subelliptic harmonic morphism.*

Proof. The identity

$$(X_a G)(x, \phi(x)) + G_\zeta(x, \phi(x))(X_a \phi)(x) = 0, \quad a = 1, 2, \tag{5}$$

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[3]Proposition 1.2.1 in [2] claims that given a smooth function $G\colon A \to \mathbb{C}$ defined on an open subset $A \subseteq \mathbb{R}^3 \times \mathbb{C}$ such that $G(x_1, x_2, x_3, z)$ is holomorphic in z and $(G_{x_1}(x, z), G_{x_2}(x, z), G_{x_3}(x, z)) \neq 0$ for any $(x, z) \in A$ with $G(x, z) = 0$ there is a smooth solution $\varphi : U \to \mathbb{C}$ on an open set $U \subseteq \mathbb{R}^3$ to the equation $G(x, \varphi(x)) = 0$, $x \in U$, such that $\Delta\varphi = 0$ and $\sum_{i=1}^3 (\varphi_{x_i})^2 = 0$ if and only if $(\Delta_x G)(x, \varphi(x)) = 0$ and $\sum_{i=1}^3 G_{x_i}(x, \varphi(x))^2 = 0$ for any $x \in U$. As a counterexample let $\varphi(x) = x_2 + \sqrt{-1}x_3$ and $G(x, z) = (1 + |x|^2)(z - \varphi(x))$. Then G satisfies the assumptions in Proposition 1.2.1 (cf. *op. cit.*, p. 6) with $A = \mathbb{R}^3 \times \mathbb{C}$ yet $(\Delta_x G)(x, \varphi(x)) = -4\varphi(x) \neq 0$ on $U = \{x \in \mathbb{R}^3 : x_2 + \sqrt{-1}x_3 \neq 0\}$.

[4]Let us set $F(x, z) = \sum_{i=1}^3 G_{x_i}(x, z)^2$ for simplicity. The formula (1.2.6) in [2], p. 6, claims that $F_z(x, \varphi(x)) = 0$ as a consequence of $F(x, \varphi(x)) = 0$, $x \in U$.

[5]Though the arguments in [2], p. 6–7, show that local solutions φ to the equations $(\Delta_x G)(x, \varphi(x)) = 0$ and $\sum_{i=1}^3 G_{x_i}(x, \varphi(x))^2 = 0$ are certainly harmonic morphisms and nothing more is needed for the further development in [2] (which remains an excellent reference for the theory of harmonic morphisms among Riemannian and semi-Riemannian manifolds). One may see `http://www.amsta.leeds.ac.uk/Pure/staff/wood/BWBook` for posted corrections.

yields $\sum_a |X_a G|^2 = |G_\zeta|^2 \sum_a |X_a \phi|^2$ hence $G_\zeta(x, \phi(x)) \neq 0$ for any $x \in U$. Similarly $\sum_a (X_a G)^2 = G_\zeta^2 \sum_a (X_a \phi)^2$ implies that $\sum_a (X_a \phi)^2 = 0$. Let us apply X_a to (5) and take the sum over a in the resulting identity. We get

$$G_\zeta(x, \phi(x))^2 H\phi(x) = \frac{\partial}{\partial \zeta}\left(\sum_a (X_a G)^2\right)_{(x,\phi(x))} = 0.$$

Now Lemma 1 follows from Corollary 1. □

3. Generalizations to CR geometry

Haec ornamenta mea.
—Valerius Maximus

A tentative generalization of Jacobi's trick to CR geometry is to look at C^∞ maps $\phi: U \subseteq \mathbb{H}_n \to \mathbb{H}_1$ and their composition with CR functions on $\mathbb{H}_1$. Let us recall that a C^1 function $v: V \to \mathbb{C}$ with $V \subseteq \mathbb{H}_1$ open is a *CR function* if

$$\overline{L} v \equiv \frac{\partial v}{\partial \overline{z}} - \sqrt{-1} z \frac{\partial v}{\partial t} = 0 \tag{6}$$

in V (and (6) are the *tangential Cauchy-Riemann equations* on $\mathbb{H}_1$). Let $\mathrm{CR}^k(V)$ denote the space of all CR functions on V of class C^k ($k \geq 1$). Let $v \in \mathrm{CR}^2(V)$ such that $\phi^{-1}(V) \neq \emptyset$. If $\phi = (F, f)$ where $F: U \to \mathbb{C}$ and $f: U \to \mathbb{R}$ then for any $X \in T(\mathbb{H}_n)$

$$X(v \circ \phi) = X(F) L v + \{X(f) - \sqrt{-1}\, \overline{F} X(F) + \sqrt{-1} F X(\overline{F})\} v_t, \tag{7}$$

$$\begin{aligned} X((Lv) \circ \phi) = {} & X(F) L^2 v + 2\sqrt{-1} X(\overline{F}) v_t \\ & + \{X(f) - \sqrt{-1}\, \overline{F} X(F) + \sqrt{-1} F X(\overline{F})\} L v_t, \end{aligned} \tag{8}$$

as $[L, T] = 0$ and $[L, \overline{L}] = -2\sqrt{-1} T$. Moreover (as $\overline{L} v_t = (\overline{L} v)_t = 0$)

$$X(v_t \circ \phi) = X(F) L v_t + \{X(f) - \sqrt{-1}\, \overline{F} X(F) + \sqrt{-1} F X(\overline{F})\} v_{tt}. \tag{9}$$

Using the identities (7)–(9) one may compute $X^2(v \circ \phi)$ hence obtain

$$\begin{aligned} H(v \circ \phi) = {} & (HF) L v + \{Hf - \sqrt{-1}\, \overline{F} HF + \sqrt{-1} F H\overline{F}\} v_t \\ & + \sum_a X_a(F)^2 (L^2 v - 2\sqrt{-1}\, \overline{F} L v_t - \overline{F}^2 v_{tt}) \\ & + \sum_a |X_a F|^2 (2\sqrt{-1} v_t + 2\sqrt{-1} F L v_t + 2|F|^2 v_{tt}) - \sum_a X_a(\overline{F})^2 F^2 v_{tt} \\ & + \sum_a X_a(f) X_a(F)(2 L v_t - 2\sqrt{-1}\, \overline{F} v_{tt}) \\ & + \sum_a X_a(f) X_a(\overline{F}) 2\sqrt{-1} F v_{tt} + \sum_a X_a(f)^2 v_{tt}. \end{aligned} \tag{10}$$

We shall prove the following

Proposition 4. *Let $U \subseteq \mathbb{H}_n$ be a connected open set and $\phi\colon U \to \mathbb{H}_1$ a continuous map such that $H\phi = 0$. Then $H(v \circ \phi) = 0$ for any CR function $v\colon V \subseteq \mathbb{H}_1 \to \mathbb{C}$ of class C^2 if and only if ϕ is constant.*

Proof. Let $v = \varphi(z)$, where φ is a holomorphic function. Then $HF = 0$, $Hf = 0$ and $H(v \circ \phi) = 0$ together with (10) yield $\sum_a X_a(F)^2 = 0$. Hence $H(v \circ \phi) = 0$ if and only if

$$\sum_a |X_a F|^2 (2\sqrt{-1} v_t + \sqrt{-1} F L v_t + 2|F|^2 v_{tt})$$
$$+ \sum_a X_a(f) X_a(F)(2L v_t - 2\sqrt{-1}\,\overline{F} v_{tt}) + \sum_a X_a(f)^2 v_{tt}$$
$$+ \sum_a X_a(f) X_a(\overline{F}) 2\sqrt{-1} F v_{tt} = 0.$$

In particular for $v = |z|^2 - \sqrt{-1}t \in \mathrm{CR}^\infty(\mathbb{H}_1)$ it is necessary that $\sum_a |X_a F|^2 = 0$ i.e. $X_a(F) = 0$, $1 \le a \le 2n$. Thus $H(v \circ \phi) = 0$ if and only if

$$\sum_a X_a(f)^2 v_{tt} = 0.$$

In particular for $v = (|z|^2 - it)^2$ it follows that $X_a(f) = 0$. Finally $[X_j, X_{j+n}] = -4T$ $(1 \le j \le n)$ yields $F_t = 0$ and $f_t = 0$. □

The (negative) result in Proposition 4 shows that the (tentative) direct generalization of the situation in Proposition 1 and Corollary 1 is not fruitful. Then what is the appropriate notion of a subelliptic harmonic morphism into a C^∞ manifold N (endowed with a preferred sheaf of functions, to play the role of harmonics in Corollary 1)? When N is a Riemannian manifold and $X = (X_1, \dots, X_m)$ is a Hörmander system on the domain $\Omega \subset \mathbb{R}^N$ the following notion is proposed in [4]. Let $\phi\colon \Omega \to N$ be a localizable map. Then ϕ is a (*weak*) *subelliptic harmonic morphism* if for any harmonic function $v\colon V \to \mathbb{R}$ with $V \subseteq N$ open the function $v \circ \phi$ is locally integrable on U for any open subset $U \subseteq \Omega$ such that $\phi(U) \subseteq V$ and $H(v \circ \phi) = 0$ in distributional sense, where H is again the Hörmander operator associated to the Hörmander system X. The definition carries over easily to the case of maps $\phi\colon M \to N$ defined on a given strictly pseudoconvex CR manifold M by merely replacing the Hörmander operator H by the sublaplacian Δ_b (cf. [5], p. 36, where the resulting notion is referred to as a *pseudoharmonic morphism*). In both cases a subelliptic harmonic morphism is actually smooth, due to i) the existence of local harmonic coordinates on N and ii) the hypoellipticity of either H or Δ_b.

