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SPECTRA AND SYMMETRIC EIGENTENSORS OF THE LICHNEROWICZ LAPLACIAN ON S^n

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Abstract

We compute the eigenvalues with multiplicities of the Lichnerowicz Laplacian acting on the space of symmetric covariant tensor fields on the Euclidian sphere S^n. The spaces of symmetric eigentensors are explicitly given.

1. Introduction

Let (M, g) be a Riemannian n-manifold. For any $p \in \mathbb{N}$, we shall denote by $\Gamma\left(\bigotimes^p T^*M\right)$, $\Omega^p(M)$ and $\mathcal{S}^pM$ the space of covariant p-tensor fields on M, the space of differential p-forms on M and the space of symmetric covariant p-tensor fields on M, respectively. Note that $\Gamma\left(\bigotimes^0 T^*M\right) = \Omega^0(M) = \mathcal{S}^0M = C^\infty(M, \mathbb{R})$, $\Omega(M) = \sum_{p=0}^n \Omega^p(M)$ and $\mathcal{S}(M) = \sum_{p\geq 0} \mathcal{S}^p(M)$.

Let D be the Levi-Civita connection associated to g; its curvature tensor field R is given by

$$R(X, Y)Z = D_{[X,Y]}Z - (D_X D_Y Z - D_Y D_X Z),$$

and the Ricci endomorphism field $r\colon TM \to TM$ is given by

$$g(r(X), Y) = \sum_{i=1}^n g(R(X, E_i)Y, E_i),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

For any $p \in \mathbb{N}$, the connection D induces a differential operator $D\colon \Gamma\left(\bigotimes^p T^*M\right) \to \Gamma\left(\bigotimes^{p+1} T^*M\right)$ given by

$$\begin{aligned} DT(X, Y_1, \dots, Y_p) &= D_X T(Y_1, \dots, Y_p) \\ &= X.T(Y_1, \dots, Y_p) - \sum_{j=1}^p T(Y_1, \dots, D_X Y_j, \dots, Y_p). \end{aligned}$$

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Its formal adjoint $D^*\colon \Gamma\big(\bigotimes^{p+1} T^*M\big) \to \Gamma\big(\bigotimes^{p} T^*M\big)$ is given by

$$D^*T(Y_1, \dots, Y_p) = -\sum_{j=1}^{n} D_{E_i} T(E_i, Y_1, \dots, Y_p),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

Recall that, for any differential p-form α, we have

$$d\alpha(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} D_{X_j} \alpha(X_1, \dots, \hat{X}_j, \dots, X_{p+1}). \tag{1}$$

We denote by δ the restriction of D^* to $\Omega(M) \oplus \mathcal{S}(M)$ and we define $\delta^*\colon \mathcal{S}^p(M) \to \mathcal{S}^{p+1}(M)$ by

$$\delta^*T(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} D_{X_j} T(X_1, \dots, \hat{X}_j, \dots, X_{p+1}).$$

Recall that the operator trace $\mathrm{Tr}\colon \mathcal{S}^p(M) \to \mathcal{S}^{p-2}(M)$ is given by

$$\mathrm{Tr}\, T(X_1, \dots, X_{p-2}) = \sum_{j=1}^{n} T(E_j, E_j, X_1, \dots, X_{p-2}),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

The Lichnerowicz Laplacian is the second order differential operator

$$\Delta_M\colon \Gamma\left(\bigotimes^{p} T^*M\right) \to \Gamma\left(\bigotimes^{p} T^*M\right)$$

given by

$$\Delta_M(T) = D^*D(T) + R(T),$$

where $R(T)$ is the curvature operator given by

$$\begin{aligned} R(T)(Y_1, \dots, Y_p) = {} & \sum_{j=1}^{p} T(Y_1, \dots, r(Y_j), \dots, Y_p) \\ & - \sum_{i<j} \sum_{l=1}^{n} \{T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p) \\ & \qquad + T(Y_1, \dots, R(Y_j, E_l)Y_i, \dots, E_l, \dots, Y_p)\}, \end{aligned}$$

where $(E_1, \ldots, E_n)$ is any local orthonormal frame and, in

$$T(Y_1, \ldots, E_l, \ldots, R(Y_i, E_l)Y_j, \ldots, Y_p),$$

E_l takes the place of Y_i and $R(Y_i, E_l)Y_j$ takes the place of Y_j.

This differential operator, introduced by Lichnerowicz in [15] pp. 26, is self-adjoint, elliptic and respects the symmetries of tensor fields. In particular, Δ_M leaves invariant $\mathcal{S}(M)$ and the restriction of Δ_M to $\Omega(M)$ coincides with the Hodge-de Rham Laplacian, i.e., for any differential p-form α,

$$\Delta_M \alpha = (d\delta + \delta d)(\alpha). \tag{2}$$

We have shown in [6] that, for any symmetric covariant tensor field T,

$$\Delta_M(T) = (\delta \circ \delta^* - \delta^* \delta)(T) + 2R(T). \tag{3}$$

Note that if $T \in \mathcal{S}(M)$ and g^l denotes the symmetric product of l copies of the Riemannian metric g, we have

$$(\mathrm{Tr} \circ \Delta_M)T = (\Delta_M \circ \mathrm{Tr})T, \tag{4}$$

$$\Delta_M(T \odot g^l) = (\Delta_M T) \odot g^l, \tag{5}$$

where $\odot$ is the symmetric product.

The Lichnerowicz Laplacian acting on symmetric covariant tensor fields is of fundamental importance in mathematical physics (see for instance [9], [20] and [22]). Note also that the Lichnerowicz Laplacian acting on symmetric covariant 2-tensor fields appears in many problems in Riemannian geometry (see [3], [5], [19], ...).

On a compact Riemannian manifold, the Lichnerowicz Laplacian Δ_M has discrete eigenvalues with finite multiplicities. For a given compact Riemannian manifold, it may be an interesting problem to determine explicitly the eigenvalues and the eigentensors of Δ_M on M.

