Osaka Mathematical Journal,
Vol. 3, No. 2, December, 1951

Potential Theory and its Applications, I.

By Zenjiro KURAMOCHI

Preface

The Riemann surface deviced as an instrument to study multiform functions of a complex variable in the z-plane, since it was defined strictly, has been one of the main subject in the study of function theory. The structure of the abstract Riemann surface has been approached chiefly from two standpoints, i. e. the topological and the metrical. But the latter is more complicated and it follows from this that the Riemann surface can be classified as zero or positive-boundary Riemann surface and so on.

From the theory of automorphic function, it is well known that there exists a one-valued meromorphic function on any given Riemann surface. It is quite natural to generalize the theorems obtained in the case when the domain of the function is the z-plane on an abstract Riemann surface.

In Chapter I, we discuss the topology of an abstract Riemann surface as done by Stoilow[1)3)] and the conformal mapping of the Riemann surface onto the unit-circle. In Chapter II, we study the behaviour of a harmonic function or meromorphic function in the neighbourhood of ideal boundary points of harmonic measure zero. Chapter III is concerned with Green function, especially, with Green function with its pole at an ideal boundary point. In Chapter VI the potential theory on the Riemann surface is discussed. The theorem of G. C. Evans in the potential theory is the most useful in the theory of the function of the z-plane. We generalize this theorem for an abstract Riemann surface under certain hypothesis, which means that the ideal boundary point is simple in a sense. The remainder of this paper is concerned with applications of G. C. Evans' theorem to the theory of function.

Chapter I.

Abstract Riemann surface.

1. Riemann surface. When a two-dimensional and orientable Hausdorff space satisfies the following conditions, it will be called a

1) The number indicates the reference at the end of this paper.

Riemann surface F.

1° F is covered by at most enumerable number of discs K_ν, which is mapped conformally and in a one-to-one manner on a circle of the z-plane by means of a local parameter.

2° When two discs K_μ, and K_ν have the common part which is cut from K_μ, and K_ν by analytic curves α_μ, α_ν, ending in two intersection points of peripheries of K_μ and K_ν, these two part can be mapped conformally and directly each other.

3° For any point of K_μ there is another disc K_ν containing the point and between them correlating relation of 2° is defined.

4° Any disc has common points with only finite number of neighbourhood discs.

5° Any two discs of F can be connected through a chain of a finite number of discs with the correlating of 2°.

Whenever any set of an infinite sequence of points in F has at least a limit point, then F is called a compact surface or closed surface, in other words if and only if F is covered by a finite number of discs.

2. **Exhausion.** F_i is a part of Riemann surface composed of a finite number of discs such that

$$F_0 \subset F_1 \subset F_2 \dots . \qquad \lim_i F_i = F.$$

The sequence of F_i is an *exhaustion.* The boundary of F_0 will be denoted by Γ_0, and F_i has a boundary composed of a finite number of closed analytic curves denoted by $\sum_j \Gamma_j^i = \Gamma^i$.

We indicate the bounded harmonic function with the boundary values 1 on Γ^i and 0 on Γ_0 defined in $F_i - F_0$, by $\omega_i(x, F_i - F_0)$; $x \in F_i - F_0$. This ω_i is monotonously decreasing with i.

If $\lim_i \omega_i(x, F_i - F_0) = 0$, then F is called a *zero-boundary Riemann surface,* otherwise F is called a *positive-boundary,* this classification does not depend on the choice of the exhäustion of F.

More generally, let F' be a part of Riemann surface with a relative boundary Γ and $F_i (i = 1. 2 \dots)$be an exhaustion of F'. Let us denote by $\omega_i(x, F_i, \Gamma)$ the bounded harmonic function in F_i with the boundary values 1 on Γ_i and 0 on Γ. If $\lim_i \omega_i(x, F_i, \Gamma) = 0$, we call that F' has a *relative zero-boundary.*

3. **Boundary of Riemman surface.** The boundary of F_i is made up of a finite number of analytic curves $\Gamma_{n_1, n_2, \dots n_i} : n_i \leqq n_i^0 : i = 1. 2. \dots$, which are closed and have no common point each other. When a curve $\Gamma_{n_1, n_2, \dots n^i}$ cuts F completely into two parts, then $\Gamma_{n_1, n_2, \dots n_i}$ will be named a *proper cut.* We assume that all of $\Gamma_{n_1, n_2, \dots n_i}$ are *proper cuts.*

Ideal boundary. One of the two parts cut by $\Gamma_{n_1, n_2, \dots n_i}$, not containing F_0 is denoted by $V_{n_1, n_2, \dots n_i}$, if $V_{n_1, n_2, \dots n_i} \supset V_{n_1, n_2, \dots n_i n_{i+1}}$, if $\bigcap_{i=\infty} V_{n_1, n_2, \dots n_i}$ has no common point in F, then we say that the sequence of $V_{n_1, n_2, \dots n_i}$ determines an ideal boundary point α, and the sequence is called a determining sequence of α, and $V_{n_1, n_2, \dots n_i}$ are called a system of neighbourhood of α.

$$\alpha = \{V_{n_i}, V_{n_1, n_2}, \dots\} = \{n_1, n_2, \dots\}$$

Two determining sequences $V_{n_1}, V_{n_1, n_2} \dots$, and $V'_{n_1}, V'_{n_1, n_2}, \dots$ are equivalent if and only if there exists for any $V_{n_1, \dots n_i}$, a certain $V'_{n_1, \dots n_j}$ such as $V_{n_1, \dots n_i} \subset V'_{n_1, \dots n_j}$, and vice versa.