Our main purpose for the remainder of this section is to prove Theorem 1. To this end, let $U \subseteq \Omega$ be a connected open set. We shall show that $V = \phi(U) \subseteq \tilde{\Omega}$ is an open subset. The proof is by contradiction. Let us assume that $V \setminus \mathring{V} \neq \emptyset$ and consider $q_0 \in V \setminus \mathring{V}$. Then $B(q_0, 1/j) \setminus V \neq \emptyset$ for any $j \geq 1$, where $B(x, r)$ denotes the Euclidean ball of radius $r > 0$ and center $x \in \mathbb{R}^M$. Let $q_j \in B(q_0, 1/j) \setminus V$, $j \geq 1$. There is $j_0 \geq 1$ such that $B(q_0, 1/j) \subseteq \tilde{\Omega}$ for any $j \geq j_0$. Summarizing, $q_j \in \tilde{\Omega} \setminus V$ for any $j \geq j_0$ and $q_j \to q_0$ as $j \to \infty$.

By a result of A. Sánchez-Calle, [34], there is a positive fundamental solution $G_Y(x, y)$ of H_Y which is C^∞ off the diagonal in $\mathbb{R}^M \times \mathbb{R}^M$ such that for any bounded subset $A \subset \mathbb{R}^M$ there exist constants $C_1 > 0$, $C_2 > 0$ and $r_0 > 0$ such that for every $x \in A$ and every $y \in A \setminus \{x\}$ with $d_Y(x, y) \leq r_0$

$$C_1 \frac{d_Y(x, y)^2}{|B_Y(x, d_Y(x, y))|} \leq G_Y(x, y) \leq C_2 \frac{d_Y(x, y)^2}{|B_Y(x, d_Y(x, y))|}. \tag{11}$$

Here d_Y is the Carnot-Carathéodory distance on $\mathbb{R}^M$ associated to the Hörmander system Y (cf. e.g. (1.9) in [13], p. 702) and $B_Y(x, r)$ is the ball of radius r with respect to d_Y. Also $|A|$ denotes the Lebesgue measure of the set A. On the other hand by a result of A. Nagel et al., [32], there are constants $a_1 > 0$ and $a_2 > 0$ such that

$$a_1 \leq \frac{|B_Y(x, \delta)|}{\Lambda(x, \delta)} \leq a_2 \tag{12}$$

for any $x \in A$, where $\Lambda(x, \delta)$ is a polynomial in δ with nonnegative coefficients

$$\Lambda(x, \delta) = \sum_I |\lambda_I(x)| \delta^{d(I)}.$$

The index I in the sum above ranges over a finite set depending on A. Let us set $E(x, \delta) = \Lambda(x, \delta)/\delta^2$ for any $x \in A$ and $\delta > 0$. As $d(I) \geq M$ for every I (cf. [32]) if $\delta = d_Y(x, y)$ then (by (11)–(12))

$$\frac{C_1 \delta^2}{|B_Y(x, \delta)|} \geq \frac{C}{E(x, \delta)} \to +\infty, \quad \delta \to 0 \ (C = C_1/a_2)$$

hence $G_Y(x, y) \to +\infty$ as $y \to x$. Gathering the information so far there is an open neighborhood $W \subseteq \tilde{\Omega}$ of q_0 such that for any $y \in W$ the function $x \mapsto G_Y(x, y)$ is strictly positive and $H_Y G(\,\cdot\,, y) = 0$ in $W \setminus \{y\}$. Also if $D = \{(x, y) \in W \times W : x = y\}$ is the diagonal then $G_Y(x, y) \to +\infty$ as $(x, y) \to D$. We may assume w.l.o.g. that $U \subseteq \phi^{-1}(W)$.

Next, we consider the sequence of functions $v_j : W \setminus \{q_j\} \to (0, +\infty)$ given by $v_j(q) = G_Y(q, q_j)$ for any $q \in W$, $q \neq q_j$. Then $H_Y(v_j) = 0$ in $W \setminus \{q_j\}$, for any $j \geq j_0$. Yet $W \setminus \{q_j\}$ is open in V. Therefore, by hypothesis, the function $u_j =$

$v_j \circ \phi \colon \phi^{-1}(W \setminus \{q_j\}) \to (0, +\infty)$ satisfies $H_X(u_j) = 0$ in $\phi^{-1}(W \setminus \{q_j\})$, and in particular in U. Let $p_0 \in U$ such that $\phi(p_0) = q_0$. Then

$$u_j(p_0) = v_j(q_0) = G_Y(q_0, q_j) \to +\infty, \quad j \to \infty,$$

hence there is a compact set $K \subset U$ such that $p_0 \in K$ and the sequence $\sup\{u_j(p) \colon p \in K\}$ is unbounded.

To end the proof of Theorem 1 we need to recall the Harnack inequality (as established by J.M. Bony, [11]). Let $\mathcal{L}(X)$ be the Lie algebra spanned by the X_i's. The *rank* of $\mathcal{L}(X)$ at a point $p \in \Omega$ is the dimension of the linear space $\{Z_p \colon Z \in \mathcal{L}(X)\}$. Let us consider the second order differential operator

$$Lu(x) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x^i} + a(x)u \tag{13}$$

satisfying the following assumptions a) $[a_{ij}(x)]$ is positive semi-definite for any $x \in \Omega$, $a(x) \leq 0$ for any $x \in \Omega$, and b) there exists a system of vector fields $\{X_1, \dots, X_m, Y\} \subset \mathcal{X}^\infty(\Omega)$ such that

$$Lu = \sum_{a=1}^{m} X_a^2 u + Yu + a(x)u. \tag{14}$$

Then

Lemma 2 (J.M. Bony, [11], p. 299). *Let us assume that the Lie algebra $\mathcal{L}(X_1, \dots, X_m)$ has rank N at any point $p \in \Omega$. Then for any compact subset $K \subset \Omega$, any point $p \in \Omega$, and any multi-index $\alpha \in \mathbb{Z}_+^N$ there is a constant $C > 0$ such that*

$$\sup\{(D^\alpha u)(x) \colon x \in K\} \leq Cu(p) \tag{15}$$

for any positive solution u to $Lu = 0$.

Under the assumptions of Lemma 2 the differential operator (13) is hypoelliptic, so one needs not specify the regularity of the solution. Let us go back to the proof of Theorem 1. By (15) (with $\alpha = 0$)

$$\sup\{u_j(x) \colon x \in K\} \leq C \inf\{u_j(x) \colon x \in K\}. \tag{16}$$

As ϕ is nonconstant $K \setminus \phi^{-1}(q_0) \neq \emptyset$. Let then $p \in K \setminus \phi^{-1}(q_0)$ and $q = \phi(p)$. We have

$$u_j(p) = v_j(q) = G_Y(q, q_j) \to G_Y(q, q_0) < \infty, \quad j \to \infty,$$

which contradicts (16). Theorem 1 is proved. A slight modification of the proof also gives

Corollary 3. *Let $\phi\colon \mathbb{H}_n \to N$ be a nonconstant subelliptic harmonic morphism from the Heisenberg group into a Riemannian manifold N of dimension $\nu \geq 2$. Then ϕ is an open mapping.*

Indeed one may replace $G_Y(x, y)$ by a fundamental solution $G(x, y)$ of the Laplace equation on N on a neighborhood $W \subseteq N$ of q_0 (so that for any $y \in W$ the function $x \mapsto G(x, y)$ is strictly positive and $\Delta_N G(\cdot, y) = 0$ in $W \setminus \{y\}$) followed by a *verbatim* repetition of the arguments in the proof of Theorem 1. Similarly we obtain

Corollary 4. *Let $\phi\colon M \to N$ be a pseudoharmonic morphism from a strictly pseudoconvex CR manifold M into a Riemannian manifold N. Then ϕ is an open mapping. Moreover if M is compact and N connected then N is compact and ϕ is surjective.*

To prove Corollary 4 we need to collect a few objects in CR and pseudohermitian geometry (cf. e.g. [16]).

Proof of Corollary 4. Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n, and $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ its Levi distribution. Let θ be a contact form on M such that the Levi form $G_\theta(X, Y) = (d\theta)(X, JY)$, $X, Y \in H(M)$, is positive definite. The horizontal gradient of a function $u \in C^1(M)$ is given by $G_\theta(\nabla^H u, X) = X(u)$ for any $X \in H(M)$. The *sublaplacian* is the second order differential operator

$$\Delta_b u = \mathrm{div}(\nabla^H u), \quad u \in C^2(M),$$

where div is the divergence with respect to the volume form $\theta \wedge (d\theta)^n$. A *pseudoharmonic morphism* is a smooth map $\phi\colon M \to N$ into a Riemannian manifold N such that $\Delta_b(v \circ \phi) = 0$ in $\phi^{-1}(V)$ for any harmonic function $v\colon V \to \mathbb{R}$ (with $V \subseteq N$ open). Let ∇ be the *Tanaka-Webster connection* of (M, θ), cf. e.g. Theorem 1.3, [16], p. 25. Let $p_0 \in M$ and let $\{X_a\colon 1 \leq a \leq 2n\}$ be a local orthonormal ($G_\theta(X_a, X_b) = \delta_{ab}$) frame of $H(M)$ defined on a connected local coordinate neighborhood $(U, x^1, \dots, x^{2n+1})$ of p_0. Let $\Gamma^a_{bc}\colon U \to \mathbb{R}$ be the C^∞ functions given by $\nabla_{X_b} X_c = \Gamma^a_{bc} X_a$. Let T be the *characteristic direction* of $d\theta$ i.e. the vector field T on M determined by $\theta(T) = 1$ and $T \,\lrcorner\, d\theta = 0$. By the *purity axiom* (cf. (1.37) in [16], p. 25)

$$[X_a, X_b] = (\Gamma^c_{ab} - \Gamma^c_{ba}) X_c - 2(d\theta)(X_a, X_b) T$$

hence $\mathcal{L}(X_1, \dots, X_{2n})$ has rank $2n + 1$. We wish to show that $\phi(U)$ is an open neighborhood of $q_0 = \phi(p_0)$. By (2.6) in [16], p. 112

$$\Delta_b u = \sum_{a=1}^{2n} X_a^2 u + Y u, \quad u \in C^2(U),$$

where $Y = -\sum_a \nabla_{X_a} X_a$. Hence the local expression of Δ_b satisfies J.M. Bony's requirements a)–b) and we may use the local chart $(x^1, \dots, x^{2n+1})$ to transplant (15) to U. In particular for any compact subset $K \subset U$ there is $C > 0$ such that

$$\sup_{x \in K} u(x) \le C u(p_0) \tag{17}$$

for any positive solution $u \in C^\infty(U)$ to $\Delta_b u = 0$. If $\phi(U)$ is not an open neighborhood of q_0 then a repetition of the arguments in the proof of Theorem 1 contradicts (17). If M is compact then $\phi(M)$ is compact, hence closed. By the first statement in Corollary 4 the set $\phi(M)$ is also open so that $\phi(M) = N$. □