Let us enumerate the cases where the spectra of Δ_M was computed:

1. Δ_M acting on $C^\infty(M, \mathbb{C})$: M is either flat torus or Klein bottles [4], M is a Hopf manifolds [1];
2. Δ_M acting on $\Omega(M)$: $M = S^n$ or $P^n(\mathbb{C})$ [10] and [11], $M = \mathbb{C}aP^2$ or $G_2/SO(4)$ [16] and [18], $M = SO(n+1)/SO(2) \times SO(n)$ or $M = Sp(n+1)/Sp(1) \times Sp(n)$ [21];
3. Δ_M acting on $\mathcal{S}^2(M)$ and M is the complex projective space $P^2(\mathbb{C})$ [22];
4. Δ_M acting on $\mathcal{S}^2(M)$ and M is either S^n or $P^n(\mathbb{C})$ [6] and [7];
5. Brian and Richard Millman give in [2] a theoretical method for computing the spectra of Lichnerowicz Laplacian acting on $\Omega(G)$ where G is a compact semisimple Lie group endowed with the biinvariant metric induced from the negative of the Killing form;
6. Some partial results where given in [12]–[14].

In this paper, we compute the eigenvalues and we determine the spaces of eigentensors of Δ_M acting on $\mathcal{S}(M)$ in the case where M is the Euclidian sphere S^n.

Let us describe our method briefly. We consider the $(n+1)$-Euclidian space $\mathbb{R}^{n+1}$ with its canonical coordinates $(x_1, \dots, x_{n+1})$. For any $k,\ p \in \mathbb{N}$, we denote by $\mathcal{S}^p H_k^\delta$ the space of symmetric covariant p-tensor fields T on $\mathbb{R}^{n+1}$ satisfying:

1. $T = \sum_{1\le i_1\le\cdots\le i_p\le n+1} T_{i_1,\dots,i_p}\, dx_{i_1} \odot \cdots \odot dx_{i_p}$ where $T_{i_1,\dots,i_p}$ are homogeneous polynomials of degree k;
2. $\delta(T) = \Delta_{\mathbb{R}^{n+1}}(T) = 0$.

The n-dimensional sphere S^n is the space of unitary vectors in $\mathbb{R}^{n+1}$ and the Euclidian metric on $\mathbb{R}^{n+1}$ induces a Riemannian metric on S^n. We denote by $i\colon S^n \hookrightarrow \mathbb{R}^{n+1}$ the canonical inclusion.

For any tensor field $T \in \Gamma\left(\bigotimes^p T^*\mathbb{R}^{n+1}\right)$, we compute $i^*(\Delta_{\mathbb{R}^{n+1}}T) - \Delta_{S^n}(i^*T)$ and get a formula (see Theorem 2.1). Inspired by this formula and having in mind the fact that $i^*\colon \sum_{k\ge 0}\mathcal{S}^p H_k^\delta \to \mathcal{S}^p S^n$ is injective and its image is dense in $\mathcal{S}^p S^n$ (see [10]), we give, for any k, a direct sum decomposition of $\mathcal{S}^p H_k^\delta$ composed by eigenspaces of Δ_{S^n}. Thus we obtain the eigenvalues and the spaces of eigentensors with its multiplicities of Δ_{S^n} acting on $\mathcal{S}(S^n)$ (see Section 4).

Note that the eigenvalues and the eigenspaces of Δ_{S^n} acting on $\Omega(S^n)$ was computed in [10] by using the representation theory. In [11], I. Iwasaki and K. Katase recover the result by a method using the restriction of harmonic tensor fields and a result in [8]. The formula obtained in Theorem 2.1 combined with the methods developed in [10] and [11] permit to present those results in a more precise form (see Section 3).

2. A relation between $\Delta_{\mathbb{R}^{n+1}}$ and Δ_{S^n}

We consider the Euclidian space $\mathbb{R}^{n+1}$ endowed with its canonical coordinates $(x_1, \dots, x_{n+1})$ and its canonical Euclidian flat Riemannian metric $\langle\ ,\ \rangle$. We denote by D be the Levi-Civita covariant derivative associated to $\langle\ ,\ \rangle$. We consider the radial vector field given by

$$\vec{r} = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

For any p-tensor field $T \in \Gamma\left(\bigotimes^p T^*\mathbb{R}^{n+1}\right)$ and for any $1 \le i < j \le p$, we denote by $i_{\vec{r},j}T$ the $(p-1)$-tensor field given by

$$i_{\vec{r},j}T(X_1, \dots, X_{p-1}) = T(X_1, \dots, X_{j-1}, \vec{r}, X_j, \dots, X_{p-1}),$$

and by $\mathrm{Tr}_{i,j}\, T$ the $(p-2)$-tensor field given by

$$\begin{aligned}&\mathrm{Tr}_{i,j}\, T(X_1, \ldots, X_{p-2})\\ &= \sum_{l=1}^{n+1} T(X_1, \ldots, X_{i-1}, E_l, X_i, \ldots, X_{j-2}, E_l, X_{j-1}, \ldots, X_{p-2}),\end{aligned}$$

where $(E_1, \ldots, E_{n+1})$ is any orthonormal basis of $\mathbb{R}^{n+1}$. Note that $\mathrm{Tr}_{i,j}\, T = 0$ if T is a differential form and $\mathrm{Tr}_{i,j}\, T = \mathrm{Tr}\, T$ if T is symmetric.

For any permutation σ of $\{1, \ldots, p\}$, we denote by T^σ the p-tensor field

$$T^\sigma(X_1, \ldots, X_p) = T(X_{\sigma(1)}, \ldots, X_{\sigma(p)}).$$

For $1 \le i < j \le p$, the transposition of (i, j) is the permutation $\sigma_{i,j}$ of $\{1, \ldots, p\}$ such that $\sigma_{i,j}(i) = j$, $\sigma_{i,j}(j) = i$ and $\sigma_{i,j}(k) = k$ for $k \ne i, j$. Let $\mathcal{T}$ denote the set of the transpositions of $\{1, \ldots, p\}$.

The sphere $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is endowed with the Euclidian metric.

Theorem 2.1. *Let T be a covariant p-tensor field on $\mathbb{R}^{n+1}$. Then,*

$$\begin{aligned}&i^*(\Delta_{\mathbb{R}^{n+1}} T)\\ &= \Delta_{S^n} i^*T + i^*\left(p(1-p)T + (2p - n + 1)L_{\vec{r}}T - L_{\vec{r}} \circ L_{\vec{r}} T - 2\sum_{\sigma\in\mathcal{T}} T^\sigma + O(T)\right),\end{aligned}$$

where $O(T)$ is given by

$$\begin{aligned}O(T)(X_1, \ldots, X_p) = {}& 2\sum_{i<j} \langle X_i, X_j\rangle\, \mathrm{Tr}_{i,j}(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)\\ &- 2\sum_{j=1}^{p} D_{X_j}(i_{\vec{r},j} T)(X_1, \ldots, \hat{X}_j, \ldots, X_p),\end{aligned}$$

where the symbol $\hat{}$ means that the term is omitted.

Proof. The proof is a massive computation in a local orthonormal frame using the properties of the Riemannian embedding of the sphere in the Euclidian space.

We choose a local orthonormal frame of $\mathbb{R}^{n+1}$ of the form $(E_1, \ldots, E_n, N)$ such that E_i is tangent to S^n for $1 \le i \le n$ and $N = (1/r)\vec{r}$ where $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.

For any vector field X on $\mathbb{R}^{n+1}$, we have

$$D_X N = \frac{1}{r}(X - \langle X, N\rangle N), \tag{6}$$

$$D_N X = [N, X] + \frac{1}{r}(X - \langle X, N\rangle N). \tag{7}$$

Let ∇ be the Levi-Civita connexion of the Riemannian metric on S^n. We have, for any vector fields X, Y tangent to S^n,

$$D_X Y = \nabla_X Y - \langle X, Y\rangle N. \tag{8}$$

Let T be a covariant p-tensor field on $\mathbb{R}^{n+1}$ and $(X_1, \dots, X_p)$ a family of vector fields on $\mathbb{R}^{n+1}$ which are tangent to S^n. A direct calculation using the definition of the Lichnerowicz Laplacian gives

$$\begin{aligned}
&\Delta_{\mathbb{R}^{n+1}}(T)(X_1, \dots, X_p) = D^* D(T)(X_1, \dots, X_p)\\
&= \sum_{i=1}^{n}\Bigg(-E_i E_i.T(X_1, \dots, X_p) + 2\sum_{j=1}^{p} E_i.T(X_1, \dots, D_{E_i}X_j, \dots, X_p)\\
&\qquad + D_{E_i}E_i.T(X_1, \dots, X_p) - \sum_{j=1}^{p} T(X_1, \dots, D_{D_{E_i}E_i}X_j, \dots, X_p)\\
&\qquad - \sum_{j=1}^{p} T(X_1, \dots, D_{E_i}D_{E_i}X_j, \dots, X_p)\\
&\qquad - 2\sum_{l<j} T(X_1, \dots, D_{E_i}X_l, \dots, D_{E_i}X_j, \dots, X_p)\Bigg)\\
&\quad - N.N.T(X_1, \dots, X_p) + 2\sum_{j=1}^{p} N.T(X_1, \dots, D_N X_j, \dots, X_p)\\
&\quad + D_N N.T(X_1, \dots, X_p) - \sum_{j=1}^{p} T(X_1, \dots, D_{D_N N}X_j, \dots, X_p)\\
&\quad - \sum_{j=1}^{p} T(X_1, \dots, D_N D_N X_j, \dots, X_p) - 2\sum_{l<j} T(X_1, \dots, D_N X_l, \dots, D_N X_j, \dots, X_p).
\end{aligned}$$

(6)–(8) make it obvious that

$$\begin{aligned}
D_{D_{E_i}E_i}X_j &= \nabla_{\nabla_{E_i}E_i}X_j - \langle\nabla_{E_i}E_i, X_j\rangle N - [N, X_j]\\
&\quad - \frac{1}{r}(X_j - \langle X_j, N\rangle N),
\end{aligned} \tag{9}$$

$$\begin{aligned}
D_{E_i}D_{E_i}X_j &= \nabla_{E_i}\nabla_{E_i}X_j - (\langle E_i, \nabla_{E_i}X_j\rangle + E_i.\langle E_i, X_j\rangle)N\\
&\quad - \frac{1}{r}\langle E_i, X_j\rangle E_i,
\end{aligned} \tag{10}$$

$$\begin{aligned}
D_N D_N X &= [N, [N, X]] + \frac{2}{r}[N, X] + \left(\frac{1}{r^2} - \frac{1}{r}\right)(X - \langle X, N\rangle N)\\
&\quad - \frac{2}{r}N.\langle X, N\rangle N.
\end{aligned} \tag{11}$$

By (8)–(10), we get easily, in restriction to S^n,

$$\begin{aligned}
&\sum_{i=1}^{n}\Bigg(2\sum_{j=1}^{p} E_i.T(X_1,\dots,D_{E_i}X_j,\dots,X_p)+D_{E_i}E_i.T(X_1,\dots,X_p)\\
&\qquad -\sum_{j=1}^{p} T(X_1,\dots,D_{D_{E_i}E_i}X_j,\dots,X_p)-\sum_{j=1}^{p} T(X_1,\dots,D_{E_i}D_{E_i}X_j,\dots,X_p)\Bigg)\\
&=\sum_{i=1}^{n}\Bigg(2\sum_{j=1}^{p} E_i.T(X_1,\dots,\nabla_{E_i}X_j,\dots,X_p)+\nabla_{E_i}E_i.T(X_1,\dots,X_p)\\
&\qquad -\sum_{j=1}^{p} T(X_1,\dots,\nabla_{\nabla_{E_i}E_i}X_j,\dots,X_p)-\sum_{j=1}^{p} T(X_1,\dots,\nabla_{E_i}\nabla_{E_i}X_j,\dots,X_p)\Bigg)\\
&\quad -2\sum_{j=1}^{p} X_j.T(X_1,\dots,\overbrace{N}^{j},\dots,X_p)+p(n+1)T(X_1,\dots,X_p)-nL_N T(X_1,\dots,X_p).
\end{aligned}$$

On other hand, also by using (8), we have

$$\begin{aligned}
&\sum_{l<j}\sum_{i=1}^{n} T(X_1,\dots,D_{E_i}X_l,\dots,D_{E_i}X_j,\dots,X_p)\\
&=\sum_{l<j}\sum_{i=1}^{n} T(X_1,\dots,D_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j} T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)\\
&=\sum_{l<j}\sum_{i=1}^{n} T(X_1,\dots,\nabla_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j} T(X_1,\dots,\overbrace{N}^{l},\dots,\nabla_{X_l}X_j,\dots,X_p)\\
&\quad -\sum_{l<j} T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)\\
&=\sum_{l<j}\sum_{i=1}^{n} T(X_1,\dots,\nabla_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j} T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)\\
&\quad -\sum_{l<j} T(X_1,\dots,\overbrace{N}^{l},\dots,D_{X_l}X_j,\dots,X_p)\\
&\quad -\sum_{l<j}\langle X_l,X_j\rangle T(X_1,\dots,\overbrace{N}^{l},\dots,\overbrace{N}^{j},\dots,X_p).
\end{aligned}$$