Clearly Theorem 1 holds for any smooth map $\phi\colon \Omega \to N$ pulling back local harmonic functions on N to solutions of $Lu = 0$ where L is given by (14) and satisfies the assumptions in Lemma 2. The moral conclusion is that one may obtain a fairly nice theory when the target manifold is Riemannian yet additional difficulties will occur for smooth maps $\phi\colon \Omega \to N$ into a strictly pseudoconvex CR manifold N pulling back harmonics of the sublaplacian (associated to a fixed contact form on N) to solutions of L. For instance, the proof of Theorem 3 in [5], p. 36 (that whenever $\nu \le m$ a pseudoharmonic morphism is a pseudoharmonic map) requires the existence of local harmonics with a prescribed gradient and hessian at a given point. While this fact is well known in Riemannian geometry (cf. Lemma 4.1 in [25], p. 221) no pseudohermitian analog is known as yet. Note that the proof of Ishihara's lemma relies on Lemma 4.2 in [25], p. 222, in elliptic theory (while Δ_b is subelliptic). Of course the case $N = \mathbb{H}_k$ may be handled as in Theorem 1 and Corollary 4.

Corollary 5. *Let M be a strictly pseudoconvex CR manifold. Let $\phi\colon M \to \mathbb{H}_k$ be a smooth map into a Heisenberg group $\mathbb{H}_k$. If ϕ pulls back local harmonics of the Hörmander operator on $\mathbb{H}_k$ to harmonics of the sublaplacian on M then ϕ is an open mapping. The same result holds if $\mathbb{H}_k$ is replaced by $\mathbb{R}^N$ endowed with an arbitrary Hörmander system $X = (X_1, \dots, X_m)$.*

Also Corollary 5 admits a direct proof based only on the Harnack inequality (it doesn't require the results in [32] and [34]). Indeed a fundamental solution of the Hörmander operator on $\mathbb{H}_k$ is given by $G(x, y) = w(xy^{-1})$ where

$$w(x) = C|x|^{-2k} = C(\|z\|^4 + t^2)^{-k/2}$$

(cf. G.B. Folland, [17]) for any $x = (z, t) \in \mathbb{H}_k$ (where $C > 0$ is a constant depending only on k). Of course $G(x, y)$ is strictly positive and tends to $+\infty$ when (x, y) tends to the diagonal hence the proof of Corollary 5 is similar to that of Corollaries 3 and 4.

We close this section with a few potential theoretic remarks. Let M be a strictly pseudoconvex CR manifold and θ a fixed contact form on M, such that the Levi form

G_θ is positive definite. Let Δ_b be the sublaplacian of (M, θ). A subset $V \subset M$ with $\partial V \neq \emptyset$ is said to be *regular* if for any $f \in C^0(\partial V)$ there is a unique $H_f^V \in C^\infty(V)$ such that i) $\Delta_b H_f^V = 0$ in V, ii) if $\tilde{H}_f^V$ is the extension to the boundary of H_f^V with f then $\tilde{H}_f^V \in C^0(\overline{V})$ and iii) if $f \geq 0$ on ∂V then $H_f^V \geq 0$ in V.

Proposition 5. *Every strictly pseudoconvex CR manifold M endowed with a contact form θ admits a base of regular open sets. Moreover the Δ_b-harmonics possess the Harnack monotone convergence property. Therefore (M, θ) is a Brelot harmonic space.*

Proof. Let $x_0 \in M$ and $\chi = (x^1, \dots, x^{2n+1})\colon U \to \mathbb{R}^{2n+1}$ a local chart on M such that 1) $x_0 \in U$ and $\chi(x_0) = 0$, 2) $\Omega = \chi(U)$ is a domain in $\mathbb{R}^{2n+1}$, and 3) there is a local G_θ-orthonormal frame $\{X_a\colon 1 \leq a \leq 2n\}$ of $H(M)$ defined on U. If $X_a = b_a^i \, \partial/\partial x^i$ then for any $u \in C^2(U)$

$$\Delta_b u = a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + a^i \frac{\partial u}{\partial x^i}, \quad a^{ij} = \sum_{a=1}^{2n} b_a^i b_a^j, \ a^k = -a^{ij} \Gamma_{ij}^k,$$

where $\Gamma_{jk}^i \in C^\infty(U)$ are the local coefficients of the Tanaka-Webster connection of (M, θ) with respect to χ. So locally the sublaplacian is a differential operator of the form (13) (with $a(x) = 0$). Let us observe that Δ_b is non totally degenerate on U (in the sense of Definition 5.1 in [11], p. 291). Indeed if $a^{ij}(x) = 0$, $1 \leq i, j \leq 2n + 1$, at some $x \in \Omega$ then $X_a(x) = 0$, $1 \leq a \leq 2n$, a contradiction. Also $\mathcal{L}(X_1, \dots, X_{2n}, Y)$ has rank $2n + 1$ at each point of U hence (by Corollary 5.2 in [11], p. 294) there is an open neighborhood of the origin $\omega \subseteq \Omega$ with $\partial \omega \neq \emptyset$ such that the Dirichlet problem for $\chi_* \Delta_b$ (the pushforward of Δ_b by χ) is uniquely and positively solvable on ω. Then $\chi^{-1}(\omega)$ is a regular open neighborhood of x_0.

To prove the second statement in Proposition 5 let $A \subseteq M$ be an open connected set and $\{u_n\}_{n \geq 1}$, $u_n \in C^\infty(A)$ be an increasing sequence of Δ_b-harmonics. Next let $u = \sup_{n \geq 1} u_n$ and $B = \{x \in A\colon u(x) = +\infty\}$. We distinguish two cases as I) $B = \emptyset$ or II) $B \neq \emptyset$. In the first case let $x_0 \in A$ and (U, χ) a local coordinate neighborhood at x_0 as above such that $U \subseteq A$. Let $v_n = u_n \circ \chi^{-1}$, $n \geq 1$, and $v = \sup_{n \geq 1} v_n$. Then $v(0) \neq \infty$ hence (by Theorem 8.2 in [11], p. 302) $v(x) \neq \infty$ for any $x \in \Omega$ and $(\chi_* \Delta_b) v = 0$ in Ω. It follows that u is finite on U and $\Delta_b u = 0$ on U and in particular in x_0. In the second case we may show that B is both open and closed in A hence $B = A$. Let $x_0 \in B$. Then, with the notations above, $v(0) = \infty$ so that (again due to the Harnack monotone convergence property on Ω) $v \equiv \infty$ on Ω hence $U \subseteq B$ i.e. B is open. To see that B is closed let $x_0 \in A \setminus B$ and (U, χ) a local chart as before ($x_0 \in U \subseteq A$). Then $v(0) \neq \infty$ hence $v(x) \neq \infty$ for any $x \in \Omega$ so that $U \subseteq A \setminus B$. $\square$

It is an open question whether the points of a strictly pseudoconvex CR manifold M are strongly polar (in the sense of [20], p. 182). As a consequence of Proposition 5 above and Theorem 5 in [20] we may conclude that

Corollary 6. *Let M and N be two strictly pseudoconvex CR manifolds endowed with the contact forms θ and θ_N. Let $\phi\colon M \to N$ be a smooth map pulling back the local harmonics of the sublaplacian on (N, θ_N) to local harmonics of the sublaplacian on (M, θ). If* i) *all points of $\phi(M)$ are strongly polar, or* ii) *ϕ is injective, then ϕ is an open mapping.*

4. Further generalizations of harmonic morphisms

4.1. Heat equation morphisms. E. Loubeau, [30], has studied heat equation morphisms (maps preserving the local solutions of the heat equation) and heat kernel morphisms (maps preserving the heat kernel). E. Loubeau's results on heat equation morphisms carry over easily to CR geometry. The *heat equation* on M is

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)u(x, t) = 0, \quad x \in M,\ t > 0, \tag{18}$$

where Δ_b is the sublaplacian associated to θ. Then $\Psi\colon M \times (0, +\infty) \to N \times (0, +\infty)$ is a *heat equation morphism* if for any open set $V \subseteq N$ and any solution $f\colon V \times (0, +\infty) \to \mathbb{R}$ to $f_t - \Delta_N f = 0$ it follows that $u = f \circ \Psi$ is a solution to (18).

Let us assume that $\Psi(x, t) = (\phi(x), h(t))$ for some smooth maps $\phi\colon M \to N$ and $h\colon (0, +\infty) \to (0, +\infty)$. Let (V, y^i) be an arbitrary local coordinate system on N, such that $U = \phi^{-1}(V) \neq \emptyset$. Let $\{X_a : 1 \le a \le 2n\}$ be a local frame of $H(M)$ defined on U. A calculation shows that

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta_b\right)(f \circ \Psi)\\ &= (f_t \circ \Psi)h' - (f_i \circ \Psi)\Delta_b\phi^i - \sum_a (f_{ij} \circ \Psi)X_a(\phi^i)X_a(\phi^j)\end{aligned} \tag{19}$$

for any $f \in C^2(V \times (0, +\infty))$, where $\phi^i = y^i \circ \phi$, $f_i = \partial f/\partial y^i$, $f_t = \partial f/\partial t$ and $f_{ij} = \partial^2 f/(\partial y^i \partial y^j)$. Let us assume that Ψ is a heat equation morphism. In particular for $f = v(x)$, where $v\colon V \to \mathbb{R}$ is a harmonic function, the identity (19) shows that $\Delta_b(v \circ \phi) = 0$ i.e. $\phi\colon M \to N$ is a subelliptic harmonic morphism, and in particular (by Theorem 6 in Appendix B) a subelliptic harmonic map. Thus

$$\Delta_b\phi^i + \sum_a (\Gamma^i_{jk} \circ \phi)X_a(\phi^j)X_a(\phi^k) = 0$$

hence (19) becomes

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)(f \circ \Psi) = (f_t \circ \Psi)h' - (f_{i,j} \circ \Psi)\sum_a X_a(\phi^i)X_a(\phi^j) \tag{20}$$

where $f_{i,j} = f_{ij} - \Gamma^k_{ij} f_k$ is the second order covariant derivative of f. Let $x_0 \in M$ and $t_0 > 0$ be fixed. Let us choose a normal coordinate system (V, y^i) on N centered at $y_0 = \phi(x_0)$. Then (by (31) in Appendix B) for any $f \in C^2$

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)(f \circ \Psi) = (f_t \circ \Psi)h' - \lambda^2(\Delta_N f) \circ \Psi \tag{21}$$

at (x_0, t_0) and hence everywhere on $M \times (0, +\infty)$. In particular for any local solution f to the heat equation on N one has $(h' - \lambda^2) f_t \circ \Psi = 0$. If for instance $\varphi(y)$ is a solution to the Poisson equation $\Delta_N \varphi = 1$ and $f(y, t) = t + \varphi(y)$ then $h'(t) = \lambda(x)^2 =$ constant. $\square$

Let $p_t(x)$ be the fundamental solution to the heat equation on the Heisenberg group i.e. $u(x, t) = (p_t * f)(x)$ solves

$$\left(\frac{\partial}{\partial t} - H\right)u(x, t) = 0, \quad u(x, 0) = f(x), \; x \in \mathbb{H}_n,$$

for $f \in L^\infty(\mathbb{H}_n)$. See A. Hulanicki, [24], where $p_t(x)$ was explicitly computed

$$\begin{aligned}\hat{p}_t(\alpha + i\beta, s) &= (\cosh 2ts)^{-n/2} \\ &\times \exp\left\{\frac{-(1/2)(|\alpha|^2 + |\beta|^2)\cosh 2ts + \sqrt{-1}(\alpha \cdot \beta)(\sinh ts)^2}{2s \cosh 2ts}\right\}\end{aligned} \tag{22}$$

for any $(\alpha + i\beta, s) \in \mathbb{H}_n$ (a hat denotes the Fourier transform). Let us recall that a heat kernel of a connected Riemannian manifold (N, g) is a C^0 function $p^N\colon N \times N \times (0, +\infty) \to \mathbb{R}$ such that $p^N(x, y, t)$ is C^2 with respect to y, C^1 with respect to t, and

$$\left(\frac{\partial}{\partial t} - \Delta_{N,y}\right)p^N = 0,$$

$$\lim_{t \to 0^+} \int_N p^N(x, y, t)\varphi(y)\, d\,\mathrm{vol}_g(y) = \varphi(x), \quad x \in N,$$

for any bounded C^0 function φ on N. A heat kernel always exists and if N is compact it is also unique (cf. e.g. M. Berger et al., [7]).

We adopt the following definition. A *heat kernel morphism* is a C^∞ map $\Phi\colon \mathbb{H}_n \times \mathbb{H}_n \times (0, +\infty) \to N \times N \times (0, +\infty)$ such that $p = p^N \circ \Phi$ where $p(x, y, t) = p_t(xy^{-1})$. Analogous to [30], p. 491–492, we show that

Proposition 6. *Let $\Phi\colon \mathbb{H}_n \times \mathbb{H}_n \times (0, +\infty) \to N \times N \times (0, +\infty)$ be a heat kernel morphism of the form $\Phi(x, y, t) = (\phi(x), \phi(y), h(t))$ for some surjective C^∞ map $\phi\colon \mathbb{H}_n \to N$ and some C^∞ function $h(t) > 0$ for $t > 0$. Then $\Psi(x, t) = (\phi(x), h(t))$ is a heat equation morphism.*

In particular ϕ: $\mathbb{H}_n \to N$ is a subelliptic harmonic morphism of constant θ_0-dilation and $h(t)$ may be explicitly determined (as in Theorem 2). To prove Proposition 6 let $v \in C^2(N \times \mathbb{R})$ be a solution to the heat equation on N. As ϕ is surjective it admits a Borel measurable one sided inverse $\psi: N \to \mathbb{H}_n$ i.e. $\phi(\psi(y)) = y$. Moreover

$$v(x, t) = \int_N p^N(x, y, t) v(y, 0)\, d\,\mathrm{vol}_g(y)$$

hence

$$\begin{aligned}\left(\frac{\partial}{\partial t} - H\right)(v \circ \Psi)(x, t) &= \int_N \left(\frac{\partial}{\partial t} - H_x\right)[p^N(\phi(x), \phi(\psi(y)), h(t))] v(y, 0)\, d\,\mathrm{vol}_g(y) \\ &= \int_N \left(\frac{\partial}{\partial t} - H_x\right)[p(x, \psi(y), t)] v(y, 0)\, d\,\mathrm{vol}_g(y) = 0.\end{aligned}$$

Proposition 6 is proved. When N is compact and the domain M of ϕ is a compact m-dimensional Riemannian manifold E. Loubeau, [30], showed that ϕ is a covering map and the cardinality each fibre $\phi^{-1}(y)$ is λ^m, cf. Theorem 3, *op. cit.*, p. 494. The proof makes use of the Minakshisundaram-Pleijel asymptotic development of the heat kernel on a Riemannian manifold (cf. e.g. M. Berger et al., [7]). When M is a CR manifold equidimensionality is of course ruled out by Theorem 6 in Appendix B. It is an open problem whether one may exploit the asymptotic development for the heat kernel on a compact strictly pseudoconvex CR manifold (cf. R. Beals et al., [6]) or the explicit form (22) of p_t (when $M = \mathbb{H}_n$) to obtain a pseudohermitian analog to Theorem 3 in [30], p. 494.

4.2. $\Box_b$-harmonic morphisms. Let M be a nondegenerate CR manifold of CR dimension n, θ a contact form on M, and T the characteristic direction of $d\theta$. A $(0, q)$-*form* on M is a complex valued differential q-form ω such that $\mathbb{C}T \oplus T_{1,0}(M) \rfloor \omega = 0$. Let $\bigwedge^{0,q}(M) \to M$ be the bundle of all $(0, q)$-forms and $\Omega^{0,q}(M) = \Gamma^\infty\left(\bigwedge^{0,q}(M)\right)$ the space of all globally defined smooth sections in $\bigwedge^{0,q}(M)$. Let

$$\overline{\partial}_b : \Omega^{0,q}(M) \to \Omega^{0,q+1}(M), \quad q \geq 0, \tag{23}$$

be the *tangential Cauchy-Riemann operator* i.e. the first order differential operator defined as follows. If $\omega \in \Omega^{0,q}(M)$ then $\overline{\partial}_b \omega$ is the unique $(0, q+1)$-form on M coinciding with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ terms). Of course $\overline{\partial}_b : \Omega^{0,0}(M) \to \Omega^{0,1}(M)$ is given by $(\overline{\partial}_b f)\overline{Z} = \overline{Z}(f)$ for any $f \in \Omega^{0,0}(M)$ and $Z \in T_{1,0}(M)$. Note that the definition of $\overline{\partial}_b f$ makes sense for any $f \in C^1(M) \otimes \mathbb{C}$. Then

$$C^\infty(M) \otimes \mathbb{C} \xrightarrow{\overline{\partial}_b} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \Omega^{0,n}(M)$$

is a cochain complex (the *tangential Cauchy-Riemann complex* of M). Let

$$H^{0,q}(M) = H^q(\Omega^{0,\bullet}(M), \overline{\partial}_b) = \mathrm{Ker}\{\overline{\partial}_b : \Omega^{0,q}(M) \to \cdot\}\big/\overline{\partial}_b \Omega^{0,q-1}(M), \quad q \geq 1,$$

be the cohomology of the tangential Cauchy-Riemann complex (the *Kohn-Rossi cohomology* of M). Let M be a compact nondegenerate CR manifold and let $\overline{\partial}_b^*$ be the formal adjoint of (23) that is

$$(\overline{\partial}_b^* \psi, \varphi) = (\psi, \overline{\partial}_b \varphi), \quad \varphi \in \Omega^{0,q}(M), \ \psi \in \Omega^{0,q+1}(M),$$

where

$$(\alpha, \beta) = \int_M G_\theta^*(\alpha, \overline{\beta})\theta \wedge (d\theta)^n, \quad \alpha, \beta \in \Omega^{0,q}(M),$$

is a the L^2 scalar product on $(0, q)$-forms. We set

$$\Box_b \varphi = \left(\overline{\partial}_b^* \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b^*\right)\varphi, \quad \varphi \in \Omega^{0,q}(M),$$

(the *Kohn-Rossi laplacian*) and $\mathcal{H}^{0,q}(M) = \mathrm{Ker}\{\Box_b : \Omega^{0,q}(M) \to \cdot\}$ (the space of all $\Box_b$-harmonic $(0, q)$-forms on M).