So we get, in restriction to S^n, since $D_N N = 0$

$$\begin{aligned}
&\Delta_{\mathbb{R}^{n+1}}(X_1,\dots,X_p) - \nabla^*\nabla T(X_1,\dots,X_p)\\
&= p(n+1)T(X_1,\dots,X_p) - nL_N T(X_1,\dots,X_p) - 2\sum_{j=1}^{p} D_{X_j}(i_{N,j}T)(X_1,\dots,\hat{X}_j,\dots,X_p)\\
&\quad + 2\sum_{l<j}\langle X_l, X_j\rangle T(X_1,\dots,\overbrace{N}^{l},\dots,\overbrace{N}^{j},\dots,X_p) - N.N.T(X_1,\dots,X_p)\\
&\quad + 2\sum_{j=1}^{p} N.T(X_1,\dots,D_N X_j,\dots,X_p) - \sum_{j=1}^{p} T(X_1,\dots,D_N D_N X_j,\dots,X_p)\\
&\quad - 2\sum_{i<j} T(X_1,\dots,D_N X_i,\dots,D_N X_j,\dots,X_p).
\end{aligned}$$

Remark that, in restriction to S^n, the following equality holds

$$\sum_{j=1}^{p} D_{X_j}(i_{N,j}T)(X_1,\dots,\hat{X}_j,\dots,X_p) = \sum_{j=1}^{p} D_{X_j}(i_{\vec{r},j}T)(X_1,\dots,\hat{X}_j,\dots,X_p).$$

Now by using (7) and (11) and by taking the restriction to S^n, we have

$$\begin{aligned}
&2\sum_{j=1}^{p} N.T(X_1,\dots,D_N X_j,\dots,X_p)\\
&= 2\sum_{j=1}^{p} N.T(X_1,\dots,[N,X_j],\dots,X_p) + 2\sum_{j=1}^{p} N\left(\frac{1}{r}\right)T(X_1,\dots,X_j,\dots,X_p)\\
&\quad - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p) + 2\sum_{j=1}^{p} N.T(X_1,\dots,X_j,\dots,X_p)\\
&= 2\sum_{j=1}^{p} N.T(X_1,\dots,[N,X_j],\dots,X_p) - 2pT(X_1,\dots,X_p) + 2pN.T(X_1,\dots,X_j,\dots,X_p)\\
&\quad - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p).
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{p} T(X_1,\dots,D_N D_N X_j,\dots,X_p)\\
&= \sum_{j=1}^{p} T(X_1,\dots,[N,[N,X_j],\dots,X_p) - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p).
\end{aligned}$$

$$\begin{aligned}
&\sum_{i<j} T(X_1, \ldots, D_N X_i, \ldots, D_N X_j, \ldots, X_p) \\
&= \sum_{i<j} T(X_1, \ldots, [N, X_i], \ldots, [N, X_j], \ldots, X_p) + \frac{p(p-1)}{2} T(X_1, \ldots, X_p) \\
&\quad + \sum_{i<j} T(X_1, \ldots, X_i, \ldots, [N, X_j], \ldots, X_p) + \sum_{i<j} T(X_1, \ldots, [N, X_i], \ldots, X_j, \ldots, X_p).
\end{aligned}$$

So we get, in restriction to S^n

$$\begin{aligned}
&-N.N.T(X_1, \ldots, X_p) + 2\sum_{j=1}^{p} N.T(X_1, \ldots, D_N X_j, \ldots, X_p) \\
&- \sum_{j=1}^{p} T(X_1, \ldots, D_N D_N X_j, \ldots, X_p) - 2\sum_{i<j} T(X_1, \ldots, D_N X_i, \ldots, D_N X_j, \ldots, X_p) \\
&= -L_N \circ L_N T(X_1, \ldots, X_p) + 2pL_N T(X_1, \ldots, X_p) - p(1+p)T(X_1, \ldots, X_p).
\end{aligned}$$

The curvature of S^n is given by

$$R(X, Y)Z = \langle X, Y\rangle Z - \langle Y, Z\rangle X$$

and

$$r(X) = (n-1)X.$$

Hence, a direct computation gives that the curvature operator is given by

$$\begin{aligned}
R(T)(X_1, \ldots, X_p) = p(n-1)T(X_1, \ldots, X_p) + 2\sum_{\sigma\in\mathcal{T}} T^\sigma(X_1, \ldots, X_p) \\
- 2\sum_{i<j}\sum_{l=1}^{n} \langle X_i, X_j\rangle T(X_1, \ldots, E_l, \ldots, E_l, \ldots, X_p).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
i^*(\Delta_{\mathbb{R}^{n+1}} T) &= \Delta_{S^n} i^* T \\
&\quad + i^*\Bigl(p(1-p)T + (2p-n)L_N T - L_N \circ L_N T - 2\sum_{\sigma\in\mathcal{T}} T^\sigma + O(T)\Bigr),
\end{aligned}$$

One can conclude the proof by remarking that

$$i^*(L_N T) = i^*(L_{\vec{r}} T)$$

and

$$i^*(L_N \circ L_N T) = -i^*(L_{\vec{r}} T) + i^*(L_{\vec{r}} \circ L_{\vec{r}} T). \qquad \square$$

Corollary 2.1. *Let α be a differential p-form on $\mathbb{R}^{n+1}$. Then*

$$i^*(\Delta_{\mathbb{R}^{n+1}}\alpha) = \Delta_{S^n} i^*\alpha + i^*((2p - n + 1)L_{\vec{r}}\alpha - L_{\vec{r}} \circ L_{\vec{r}}\alpha - 2d i_{\vec{r}}\alpha)$$

Corollary 2.2. *Let T be a symmetric p-tensor field on $\mathbb{R}^{n+1}$. Then*

$$\begin{aligned} i^*(\Delta_{\mathbb{R}^{n+1}}T) = \Delta_{S^n} i^*T + i^*(2p(1 - p)T + (2p - n + 1)L_{\vec{r}}T - L_{\vec{r}} \circ L_{\vec{r}}T \\ - 2\delta^*(i_{\vec{r}}T) + 2\,\mathrm{Tr}(T) \odot \langle\ ,\ \rangle), \end{aligned}$$

where $\odot$ is the symmetric product.

3. Eigenvalues and eigenforms of Δ_{S^n} acting on $\Omega(S^n)$

In this section, we will use Corollary 2.1 and the results developed in [10] to deduce the eigenvalues and the spaces of eigenforms of Δ_{S^n} acting on $\Omega^*(S^n)$. We recover the results of [10] and [11] in a more precise form.