Given two CR manifolds M and N endowed with the contact forms θ and θ_N a *pseudohermitian* map is a smooth CR map $\phi: M \to N$ (i.e. $(d_x\phi)T_{1,0}(M)_x \subseteq T_{1,0}(N)_{\phi(x)}$ for any $x \in M$) such that $\phi^*\theta_N = c\theta$ for some $c \in \mathbb{R} \setminus \{0\}$. Given a pseudohermitian map $\phi : M \to N$ of nondegenerate CR manifolds, by the axiomatic description of the tangential Cauchy-Riemann operator

$$\phi^* \overline{\partial}_b^N \varphi = \overline{\partial}_b(\phi^* \varphi), \quad \varphi \in \Omega^{0,q}(N), \tag{24}$$

hence there is a naturally induced linear map at the level of the Kohn-Rossi cohomology $\phi^* : H^{0,q}(N) \to H^{0,q}(M)$, $q \geq 1$.

A smooth map $\phi : M \to N$ is said to be a $\Box_b$*-harmonic morphism* if the pullback by ϕ of any local $\Box_b^N$-harmonic function on N is a local $\Box_b$-harmonic function on M. We shall prove the following

Theorem 3. *Let $\phi : M \to N$ be a pseudohermitian map of a compact strictly pseudoconvex CR manifold M into a compact strictly pseudoconvex real hypersurface $N \subset \mathbb{C}^M$. Let $\phi^* : H^{0,1}(N) \to H^{0,1}(M)$ the induced map on Kohn-Rossi cohomology. If ϕ is a submersive $\Box_b$-harmonic morphism then ϕ^* is injective.*

Theorem 3 is a pseudohermitian analog to Proposition 4.3.11 in [2]. p. 113. Here by a *submersive* map we mean a surjective smooth map $\phi : M \to N$ which is a submersion at each point $x \in M \setminus C_\phi$, where $C_\phi = \{x \in M : d_x\phi = 0\}$. It is an open problem whether this assumption might be dropped i.e. whether an analog to the Fuglede-Ishihara theorem holds for $\Box_b$-harmonic morphisms. To prove Theorem 3 we shall need

Lemma 3. *Let $\phi: M \to N$ be a pseudohermitian map of a nondegenerate CR manifold M and a nondegenerate real hypersurface $N \subset \mathbb{C}^{m+1}$ whose Levi form has at least $p_0 = 2$ positive eigenvalues. Then ϕ is a $\Box_b$-harmonic morphism if and only if ϕ pulls back the $\Box_b^N$-harmonic $(0,1)$-forms on N to $\Box_b$-harmonic $(0,1)$-forms on M i.e. $\phi^*\mathcal{H}^{0,1}(N) \subseteq \mathcal{H}^{0,1}(M)$.*

Proof. Let $\varphi \in \mathcal{H}^{0,1}(N)$ be a $\Box_b^N$-harmonic form on N. We wish to show that $\phi^*\varphi$ is $\Box_b$-harmonic. The well known identity

$$\left(\Box_b^N \varphi, \varphi\right) = \left\|\overline{\partial}_b^N \varphi\right\|^2 + \left\|\left(\overline{\partial}_b^N\right)^* \varphi\right\|^2$$

implies that φ is $\overline{\partial}_b^N$-closed. For any $x_0 \in N$ we set

$$H^{0,q}(N, x_0) = \varinjlim_{\substack{x_0 \in V \subset \mathbb{C}^{m+1} \\ V \text{ open}}} H^{0,q}(N \cap V).$$

By a result of M. Nacinovich, [31], $H^{0,q}(N, x_0) = 0$ for any $1 \leq q < p_0$ (the *Poincaré lemma* for the $\overline{\partial}_b^N$-complex, cf. Proposition 11 in [31], p. 468). Let $x_0 \in N$. Under the assumptions in Lemma 3 one has $H^{0,1}(N, x_0) = 0$. Let then $v: V \to \mathbb{C}$ a smooth function defined on an open neighborhood $V \subseteq N$ of x_0 such that $\varphi = \overline{\partial}_b^N v$ on V. As φ is also $\left(\overline{\partial}_b^N\right)^*$-closed it follows that v is a $\Box_b^N$-harmonic function. Yet ϕ is a $\Box_b$-harmonic morphism hence $v \circ \phi$ is $\Box_b$-harmonic in $\phi^{-1}(V)$. Then (by (24))

$$\overline{\partial}_b(\phi^*\varphi) = \overline{\partial}_b^2(v \circ \phi) = 0,$$
$$\overline{\partial}_b^*(\phi^*\varphi) = \overline{\partial}_b^*\overline{\partial}_b(v \circ \phi) = \Box_b(v \circ \phi) = 0,$$

i.e. $\Box_b(\phi^*\varphi) = 0$. Viceversa let $v: V \to \mathbb{C}$ be a local $\Box_b^N$-harmonic function on N. We wish to show that $v \circ \phi$ is $\Box_b$-harmonic in $U = \phi^{-1}(V)$. Let us set $\varphi = \overline{\partial}_b^N v$. Then

$$\Box_b^N \varphi = \overline{\partial}_b^N \left(\overline{\partial}_b^N\right)^* \overline{\partial}_b^N v = \overline{\partial}_b^N \Box_b^N v = 0$$

i.e. $\varphi \in \mathcal{H}^{0,1}(V)$. Yet $\phi^*\mathcal{H}^{0,1}(V) \subseteq \mathcal{H}^{0,1}(U)$ hence $\phi^*\varphi$ is $\overline{\partial}_b^*$-closed so that (by (24))

$$0 = \overline{\partial}_b^* \phi^* \varphi = \overline{\partial}_b^* \overline{\partial}_b (v \circ \phi) = \Box_b(v \circ \phi)$$

hence ϕ is a $\Box_b$-harmonic morphism. $\Box$

Proof of Theorem 3. Let $C \in H^{0,1}(N)$ such that $\phi^* C = 0$. By a result of J.J. Kohn, [28], C has a $\Box_b^N$-harmonic representative $\varphi \in \mathcal{H}^{0,1}(N)$. Yet ϕ is a $\Box_b$-harmonic morphism hence $\phi^*\varphi$ is a $\Box_b$-harmonic representative of $\phi^* C \in H^{0,1}(M)$. By the uniqueness statement in Corollary 7.8 in [28], p. 93, $\phi^* C = 0$ yields $\phi^*\varphi = 0$. As $\phi\colon M \setminus C_\phi \to N$ is a submersion $\varphi_{\phi(x)} = 0$ for any $x \in M \setminus C_\phi$. That is to say the (0, 1)-form φ vanishes at each point of N which is a regular value of ϕ. As ϕ is surjective the set of its regular values is dense in N (by Sard's theorem, cf. e.g. Theorem 3.8 in [12], p. 29) hence by continuity φ vanishes everywhere on N. $\square$

4.3. $\mathcal{L}$-harmonic morphisms. In this section we adopt the point of view in [8], pp. 113–114 (cf. also G.B. Folland, [18]). Precisely a second order partial differential operator $\mathcal{L} = \sum_{j=1}^p X_j^2$ is said to be a real *sublaplacian* on $\mathbb{R}^n$ if it satisfies the following two axioms 1) there is a group structure $\circ$ on $\mathbb{R}^n$ making $\mathbb{G} = (\mathbb{R}^n, \circ)$ into a Lie group such that each X_j is a first order differential operator with smooth real valued coefficients and X_j is left invariant, 2) the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is stratified and nilpotent i.e. there is an integer $r \geq 1$ and there are linear subspaces $V_j \subset \mathfrak{g}$, $1 \leq i \leq r$, such that $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ and i) $[V_1, V_j] = V_{j+1}$, $1 \leq j \leq r-1$, ii) $[V_j, V_r] = 0$, $1 \leq j \leq r$, and $\{X_1, \ldots, X_p\}$ is a basis of V_1 (as a real linear space). Then $\mathbb{G}$ is a *Carnot group* and the smallest integer $r \geq 1$ as above is its *step*.

By a result of L. Gallardo, [21], there is a homogeneous norm $|\cdot|$ on $\mathbb{G}$ and a constant $c_Q > 0$ such that

$$\Gamma(x, y) = c_Q |x^{-1} \circ y|^{2-Q}, \quad x, y \in \mathbb{G},$$

is a fundamental solution for $\mathcal{L}$, where $Q = \sum_{j=1}^r j m_j$ is the *homogeneous dimension* of $\mathbb{G}$ (here $m_j = \dim_{\mathbb{R}} V_j$).

We adopt the following definition. Let N be a Riemannian manifold. We say that N has the *Liouville property* if any harmonic function $f\colon N \to [0, +\infty)$ is constant. For instance any closed (i.e. compact, without boundary) Riemannian manifold has the Liouville property (as an elementary consequence of the Hopf maximum principle). Also, if N is a complete Riemannian manifold of nonnegative Ricci curvature then any bounded harmonic function on N is a constant (cf. S.-T. Yau, [38]). An extension of the Liouville property to the case of the Hörmander operator on the Heisenberg group was proved by A. Korányi and N.K. Stanton, [29].

Let $\mathcal{L}$ be a real sublaplacian as above. Recently A. Bonfiglioli et al., [8], have shown that for any $p \in (Q/2, +\infty]$ there exist constants $C > 0$ and $\theta > 0$ (depending only on $\mathcal{L}$ and p) such that

$$\sup_{|x| \leq r} u(x) \leq C \left\{ \inf_{|x| \leq r} u(x) + r^{2-Q/p} \|\mathcal{L} u\|_{L^p(D(0, \theta r))} \right\} \tag{25}$$

for any C^2 function $u\colon \mathbb{R}^n \to [0, +\infty)$ and any $r > 0$. Here $D(x, r) = \{y \in \mathbb{R}^n \colon |x^{-1} \circ y| \leq r\}$. The (Harnack type) inequality (25) is easily seen to imply an extension of the Liouville property to $\mathbb{G}$, relative to $\mathcal{L}$-harmonics (cf. [8], p. 112). It is a natural question whether the Liouville property pushes forward via a harmonic morphism.