Let $\bigwedge^p H_k$ be the space of all coclosed harmonic homogeneous p-forms of degree k on $\mathbb{R}^{n+1}$. A differential form α belongs to $\bigwedge^p H_k$ if $\delta(\alpha) = 0$ and α can be written

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq n+1} \alpha_{i_1 \cdots i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $\alpha_{i_1 \cdots i_p}$ are harmonic polynomial functions on $\mathbb{R}^{n+1}$ of degree k. For any $\alpha \in \bigwedge^p H_k$, we have

$$L_{\vec{r}}\alpha = d i_{\vec{r}}\alpha + i_{\vec{r}}\, d\alpha = (k + p)\alpha. \tag{12}$$

We have (see [10])

$$i^*\colon \sum_{k \geq 0} \bigwedge^p H_k \to \Omega^p(S^n)$$

is injective and its image is dense.

For any $\alpha \in \bigwedge^p H_k$, we put

$$\omega(\alpha) = \alpha - \frac{1}{p + k}\, d i_{\vec{r}}\alpha. \tag{13}$$

Lemma 3.1. *We get a linear map* $\omega\colon \bigwedge^p H_k \to \bigwedge^p H_k$ *which is a projector, i.e.,* $\omega \circ \omega = \omega$. *Moreover,*

$$\operatorname{Ker}\omega = d\left(\bigwedge^{p-1} H_{k+1}\right), \quad \operatorname{Im}\omega = \bigwedge^{p} H_k \cap \operatorname{Ker} i_{\vec{r}},$$

and hence

$$\bigwedge^{p} H_k = \bigwedge^{p} H_k \cap \operatorname{Ker} i_{\vec{r}} \oplus d\left(\bigwedge^{p-1} H_{k+1}\right).$$

The following lemma is an immediate consequence of Corollary 2.1 and (12).

Lemma 3.2. 1. *For any* $\alpha \in \bigwedge^p H_k \cap \operatorname{Ker} i_{\vec{r}}$, *we have*

$$\Delta_{S^n} i^*\alpha = (k+p)(k+n-p-1)i^*\alpha.$$

2. *For any* $\alpha \in d\left(\bigwedge^{p-1} H_{k+1}\right)$, *we have*

$$\Delta_{S^n} i^*\alpha = (k+p)(k+n-p+1)i^*\alpha.$$

REMARK 3.1. We have

$$(k+p)(k+n-p-1) = (k'+p)(k'+n-p+1) \Leftrightarrow k = k'+1$$

and

$$n = 2p.$$

The following table gives explicitly the spectra of Δ_{S^n} and the spaces of eigenforms with its multiplicities. The multiplicity was computed in [11].

Table I.

p	The eigenvalues	The space of eigenforms	Multiplicity
$p = 0$	$k(k+n-1)$, $k \in \mathbb{N}$	$\bigwedge^0 H_k$	$\frac{(n+k-2)!\,(n+2k-1)}{k!\,(n-1)!}$
$1 \le p \le n$, $n \ne 2p$	$(k+p)(k+n-p-1)$, $k \in \mathbb{N}^*$	$\omega\left(\bigwedge^p H_k\right)$	$\frac{(n+k-1)!\,(n+2k-1)}{p!\,(k-1)!\,(n-p-1)!\,(n+k-p-1)(k+p)}$
	$(k+p)(k+n-p+1)$, $k \in \mathbb{N}$	$d\left(\bigwedge^{p-1} H_{k+1}\right)$	$\frac{(n+k)!\,(n+2k+1)}{(p-1)!\,k!\,(n-p)!\,(n+k-p+1)(k+p)}$
$1 \le p \le n$, $n = 2p$	$(k+p)(k+p+1)$, $k \in \mathbb{N}$	$\omega\left(\bigwedge^p H_{k+1}\right) \oplus d\left(\bigwedge^{p-1} H_{k+1}\right)$	$\frac{2(2p+k)!\,(2p+2k+1)}{p!\,(p-1)!\,k!\,(k+p+1)(k+p)}$

4. Eigenvalues and eigentensors of Δ_{S^n} acting on $\mathcal{S}(S^n)$

This section is devoted to the determination of the eigenvalues and the spaces of eigentensors of Δ_{S^n} acting on $\mathcal{S}(S^n)$.

Let $\mathcal{S}^p P_k$ be the space of $T \in \mathcal{S}^p(\mathbb{R}^{n+1})$ of the form

$$T = \sum_{1\leq i_1\leq\cdots\leq i_p\leq n+1} T_{i_1\cdots i_p}\, dx_{i_1}\odot\cdots\odot dx_{i_p},$$

where $T_{i_1\cdots i_p}$ are homogeneous polynomials of degree k. We put

$$\mathcal{S}^p H_k^\delta = \mathcal{S}^p P_k \cap \operatorname{Ker}\Delta_{\mathbb{R}^{n+1}} \cap \operatorname{Ker}\delta$$

and

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^\delta \cap \operatorname{Ker}\operatorname{Tr}.$$

In a similar manner as in [10] Lemma 6.4 and Corollary 6.6, we have

$$\mathcal{S}^p P_k = \mathcal{S}^p H_k^\delta \oplus (r^2\mathcal{S}^p P_{k-2} + dr^2\odot\mathcal{S}^{p-1}P_{k-1}), \tag{14}$$

and

$$i^*\colon \sum_{k\geq 0}\mathcal{S}^p H_k^\delta \to \mathcal{S}^p S^n$$

is injective and its image is dense in $\mathcal{S}^p S^n$.

Now, for any $k\geq 0$, we proceed to give a direct sum decomposition of $\mathcal{S}^p H_k^\delta$ consisting of eigenspaces of Δ_{S^n} and, hence, we determine completely the eigenvalues of Δ_{S^n} acting on $\mathcal{S}^p(S^n)$. This will be done in several steps.

At first, we have the following direct sum decomposition:

$$\mathcal{S}^p H_k^\delta = \mathcal{S}^p H_k^{\delta 0} \oplus \bigoplus_{l=1}^{[p/2]} \mathcal{S}^{p-2l} H_k^{\delta 0}\odot\langle\ ,\ \rangle^l, \tag{15}$$

where $\langle\ ,\ \rangle^l$ is the symmetric product of l copies of $\langle\ ,\ \rangle$.

The task is now to decompose $\mathcal{S}^p H_k^{\delta 0}$ as a sum of eigenspaces of Δ_{S^n} and get, according to (5), all the eigenvalues. This decomposition needs some preparation.

Lemma 4.1. *Let $T\in\mathcal{S}^p P_k$ and $h\in\mathbb{N}^*$. Then we have the following formulas*:
1. $\delta^*(i_{\vec r}T) - i_{\vec r}\delta^*(T) = (p-k)T$;
2. $\delta^{*(h)}(i_{\vec r}T) - i_{\vec r}\delta^{*(h)}(T) = h(p-k+h-1)\delta^{*(h-1)}(T)$;
3. $\delta^*(i_{\vec r^h}T) - i_{\vec r^h}\delta^*(T) = h(p-k-h+1)i_{\vec r^{h-1}}T$,

where $i_{\vec{r}^h} = \overbrace{i_{\vec{r}} \circ \cdots \circ i_{\vec{r}}}^{h}$ *and* $\delta^{*(h)} = \overbrace{\delta^* \circ \cdots \circ \delta^*}^{h}$.

Proof. The first formula is easily verified and the others follow by induction on h. $\square$

Note that the spaces $\mathcal{S}^p H_k^{\delta 0}$ are invariant by δ^* and $i_{\vec{r}}$; this is a consequence of the following formulas which one can check easily. For any symmetric tensor field T on $\mathbb{R}^{n+1}$, we have

$$\Delta_{\mathbb{R}^{n+1}}(i_{\vec{r}} T) = i_{\vec{r}} \Delta_{\mathbb{R}^{n+1}}(T) + 2\delta T, \tag{16}$$

$$\delta(i_{\vec{r}} T) = i_{\vec{r}} \delta(T) - \mathrm{Tr}(T), \tag{17}$$

$$\mathrm{Tr}(\delta^*(T)) = -2\delta(T) + \delta^*(\mathrm{Tr}(T)), \tag{18}$$

$$\mathrm{Tr}(i_{\vec{r}} T) = i_{\vec{r}} \,\mathrm{Tr}(T). \tag{19}$$

Now the desired decomposition of $\mathcal{S}^p H_k^{\delta 0}$ is based on the following algebraic lemma.

Lemma 4.2. *Let V be a finite dimensional vectorial space, ϕ and ψ are two endomorphisms of V and $(A_k^p)_{k,\, p \in \mathbb{N} \cup \{-1\}}$ a family of vectorial subspaces of V such that*:

1. *for any $p, k \in \mathbb{N}$, $A_{-1}^p = A_k^{-1} = 0$;*
2. *for any $p, k \in \mathbb{N}$, $\phi(A_k^p) \subset A_{k-1}^{p+1}$ and $\psi(A_k^p) \subset A_{k+1}^{p-1}$;*
3. *for any $p, k \in \mathbb{N}$ and for any $a \in A_k^p$,*

$$\phi \circ \psi(a) - \psi \circ \phi(a) = (p - k)a.$$

Then:

(i) *for any $k < p$, $\psi \colon A_k^p \to A_{k+1}^{p-1}$ is injective*;

(ii) *for $k \leq p$, we have*

$$A_k^p = (A_k^p \cap \mathrm{Ker}\,\phi) \oplus \psi\left(A_{k-1}^{p+1}\right)$$

and

$$A_k^p = \bigoplus_{l=0}^{k} \psi^l \left(A_{k-l}^{p+l} \cap \mathrm{Ker}\,\phi\right).$$

Proof. Note that one can deduce easily, by induction, that for any $l \in \mathbb{N}^*$ and for any $a \in A_k^p$

$$\phi^l \circ \psi(a) - \psi \circ \phi^l(a) = l(p - k + l - 1)\phi^{l-1}(a), \tag{20}$$

$$\psi^l \circ \phi(a) - \phi \circ \psi^l(a) = l(k - p + l - 1)\psi^{l-1}(a). \tag{21}$$

(i) Let $a \in A_k^p$ such that $\psi(a) = 0$. From (20) and since $p - k > 0$, for any $l \geq 0$, if $\phi^l(a) = 0$ then $\phi^{l-1}(a) = 0$. Now, since $\phi^l(a) \in A_{k-l}^{p+l}$ and since $A_{-1}^{p+l} = 0$, we have, for any $l \geq k+1$, $\phi^l(a) = 0$ which implies, by induction, that $a = 0$ and hence $\psi : A_k^p \to A_{k+1}^{p-1}$ is injective.

(ii) Suppose that $k \leq p$. We define $P_k^p : A_k^p \to A_k^p$ as follows

$$\begin{cases} P_k^p(a) = \displaystyle\sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(a) \\ \alpha_0 = 1 \text{ and } \alpha_s - (s+1)(k-p-s-2)\alpha_{s+1} = 0 \quad \text{for} \quad 1 \leq s \leq k-1. \end{cases}$$

P_k^p satisfies

$$P_k^p \circ P_k^p = P_k^p, \quad \operatorname{Ker} P_k^p = \psi\left(A_{k-1}^{p+1}\right)$$

and

$$\operatorname{Im} P_k^p = A_k^p \cap \operatorname{Ker} \phi.$$

Indeed, let $a \in A_{k-1}^{p+1}$. We have

$$\begin{aligned} P_k^p(\psi(a)) &= \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(\psi(a)) \\ &\overset{(20)}{=} \sum_{s=0}^{k} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=0}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &\overset{\phi^k(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=1}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &= \sum_{s=0}^{k-1} (\alpha_s + (s+1)(p-k+s+2)\alpha_{s+1})\psi^{s+1} \circ \phi^s(a) \\ &= 0. \end{aligned}$$

Conversely, since $P_k^p(a) = a + \sum_{s=1}^{k} \alpha_s \psi^s \circ \phi^s(a)$, we deduce that $P_k^p(a) = 0$ implies that $a \in \psi(A_{k-1}^{p+1})$, so we have shown that $\operatorname{Ker} P_k^p = \psi(A_{k-1}^{p+1})$. The relation $P_k^p \circ P_k^p = P_k^p$ is a consequence of the definition of P_k^p and $P_k^p \circ \psi = 0$.