A continuous map $\phi\colon \mathbb{G} \to N$ of a Carnot group $\mathbb{G}$ as above into a Riemannian manifold N is said to be a *$\mathcal{L}$-harmonic morphism* if for any local harmonic function $v\colon V \subseteq N \to \mathbb{R}$ with $\phi^{-1}(V) \neq \emptyset$ one has $\mathcal{L}(v \circ \phi) = 0$ in $\phi^{-1}(V)$ (in distributional sense). As $|\cdot| \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C^0(\mathbb{R}^n)$ it follows that $\Gamma(x, y)$ is C^∞ away from the diagonal, hence $\mathcal{L}$ is hypoelliptic. This fact together with the existence of harmonic local coordinates on N implies that any $\mathcal{L}$-harmonic morphism is actually smooth. We shall show that

Theorem 4. *Let $\mathcal{L}$ be a real sublaplacian on $\mathbb{R}^n$ and N a Riemannian manifold. Let $\phi\colon \mathbb{G} \to N$ be a nonconstant $\mathcal{L}$-harmonic morphism. Then its image $A = \phi(\mathbb{R}^n)$ is an open set and any harmonic function $v\colon A \to [0, +\infty)$ is a constant.*

Proof. Let $x_0 \in \mathbb{G}$ and $y_0 = \phi(x_0)$. We shall show that $y_0 \in \mathring{A}$. The proof is by contradiction. If $y_0 \in A \setminus \mathring{A}$ there is a sequence $y_k \in N \setminus A$ such that $y_k \to y_0$ as $k \to \infty$. Let G be a positive Green function on a neighborhood V of y_0 in N. Thus for each $y \in V$ the function $x \mapsto G(x, y)$ is strictly positive and Δ_N-harmonic in $V \setminus \{y\}$. Also $G(x, y) \to \infty$ as (x, y) goes to the diagonal. Let $\Omega \subseteq \mathbb{R}^n$ be a domain such that $x_0 \in \Omega \subseteq \phi^{-1}(V)$. The functions $f_k(x) = G(x, y_k)$ are strictly positive and Δ_N-harmonic in $V \setminus \{y_k\}$. Note that $\phi(\Omega) \subseteq V \setminus \{y_k\}$ for any $k \in \{1, 2, \dots\}$. As ϕ is a $\mathcal{L}$-harmonic morphism the functions $H_k(x) = f_k \circ \phi$ are strictly positive and $\mathcal{L}$-harmonic in $\phi^{-1}(V \setminus \{y_k\})$ and in particular in Ω. Moreover

$$H_k(x_0) = f_k(y_0) = G(y_0, y_k) \to \infty, \quad k \to \infty.$$

Note that $0^{-1} = 0 \in \mathbb{G}$ as a consequence of the existence of dilations $\delta_a\colon \mathbb{G} \to \mathbb{G}$ $(a > 0)$ cf. [8], p. 114. Then $L_x(D(0,r)) = D(x,r)$ where L_x is the left translation $L_x(y) = x \circ y$, for any $x, y \in \mathbb{G}$. Let us apply the estimate (25) for $u = v \circ L_{x_0}^{-1}$ so that to get (by the left invariance of $\mathcal{L}$ and a change of variable under the integral sign)

$$\sup_{y \in D(x_0,r)} v(y) \leq C \left\{ \inf_{y \in D(x_0,r)} v(y) + r^{2-Q/p} \left(\int_{D(x_0,\theta r)} |\mathcal{L} v(z)|^p \, |J_{x_0}(z)| \, dz \right)^{1/p} \right\} \tag{26}$$

where J_x is the determinant of the Jacobian of L_x^{-1}. In particular let $v = H_k$ and let $r > 0$ such that $D(x_0, R) \subseteq \Omega$ where $R = \max\{r, \theta r\}$ (so that the integral in (26) vanishes). Let K be a compact set such that $x_0 \in K \subset D(x_0, r)$ and $\mathring{K} \neq \emptyset$. As ϕ is nonconstant there is $a \in K \setminus \phi^{-1}(y_0)$ so that

$$H_k(a) = f_k(\phi(a)) = G(\phi(a), y_k) \to G(\phi(a), y_0) < \infty$$

as $k \to \infty$. Finally

$$\begin{aligned}\sup\{H_k(x)\colon x \in K\} &\leq \sup_{x\in D(x_0,r)} H_k(x)\\ &\leq C \inf_{x\in D(x_0,r)} H_k(x) \leq C \inf\{H_k(x)\colon x \in K\}\\ &\leq C H_k(a) < \infty\end{aligned}$$

a contradiction. Thus A is an open set. Let $f \in C^\infty(A)$ such that $f \geq 0$ and $\Delta_N f = 0$ in A. We set $v = f - \inf_{y\in A} f(y)$ so that $u = v \circ \phi$ is nonnegative and $\mathcal{L}$-harmonic in $\mathbb{R}^n$. Then $\sup_A v = \sup_{\mathbb{R}^n} u \leq C \inf_{\mathbb{R}^n} u = C \inf_A v = 0$ hence f is constant on A. As a consequence of Theorem 4 □

Corollary 7. *Let N be a Riemannian manifold. If there is a surjective $\mathcal{L}$-harmonic morphism from a Carnot group into N then N has the Liouville property.*

4.4. CR-pluriharmonic morphisms. Let M be a CR manifold. A C^1 function $u\colon M \to \mathbb{R}$ is *CR-pluriharmonic* if for any $x \in M$ there is an open neighborhood U of x in M and a C^1 function $v\colon U \to \mathbb{R}$ such that $\overline{\partial}_b(u + \sqrt{-1}v) = 0$ in U i.e. $u + \sqrt{-1}v$ is a CR function. As CR functions may be thought of as boundary values of holomorphic functions, it is natural to think of CR-pluriharmonic functions as boundary values of pluriharmonic functions. The former are also several complex variables analogs of harmonic functions. This prompts the following natural generalization of harmonic morphisms. Let N be a CR manifold. Let $\mathcal{PM}(N)$ be the class of all C^0 maps $\phi\colon U \subseteq \mathbb{H}_n \to N$ (U open) such that $u \circ \phi$ is a weak CR-pluriharmonic function, for any CR-pluriharmonic function $u\colon V \to \mathbb{R}$ with $V \subseteq N$ open and $\phi^{-1}(V) \neq \emptyset$. A function $f \in L^1_{\rm loc}(\mathbb{H}_n)$ is a weak solution to the tangential Cauchy-Riemann equations $\overline{\partial}_b f = 0$ (a *weak CR function*) if

$$\int_{\mathbb{H}_n} f(x)(L_\alpha\varphi)(x)\, dx = 0, \quad 1 \leq \alpha \leq n,$$

for any $\varphi \in C_0^\infty(\mathbb{H}_n)$. A *weak* CR-pluriharmonic function is locally the real part of a weak CR function. The properties of the class $\mathcal{PM}(N)$ are unknown as yet. Of course one may replace $\mathbb{H}_n$ in the above definition by just any CR manifold.

Appendix A. Subelliptic harmonic maps

Let $\Omega \subset \mathbb{R}^N$ be a domain and $X = (X_1, \ldots, X_m)$ a Hörmander system defined on an open neighborhood of $\overline{\Omega}$. A *subelliptic harmonic map* is a smooth solution $\phi\colon \Omega \to N$ to

$$H_X\phi^i + \sum_{a=1}^m (\Gamma^i_{jk} \circ \phi) X_a(\phi^j) X_a(\phi^k) = 0, \quad 1 \leq i \leq \nu. \tag{27}$$

Here N is a ν-dimensional Riemannian manifold. Also if (V, y^i) is a local coordinate system on N (such that $\phi^{-1}(V) \neq \emptyset$) then $\phi^i = y^i \circ \phi$ and Γ^i_{jk} are the local coefficients of the Levi-Civita connection on N. The system (27) (the *subelliptic harmonic map system*, written briefly $H_N\phi = 0$) is a nonlinear subelliptic system of variational origin. Indeed (27) are the Euler-Lagrange equations of the variational principle $\delta E_X(\phi) = 0$ where

$$E_X(\phi) = \frac{1}{2}\int_\Omega \sum_{a=1}^m (g_{ij} \circ \phi) X_a(\phi^i) X_a(\phi^j)\, dx$$

and dx is the Lebesgue measure on $\mathbb{R}^N$. Also g_{ij} are the local components of the Riemannian metric g on N. Although the equations (27) are nonlinear an appropriate notion of weak solution is available. The relevant function spaces for subelliptic variational problems are $W^{1,2}(\Omega, X) = \{u \in L^2(\Omega)\colon X_a u \in L^2(\Omega),\ 1 \leq a \leq m\}$ (the derivatives $X_a u$ are meant in distributional sense) with the norm $\|u\|_{W^{1,2}} = \left(\|u\|^2_{L^2} + \sum_{a=1}^{2n} \|X_a u\|^2_{L^2}\right)^{1/2}$, cf. e.g. C.-J. Xu, [36]. Let $W^{1,2}_0(\Omega, X)$ be the completion of $C^\infty_0(\Omega)$ with respect to $\|\cdot\|_{W^{1,2}}$. Let us assume that N may be covered by one coordinate chart $\chi = (y^1, \ldots, y^\nu)\colon N \to \mathbb{R}^\nu$. Then $W^{1,2}_X(\Omega, N)$ consists of all maps $\phi\colon \Omega \to N$ such that $\phi^i \in W^{1,2}(\Omega, X)$ for any $1 \leq i \leq \nu$. A *weak solution* to (27) is a map $\phi \in W^{1,2}_X(\Omega, N)$ such that

$$\sum_{a=1}^m \int_\Omega \{X_a(\phi^i) X_a(\varphi) + (\Gamma^i_{jk} \circ \phi) X_a(\phi^j) X_a(\phi^k)\varphi\}\, dx = 0$$

forany $\varphi \in C^\infty_0(\Omega)$. Given a domain $\omega \subset \mathbb{R}^N$ such that $\overline{\omega} \subset \Omega$ J. Jost and C.-J. Xu, [27], considered the Dirichlet problem

$$H_N\phi = 0 \quad \text{in} \quad \omega, \quad \phi = f \quad \text{on} \quad \partial\omega, \tag{28}$$

with $f \in C^0(\overline{\omega}, N) \cap W^{1,2}_X(\omega, N)$, such that $f(\overline{\omega}) \subseteq B(p, \mu)$ for some regular ball $B(p, \mu) \subset N$, and exploited the variational origin of the system (27) in order to prove the existence of weak solutions to (28) i.e. weak solutions ϕ to $H_N\phi = 0$ such that

$$\phi - f \in W^{1,2}_{X,0}(\omega, N), \quad \phi(\overline{\omega}) \subseteq B(p, \mu).$$