Note that $\phi(a) = 0$ implies that $P_k^p(a) = a$ and hence $A_k^p \cap \operatorname{Ker}\phi \subset \operatorname{Im} P_k^p$. Conversely, let $a \in A_k^p$, we have

$$\begin{aligned}
\phi \circ P_k^p(a) &= \sum_{s=0}^{k} \alpha_s \phi \circ \psi^s \circ \phi^s(a) \\
&\overset{(21)}{=} \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=0}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&\overset{\phi^{k+1}(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=1}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&= \sum_{s=0}^{k-1} (\alpha_s - (s+1)(k-p-s-2)\alpha_{s+1})\psi^s \circ \phi^{s+1}(a) \\
&= 0.
\end{aligned}$$

We conclude that P_k^p is a projector, $\operatorname{Ker} P_k^p = \psi\left(A_{k-1}^{p+1}\right)$ and $A_k^p \cap \operatorname{Ker}\phi = \operatorname{Im} P_k^p$ and we deduce immediately that $A_k^p = \psi\left(A_{k-1}^{p+1}\right) \oplus A_k^p \cap \operatorname{Ker}\phi$. The same decomposition holds for A_{k-1}^{p+1} and, since $\psi\colon A_{k-1}^{p+1} \to A_k^p$ is injective, we get

$$A_k^p = \psi \circ \psi\left(A_{k-2}^{p+2}\right) \oplus \psi\left(A_{k-1}^{p+1} \cap \operatorname{Ker}\phi\right) \oplus A_k^p \cap \operatorname{Ker}\phi.$$

We proceed by induction and we get the desired decomposition. □

According to Lemma 4.1, the hypothesis of Lemma 4.2 are satisfied by the spaces $\mathcal{S}^p H_k^{\delta 0}$ and the operators δ^* and $i_{\vec{r}}$. So we get, in a first time,

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker}\delta^* \oplus i_{\vec{r}}\left(\mathcal{S}^{p+1} H_{k-1}^{\delta 0}\right), \quad \text{if} \quad k \leq p, \tag{22}$$

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}} \oplus \delta^*\left(\mathcal{S}^{p-1} H_{k+1}^{\delta 0}\right), \quad \text{if} \quad k \geq p, \tag{23}$$

and, in a second time, the desired decomposition of $\mathcal{S}^p H_k^{\delta 0}$.

Lemma 4.3. *We have*:

1. *If* $k \leq p$

$$\mathcal{S}^p H_k^{\delta 0} = \bigoplus_{l=0}^{k} i_{\vec{r}^l}\left(\mathcal{S}^{p+l} H_{k-l}^{\delta 0} \cap \operatorname{Ker}\delta^*\right);$$

2. *If* $k \geq p$

$$\mathcal{S}^p H_k^{\delta 0} = \bigoplus_{l=0}^{p} \delta^{*l}\left(\mathcal{S}^{p-l} H_{k+l}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\right);$$

3. *If $k = p$, for any $0 \le l \le p$,*

$$\mathcal{S}^p H_p^{\delta 0} = \bigoplus_{l=0}^{p} i_{\vec{r}^l}\left(\mathcal{S}^{p+l} H_{p-l}^{\delta 0} \cap \operatorname{Ker} \delta^*\right) = \bigoplus_{l=0}^{p} \delta^{*l}\left(\mathcal{S}^{p-l} H_{p+l}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\right).$$

Now, we use Corollary 2.2 to show that the decompositions of $\mathcal{S}^p H_k^{\delta 0}$ given in Lemma 4.3 are composed by eigenspaces of Δ_{S^n}.

Theorem 4.1. *We have*:

1. *If $k \le p$, for any $0 \le q \le k$ and any $T \in i_{\vec{r}^{(k-q)}}(\mathcal{S}^{p+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*)$,*

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T;$$

2. *If $k \ge p$, for any $0 \le q \le p$ and for any $T \in \delta^{*(p-q)}(\mathcal{S}^q H_{k+p-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}})$,*

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T.$$

Proof. 1. Let $T = i_{\vec{r}^{(k-q)}}(T_0)$ with $T_0 \in \mathcal{S}^{p+k-q} H_q^{\delta 0} \cap Ker\, \delta^*$. We have from Corollary 2.2

$$\begin{aligned}\Delta_{S^n} i^* T = i^*(&2p(p-1)T + (n-2p-1)L_{\vec{r}} T + L_{\vec{r}} \circ L_{\vec{r}} T \\ &+ 2\delta^*(i_{\vec{r}} T) - 2 \operatorname{Tr}(T) \odot \langle\ ,\ \rangle).\end{aligned}$$

We have

$$\operatorname{Tr} T = 0, \quad L_{\vec{r}} T = (k+p)T$$

and

$$L_{\vec{r}} \circ L_{\vec{r}} T = (k+p)^2 T.$$

Moreover, by using Lemma 4.1, we have

$$\begin{aligned} 2\delta^*(i_{\vec{r}} T) &= 2\delta^*(i_{\vec{r}^{(k-q+1)}} T_0) \\ &\overset{\delta^*(T_0)=0}{=} 2(k-q+1)(p+k-q-q-k+q-1+1) i_{\vec{r}^{(k-q)}} T_0 \\ &= 2(k-q+1)(p-q)T. \end{aligned}$$

Hence

$$\Delta_{S^n} i^* T = (2p(p-1) + (n-2p-1)(k+p) + (k+p)^2 + 2(p-q)(k-q+1)) i^* T.$$

One can deduce the desired relation by remarking that

$$2p(p-1) + 2(p-q)(k-q+1) = 2(k+p)(p-q) + 2q(q-1).$$

2. This follows by the same calculation as 1. □

From the fact that

$$i^*: \sum_{k\geq 0} \mathcal{S}^p H_k^\delta \to \mathcal{S}^p S^n$$

is injective and its image is dense in $\mathcal{S}^p S^n$, from (15), and from Lemma 4.3 and Theorem 4.1, note that we have actually proved that the eigenvalues of Δ_{S^n} acting on $\mathcal{S}^p S^n$ belongs to

$$\Big\{(k+p-2l)(n+p+k-2l-2q-1)+2q(q-1),$$
$$k \in \mathbb{N},\ 0 \leq l \leq \left[\frac{p}{2}\right],\ 0 \leq q \leq \min(k,\ p-2l)\Big\}.$$

Our next goal is to sharpen this result by computing $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$ if $k \leq p$ and $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$ if $k \geq p$.

Lemma 4.4. *We have the following formulas*:
1. $\dim \mathcal{S}^p H_k^\delta = \dim \mathcal{S}^p P_k - \dim \mathcal{S}^p P_{k-2} - \dim \mathcal{S}^{p-1} P_{k-1} + \dim \mathcal{S}^{p-1} P_{k-3}$,
2. $\dim \mathcal{S}^p H_k^{\delta 0} = \dim \mathcal{S}^p H_k^\delta - \dim \mathcal{S}^{p-2} H_k^\delta$,
3. $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p+1} H_{k-1}^{\delta 0}$ $(k \leq p)$,
4. $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p-1} H_{k+1}^{\delta 0}$ $(k \geq p)$.