Moreover any bounded weak solution ϕ to $H_N\phi = 0$ such that $\phi(\overline{\omega}) \subseteq B(p, \mu)$ (for some regular ball $B(p, \mu) \subset N$) may be shown (again cf. [27]) to be continuous in ω. On the other hand, by a result of C.-J. Xu and C. Zuily, [37], interior continuity $\phi \in C^0(\omega)$ of weak solutions to a class of quasi-linear subelliptic systems (including (27)) implies smoothness $\phi \in C^\infty(\omega)$, thus settling the problem of the existence of subelliptic harmonic maps. See also Z.-R. Zhou, [39], P. Hájlasz and P. Strzelecki, [22]. Subelliptic harmonic maps turn out to be local manifestations of *pseudoharmonic* *maps* i.e. smooth critical points $\phi\colon M \to N$ of the functional

$$E(\phi) = \frac{1}{2}\int_M \operatorname{trace}_{G_\theta}(\phi^* g)\theta \wedge (d\theta)^n$$

where M is a compact strictly pseudoconvex CR manifold of CR dimension n and θ is a contact form on M, cf. E. Barletta et al., [3]. To simplify terminology these are referred to as subelliptic harmonic maps, as well.

We end Appendix A with a remark on the unique continuation principle for harmonic maps, due to J.H. Sampson, [33], that two harmonic maps coinciding on an open subset must coincide everywhere. We conjecture that given two subelliptic harmonic maps $\phi, \psi\colon M \to N$, from a strictly pseudoconvex CR manifold M into a Riemannian manifold N, if there is a nonempty open subset $A \subseteq M$ such that $\phi(x) = \psi(x)$ for any $x \in A$ then $\phi(x) = \psi(x)$ for any $x \in M$. The proof of J.H. Sampson's result (cf. *op. cit.*) relies on a result of N. Aronszajn, [1], that solutions to an elliptic inequality

$$|Au(x)|^2 \leq M\left\{\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x^i}(x)\right|^2 + |u(x)|^2\right\} \tag{29}$$

in a domain $\Omega \subset \mathbb{R}^n$ vanish identically provided they have a zero $x_0 \in \Omega$ of infinite order in the 1-mean. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and A an elliptic operator of the form

$$Au = \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^i}\left(\sqrt{a}\,a^{ij}\frac{\partial u}{\partial x^j}\right),$$

with $a = \det[a_{ij}]$ and $[a_{ij}] = [a^{ij}]^{-1}$. Here $a^{ij} \in C^{2,1}(\Omega)$ i.e. the coefficients a^{ij} are of class C^2 with second derivatives Lipschitzian, and $[a^{ij}(x)]$ is positive definite at each $x \in \Omega$. Let $r(x)$ be the geodesic distance from a point $x_0 \in \Omega$ associated to the Riemannian metric $g = a_{ij}\,dx^i \odot dx^j$ on Ω. A crucial ingredient in the proof of N. Aronszajn's result (cf. [1], p. 237) is to consider the conformally equivalent metric $\tilde{g} = e^{-2\nu r^2} g$ and relate the new geodesic distance function $\tilde{r}(x)$ to $r(x)$. By classical results in Riemannian geometry (cf. e.g. G. de Rham, [15], p. 134) there is a continuous function $\rho\colon \Omega \to (0, +\infty)$ such that the exponential mapping $\exp_{x_0}\colon B(\rho(x_0)) \to \Omega$ (associated to g) is a diffeomorphism of $B(\rho(x_0))$ onto its image U_{x_0}. Here $B(\rho) = \{w \in T_{x_0}(\Omega)\colon \|w\| < \rho\}$, $\|w\|^2 = g_{x_0}(w, w)$. Let $x \in U_{x_0}$ and let $\gamma\colon [0,1] \to U_{x_0}$ be the unique geodesic of g of initial conditions $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v$ where $\exp_{x_0} v = x$. If γ where a geodesic of $\tilde{g}$ as well then

$$\tilde{r}(x) = \int_0^1 \tilde{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\,dt = \int_0^1 e^{-\nu r(\gamma(t))^2}\|v\|\,dt = \int_0^{r(x)} e^{-\nu\sigma^2}\,d\sigma$$

which is the identity (2.3) in [1], p. 237. Indeed it is claimed there that g and $\tilde{g}$ have the same geodesics issuing at x_0. However the claim turns out to be false, as easily shown by the following simple example. Let $n = 2$ and $a_{ij} = \delta_{ij}$ the Euclidean metric. Let us consider the conformally equivalent metric $\tilde{a}_{ij} = e^{-(x^2+y^2)}\delta_{ij}$. The equations of geodesics of the new metric are

$$\frac{d^2z}{dt^2} - \overline{z(t)}\left(\frac{dz}{dt}\right)^2 = 0 \tag{30}$$

where $z = x + iy \in \mathbb{C}$ and $z(t) = \zeta t$ is a solution to (30) if and only if $\zeta = 0$. In general we may show

Theorem 5. *Let $\gamma : [0, 1] \to U_{x_0}$ be a geodesic of g issuing at x_0. Then γ is a geodesic of $\tilde{g} = e^{-2\nu r^2} g$ if and only if γ is constant.*

Proof. That is g and $\tilde{g}$ have no common geodesics issuing at x_0 except for points. To prove Theorem 5 let $(x^1, \dots, x^n)$ be the Cartesian coordinates on Ω and $y^i : T(\Omega) \to \mathbb{R}$ the induced fibre coordinates i.e. $v = y^i(v)(\partial/\partial x^i)_{x_0}$ for any $v \in T_{x_0}(\Omega)$. The geodesic distance from x_0 to $x \in U_{x_0}$ is given by

$$r(x) = (a_{ij}(x_0)y^i(v)y^j(v))^{1/2}, \quad v = \exp_{x_0}^{-1}(x).$$

Let $\varphi = (u^1, \dots, u^n) : U_{x_0} \to \mathbb{R}^n$ be normal coordinates at x_0 i.e. $\varphi(x) = y^i(\exp_{x_0}^{-1}(x))e_i$ where $\{e_1, \dots, e_n\}$ is the canonical linear basis in $\mathbb{R}^n$. Then the local expression of r with respect to φ is $(r \circ \varphi^{-1})(\xi) = (g_{ij}(x_0)\xi^i\xi^j)^{1/2}$ for any $\xi \in \varphi(U_{x_0})$. The Christoffel symbols of g and $\tilde{g}$ are related by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} - 2\nu r(\delta^i_j r_k + \delta^i_k r_j - g_{jk} r^i)$$

where $r_i = \partial r/\partial u^i$ and $r^i = g^{ij} r_j$. If $\gamma(t)$ is a geodesic of g of initial conditions (x_0, v) then $\gamma^i(t) = y^i(v)t$ in normal coordinates. Then $r_i(\gamma(t)) = \|v\|^{-1} a_{ij}(x_0) y^j(v)$ so that

$$\begin{aligned}\frac{d^2\gamma^i}{dt^2} + \tilde{\Gamma}^i_{jk}(\gamma(t))\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} &= -2\nu r(\gamma(t))\left\{2\frac{d\gamma^i}{dt} r_j(\gamma(t))\frac{d\gamma^j}{dt} - \|\dot{\gamma}(t)\|^2 r^i(\gamma(t))\right\}\\ &= -2\nu r(\gamma(t))\|v\| y^j(v)\{2\delta^i_j - g^{ik}(\gamma(t))a_{jk}(x_0)\}\end{aligned}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $\partial_i = \partial/\partial u^i$. Let us assume that γ is a geodesic of $\tilde{g}$ as well. Then $y^j(v)\{2\delta^i_j - g^{ik}(\gamma(t))a_{jk}(x_0)\} = 0$ and contraction with $g_{il}(\gamma(t))y^l(v)$ gives $\|v\| = 0$. □

Appendix B. A theorem of E. Barletta

The scope of this appendix is to restate a result by E. Barletta, [5], and add a few elementary consequences.

Theorem 6. *Let M be a connected strictly pseudoconvex CR manifold, of CR dimension n, endowed with a contact form θ such that the Levi form G_θ is positive definite. Let N be a ν-dimensional Riemannian manifold.* i) *Any pseudoharmonic morphism is a subelliptic harmonic map and there is a C^0 function*[6] $\lambda\colon M \to [0, +\infty)$ *such that λ^2 is C^∞ and*

$$G_\theta(\nabla^H \phi^i, \nabla^H \phi^j)_x = \lambda(x)^2 \delta^{ij}, \quad 1 \le i, j \le \nu, \tag{31}$$

for any $x \in M$ and any local system of normal coordinates (V, y^i) on N at x (here $\phi^i = y^i \circ \phi$). ii) *Viceversa, any subelliptic harmonic map $\phi\colon M \to N$ satisfying* (31) *is a pseudoharmonic morphism.* iii) *As a consequence of* (31) *if $\nu > 2n$ then there are no nonconstant pseudoharmonic morphisms from M into N while if $\nu \le 2n$ then for any $x \in M$ such that $\lambda(x) \ne 0$ there is an open neighborhood $U \subseteq M$ such that $\phi\colon U \to N$ is a submersion.* iv) *For any pseudoharmonic morphism $\phi\colon M \to N$ and any $f \in C^2(N)$*

$$\Delta_b(f \circ \phi) = \lambda^2 (\Delta_N f) \circ \phi \tag{32}$$

where Δ_N is the Laplace-Beltrami operator on N.

We take the opportunity to correct a missprint[7] in [5]. Also the fact that the converse holds (cf. the second statement in Theorem 6 above) is not emphasized (in Theorem 1 of [5], p. 36). We may state

Lemma 4. *Let $\phi\colon M \to N$ be a smooth map and (V, y^i) a local coordinate system on N such that $\phi^{-1}(V) \ne \emptyset$. Then*

$$\begin{aligned} \Delta_b(v \circ \phi) = (v_i \circ \phi)\left\{ \Delta_b \phi^i + \sum_a (\Gamma^i_{jk} \circ \phi) X_a(\phi^j) X_a(\phi^k) \right\} \\ + \sum_a (v_{i,j} \circ \phi) X_a(\phi^i) X_a(\phi^j) \end{aligned} \tag{33}$$

for any C^2 function $v\colon V \to \mathbb{R}$ and any local orthonormal ($G_\theta(X_a, X_b) = \delta_{ab}$) frame $\{X_a\colon 1 \le a \le 2n\}$ of $H(M)$ on $\phi^{-1}(V)$. Here Γ^i_{jk} are the local coefficients of the Levi-Civita connection of N. Also $v_i = \partial v/\partial y^i$, $v_{i,j} = v_{ij} - \Gamma^k_{ij} v_k$ and $v_{ij} = \partial^2 v / \partial y^i \partial y^j$.

Lemma 5. *Let $C_i, C_{ij} \in \mathbb{R}$, $1 \le i, j \le \nu$, such that $C_{ij} = C_{ji}$ and $\sum_{i=1}^{\nu} C_{ii} = 0$. Let $y_0 \in N$ and let (V, y^i) be a local system of normal coordinates on N at y_0 such*

[6] In the notations of [5], p. 36 and p. 46, the θ-dilation of ϕ is $\sqrt{\lambda}$.

[7] In [5], p. 36, the dimension ν is compared to the CR dimension n rather than the rank $2n$ of the Levi distribution.

that $y^i(y_0) = 0$. *Then there is a harmonic function* $v\colon V \to \mathbb{R}$ *such that*

$$v_i(y_0) = C_i, \quad v_{ij}(y_0) = C_{ij}, \quad 1 \le i,\, j \le \nu.$$

The proof of Lemma 4 is a straightforward calculation. Lemma 5 is due to T. Ishihara, [25]. Let us prove Theorem 6. Let $i_0 \in \{1, \dots, \nu\}$ be a fixed index. Let us fix a point $x_0 \in M$, choose a normal coordinate system on N at $y_0 = \phi(x_0)$ and the constants $C_i = \delta_{ii_0}$ and $C_{ij} = 0$, and apply Lemma 5 to produce a harmonic function $v\colon V \to \mathbb{R}$ such that $v_i(y_0) = \delta_{ii_0}$ and $v_{ij}(y_0) = 0$. As the Levi-Civita connection is torsion free one has $\Gamma^i_{jk}(x_0) = 0$ hence (31) in Lemma 4 together with $\Delta_b(v \circ \phi)_{x_0} = 0$ gives

$$\left(\Delta_b \phi^{i_0} + \sum_a (\Gamma^{i_0}_{jk} \circ \phi) X_a(\phi^j) X_a(\phi^k)\right)_{x_0} = 0$$

i.e. ϕ is a subelliptic harmonic map. To prove (31) we need the following

Lemma 6. *Let us consider the* C^∞ *functions* $X^{ij}\colon V \to \mathbb{R}$ *given by* $X^{ij} = \sum_{a=1}^{2n} X_a(\phi^i) X_a(\phi^j)$ *for* $1 \le i,\, j \le \nu$. *Then*

$$X^{ij}(x_0) = X^{11}(x_0)\delta^{ij}. \tag{34}$$

Moreover there is a C^0 *function* $\lambda\colon M \to [0, +\infty)$ *such that* $\lambda^2|_V = X^{11}$.

The function λ furnished by Lemma 6 is called the *θ-dilation* of the pseudo-harmopnic morphism ϕ.

Proof of Lemma 6. Let us choose the constants $C_{ij} \in \mathbb{R}$ such that $C_{ij} = C_{ji}$ and $\sum_i C_{ii} = 0$ and apply Lemma 5 to produce a harmonic function $v\colon V \to \mathbb{R}$ such that $v_i(y_0) = 0$ and $v_{ij}(y_0) = C_{ij}$. Then (by (33))

$$\sum_a C_{ij} X_a(\phi^i)_{x_0} X_a(\phi^j)_{x_0} = 0$$

which may be written as

$$\sum_{i \ne j} C_{ij} X^{ij}(x_0) + \sum_i C_{ii}\{X^{ii}(x_0) - X^{11}(x_0)\} = 0. \tag{35}$$

Let $i_0 \in \{2, \dots, \nu\}$ be a fixed index and choose the constants $C_{ij} \in \mathbb{R}$ as

$$i \ne j \implies C_{ij} = 0, \quad C_{ii} = \begin{cases} 1, & i = i_0, \\ -1, & i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then (by (35))

$$X^{i_0 i_0}(x_0) - X^{11}(x_0) = 0$$

i.e. $X^{11}(x_0) = \cdots = X^{\nu\nu}(x_0)$ and (35) may be written

$$\sum_{i \neq j} C_{ij} X^{ij}(x_0) = 0.$$

At this point we fix two indices $i_0, j_0 \in \{1, \ldots, \nu\}$, $i_0 \neq j_0$, and choose the constants

$$C_{ij} = \begin{cases} 1, & i = i_0 \quad \text{and} \quad j = j_0, \\ 0, & \text{otherwise,} \end{cases}$$

so that to obtain $X^{i_0 j_0}(x_0) = 0$. Summing up, we proved that $X^{ij}(x_0) = X^{11}(x_0)\delta^{ij}$. Next let us set $\lambda_V^2 = X_{11} = \sum_a X_a(\phi^1)^2$. Then $\lambda_V \in C^0(V)$ and $\lambda_V^2 \in C^\infty(V)$. Contraction of i and j in (34) leads to

$$\nu\lambda_V = \sum_{a,i} X_a(\phi^i)^2.$$

If (V', y'^i) is another normal coordinate system centered at $y_0 = \phi(x_0)$ then the local coordinate transformation is an orthogonal transformation

$$y'^i = a^i_j y^j, \quad 1 \leq i \leq \nu, \ ([a^i_j] \in \mathrm{O}(\nu))$$

hence $\sum_a X_{a,i}(\phi'^i)^2 = \sum_{a,i} X_a(\phi^i)^2$ on $V \cap V'$ i.e. the functions λ_V glue up to a (globally defined) continuous function $\lambda\colon M \to [0, +\infty)$ such that $\lambda|_V = \lambda_V$. Lemma 6 is proved. $\square$

Clearly (34) may be written as (31). Let $x_0 \in M$ be an arbitrary point and let (V, y^i) be a normal coordinate neighborhood on N, centered at $y_0 = \phi(x_0)$. Let $\{X_a\colon 1 \leq a \leq 2n\}$ be a local orthonormal framme of $H(M)$ on $\phi^{-1}(V)$. Let us consider the vectors

$$\xi^i = (X_1(\phi^i)_{x_0}, \ldots, X_{2n}(\phi^i)_{x_0}) \in \mathbb{R}^{2n}, \quad 1 \leq i \leq \nu.$$

Then (by Lemma 6)

$$\xi^i \cdot \xi^j = \lambda(x_0)^2 \delta^{ij}$$

where the dot denotes the Euclidean inner product on $\mathbb{R}^{2n}$ i.e. the vectors ξ^i, $1 \leq i \leq \nu$, are mutually orthogonal. To complete the proof of Theorem 6 let us assume that $\nu > 2n$. It follows that $\xi^{i_0} = 0$ for some $i_0 \in \{1, \ldots, \nu\}$ hence $\lambda(x_0) = 0$. This yields (by (31)) $(\nabla^H \phi^i)(x_0) = 0$ and in particular each ϕ^i is a real valued CR function on $\phi^{-1}(V)$ hence ϕ^i = constant (as M is nondegenerate).

Let us assume that $\nu \leq 2n$ and let $x_0 \in M$ such that $\lambda(x_0) \neq 0$. Then the vectors $\{\xi^i \colon 1 \leq i \leq \nu\}$ are linearly independent hence $\mathrm{rank}((\xi^1)^t, \dots, (\xi^\nu)^t) = \nu$ and in particular $\mathrm{rank}(d_{x_0}\phi) = \nu$ i.e. ϕ is a submersion on some neighborhood of x_0.

Let $\phi\colon M \to N$ be a a subelliptic harmonic map (a pseudoharmonic map, according to the terminology in [3]) satisfying (31). Then (by (33))

$$\Delta_b(v \circ \phi)_{x_0} = \sum_a v_{ij}(y_0) X_a(\phi^i)_{x_0} X_a(\phi^j)_{x_0} = \lambda(x_0)^2 \sum_i v_{ii}(y_0)$$

for any $v \in C^2(V)$. Also $\Delta_N v = h^{ij}(v_{ij} - \Gamma^k_{ij} v_k)$ (where h^{ij} are the local coefficients of the metric tensor on N) hence $\sum_i v_{ii}(y_0) = (\Delta_N v)(y_0)$. This proves (32). Finally if $v\colon V \to \mathbb{R}$ is a harmonic function then $\Delta_b(v \circ \phi)_{x_0} = 0$ i.e. ϕ is a subelliptic harmonic morphism.

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