Note that we use the convention that $\mathcal{S}^p P_k = \mathcal{S}^p H_k^\delta = \mathcal{S}^p H_k^{\delta 0} = 0$ *if* $k < 0$ *or* $p < 0$.

Proof. 1. The formula is a consequence of (14), the relation

$$(r^2 \mathcal{S}^p P_{k-2}) \cap (dr^2 \odot \mathcal{S}^{p-1} P_{k-1}) = r^2 (dr^2 \odot \mathcal{S}^{p-1} P_{k-3})$$

and the fact that $dr^2 \odot .: \mathcal{S}^p P_k \to \mathcal{S}^{p+1} P_{k+1}$ is injective.

2. The formula is a consequence of (15).
3. The formula is a consequence of (22) and Lemma 4.2.
4. The formula is a consequence of (23) and Lemma 4.2. □

A straightforward calculation using Lemma 4.4 and the formula

$$\dim \mathcal{S}^p P_k = \frac{(n+p)!}{n!\,p!} \frac{(n+k)!}{n!\,k!}$$

gives $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$ if $k \leq p$ and $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$ if $k \geq p$. We summarize the results on the following table.

Table II.

Space	Dimension	Conditions on k and p
$\mathcal{S}^0 H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-2)!\,(n+2k-1)}{k!\,(n-1)!}$	$k \geq 0$
$\mathcal{S}^p H_0^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+p-2)!\,(n+2p-1)}{p!\,(n-1)!}$	$p \geq 0$
$\mathcal{S}^1 H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-3)!\,k(n+2k-1)(n+k-1)}{(n-2)!\,(k+1)!}$	$k \geq 1$
$\mathcal{S}^p H_1^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+p-3)!\,p(n+2p-1)(n+p-1)}{(n-2)!\,(p+1)!}$	$p \geq 1$
$\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+k-4)!\,(n+p-3)!\,(n+p+k-2)}{k!\,(p+1)!\,(n-1)!\,(n-2)!} \times$ $(n-2)(n+2k-3)(n+2p-1)(p-k+1)$	$2 \leq k \leq p$
$\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-3)!\,(n+p-4)!\,(n+p+k-2)}{(k+1)!\,p!\,(n-1)!\,(n-2)!} \times$ $(n-2)(n+2k-1)(n+2p-3)(k-p+1)$	$k \geq p \geq 2$

REMARK 4.1. Note that, for $n = 2$, we have

$$\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*) = 0 \quad \text{for} \quad 2 \leq k \leq p,$$
$$\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) = 0 \quad \text{for} \quad k \geq p \geq 2.$$

For simplicity we introduce the following notations.

$$S_0 = \left\{(k, l, q) \in \mathbb{N}^3,\ 0 \leq l \leq \left[\frac{p}{2}\right],\ 0 \leq k \leq p - 2l,\ 0 \leq q \leq k\right\},$$
$$S_1 = \left\{(k, l, q) \in \mathbb{N}^3,\ 0 \leq l \leq \left[\frac{p}{2}\right],\ k > p - 2l,\ 0 \leq q \leq p - 2l\right\},$$
$$V_{q,l}^k = i_{\vec{r}^{k-q}}\big(\mathcal{S}^{p-2l+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*\big) \odot \langle\ ,\ \rangle^l \quad \text{for} \quad (k, l, q) \in S_0,$$
$$W_{q,l}^k = \delta^{*(p-2l-q)}\big(\mathcal{S}^q H_{p-2l+k-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\big) \odot \langle\ ,\ \rangle^l \quad \text{for} \quad (k, l, q) \in S_1.$$

Let us summarize all the results above.

Theorem 4.2. 1. *For $n = 2$, we have*:
(a) *The set of the eigenvalues of Δ_{S^2} acting on $\mathcal{S}^p S^2$ is*

$$\left\{(k + p - 2l)(p + k - 2l + 1), \quad k \in \mathbb{N},\ 0 \leq l \leq \left[\frac{p}{2}\right]\right\};$$

(b) *The eigenspace associated to the eigenvalue* $\lambda(k,l) = (k+p-2l)(k+p-2l+1)$ *is given by*

$$V_{\lambda(k,l)} = \begin{cases} \displaystyle\bigoplus_{a=0}^{\min(l,[k/2])} \left(V_{0,l-a}^{k-2a} \oplus V_{1,l-a}^{k+1-2a}\right) & \textit{if} \quad 0 \le k \le p-2l, \\ \displaystyle\bigoplus_{a=0}^{\min(l,[k/2])} \left(W_{0,l-a}^{k-2a} \oplus W_{1,l-a}^{k+1-2a}\right) & \textit{if} \quad k > p-2l; \end{cases}$$

(c) *The multiplicity of* $\lambda(k,l)$ *is given by*

$$m(\lambda(k,l)) = 2\left(\min\left(l, \left[\frac{k}{2}\right]\right) + 1\right)(1+2p+2k-4l).$$

2. *For* $n \ge 3$, *we have*:
(a) *The set of the eigenvalues of* Δ_{S^n} *acting on* $\mathcal{S}^p S^n$ *is*

$$\left\{(k+p-2l)(n+p+k-2l-2q-1) + 2q(q-1),\right.$$
$$\left. k \in \mathbb{N},\ 0 \le l \le \left[\frac{p}{2}\right],\ 0 \le q \le \min(k,\ p-2l)\right\};$$

(b) *The space*

$$\mathcal{P} = \sum_{k\ge 0} \mathcal{S}^p H_k^\delta = \left(\bigoplus_{(k,l,q)\in S_0} V_{q,l}^k\right) \oplus \left(\bigoplus_{(k,l,q)\in S_1} W_{q,l}^k\right)$$

is dense in $\mathcal{S}^p S^n$ *and, for any* $(k,q,l) \in S_0$ (*resp.* $(k,q,l) \in S_1$), $V_{q,l}^k$ (*resp.* $W_{q,l}^k$) *is a subspace of the eigenspace associated to the eigenvalue* $(k+p-2l)(n+p+k-2l-2q-1)+2q(q-1)$;
(c) *The dimensions of* $V_{q,l}^k$ *and* $W_{q,l}^k$ *are given in Table II since*

$$\dim V_{q,l}^k = \dim(\mathcal{S}^{p-2l+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*) \quad \textit{for} \quad (k,l,q) \in S_0,$$
$$\dim W_{q,l}^k = \dim(\mathcal{S}^q H_{p-2l+k-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) \quad \textit{for} \quad (k,l,q) \in S_1.$$

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