We say that two equivalent sequences determine the same ideal boundary point. In the sequel we use for simplicity $V_{n_1, n_2, \dots n_i}$ defined by the exhaustion as the neighbourhood system $V_{n_1, \dots n_i} \dots$.

Let $a_1 . a_2 . \dots a_j \in F$ and $\alpha = (n_1, n_2, \dots)$ be a sequence of points of F and an ideal boundary point. If for any given i there is a number j_0 such that

$$a_j \in V_{n_1, \dots n_i}, \quad ; \quad (j \geqq j_0),$$

then we say that the sequence of a_j *converges to* α. It is clear that any subsequence of a_j converges also to α.

We say that all boundary points constitute a *boundary point set* R. From the definition, if infinitely many points a_j of F have no limit point in F, then a_j converges to R, and there exists at least a point $\alpha = (n_1, n_2, \dots)$ and the subsequence such as $\lim_i a_{n_i} = \alpha$.

Limit of the ideal boundary points: Let

$$\begin{aligned} \alpha^p &= (V^p_{n_1}, V^p_{n_1 n_2}, \dots) \\ &\dots\dots\dots \qquad\qquad p = 1, 2, 3 \dots \\ \alpha &= (V_{n_1}, V_{n_1 n_2}, \dots) \end{aligned}$$

be ideal boundary points, if for any given i, there exist $p_0(i)$ such as for $p > p_0(i)$ there holds $V^p_{n_1, \dots n_s} \subset V_{n_1, \dots n_i}$; where s depends on p; $s = s(p)$, then we say that $\lim_p \alpha^p = \alpha$.

3. **Theorem 1.** *In this topology the boundary set R is compact and closed.*

Proof. Let

$$\begin{aligned} \alpha_1 &= (V^1_{n_1}, V^1_{n_1, n_2}, \dots\dots\dots) \\ \alpha_2 &= (V^2_{n_1}, V^2_{n_1, n_2}, \dots\dots\dots) \\ &\dots\dots\dots \\ \alpha_i &= (V^i_{n_1}, V^i_{n_1, n_2}, \dots\dots\dots) \end{aligned}$$

be an infinite number of ideal boundary points. Since there is only a finite number of $V_{n_1}: 0 \leq n_1 \leq n_1^0$, then there exists at least one $V^*_{n_1}$ such as $V^*_{n_1}$ contains an infinite subsequence $\{\alpha_i^2\}$ of $\{\alpha_i^1\}$. and there exists at least one $V^*_{n_1, n_2}$ such as $V^*_{n_1, n_2}$ contains an infinite subsequence $\{\alpha_i^1\}$ of $\{\alpha_i^1\}$. Thus we have

$$\alpha_1^1,\ \alpha_2^1 \quad \cdots\cdots\cdots \quad \in V^*_{n_1}$$
$$\alpha_1^2,\ \alpha_2^2 \quad \cdots\cdots\cdots \quad \in V^*_{n_1,n_2}$$
$$\cdots\cdots\cdots$$

Put $$\alpha^* = (V^*_{n_1},\ V^*_{n_1 n_2}, \ldots \quad \ldots).$$

It is clear that $\lim_i \alpha_i^i = \alpha^*$ and $\alpha^* \in R$.

4. Let F be a relative zero-boundary Riemann surface with a relative boundary Γ, and $\Gamma_{n_1, n_2, \ldots n_i}$ be the proper cuts defined in 3. $G_i = V_{n_1, n, \ldots n_i} - V_{n_1, \ldots n_{i+1}}$ is a tube with oan boundary curve $\Gamma_{n_1, \ldots n_i} \in \Gamma_i$ and k boundary curves $\Gamma_{n_1, \ldots n_{i+1}}; 1 \leq k < \infty$ contained in Γ_{i+1} and has genus g_i.

In G_i we make the conjugate loopcuts $\gamma_2, \gamma_2', \ldots \gamma_i, \gamma_2'$ corresponding to g_i and γ_i. Let us cut G_i along $\gamma_2, \gamma_{2'} \ldots \gamma_i'$ and γ_i, then G_i becomes a $\kappa+2$ multiply connected domain G_i', and take a point p on γ_1' and q_j; $j = 1.2 \ldots$ k on $\Gamma_{\underbrace{n_1, \ldots n_j}_{i}}$ and connect q^j and p with an analytic curve $\widehat{q_j p}$ for every j. After cutting G_i' along them, G_i' becomes a simply connected domain G_i''. In all G_i, if we construct a system of cuts, then F becomes a simply connected domain denoted by $\widetilde{F}$ and $V_{n_1, \ldots n_i}$ becomes a simply connected domain $\widetilde{V}_{n_1, \ldots n_i}$ and further every boundary curve $\Gamma_{n \ldots n_i}$ has only one intersecting point with the system of cuts.

We map $\widetilde{F}$ on to the unit-circle, making use of the universal covering surface F^∞ of F. In this mapping the boundary Γ of F corresponds on the system of arcs with linear measure 2π on the periphery $|z| = 1$, and the ideal boundary point set will be transformed to the linear measure zero set on $|z| = 1$, and F is mapped on to a system of equivalent fundamental domains. Let us take one of them which is the fundamental domain containing $z = z_0$, enclosed by the images of loopcuts and the images of cuts and denote it by D_0. In this mapping $\Gamma_{n_1, \ldots n_i}$ is transformed to the curve connecting two equivalent points. In the sequel we use the same notation with under-line for the image in $|z| < 1$ mapped conformally of the figure (point, curve, domain) in F.

In the fundamental domain, ideal boundary point α corresponds in one-to-one manner on the set $\underline{R}$ on $|z| = 1$.

1) *The image of curve l passing $a_1, a_2 \ldots$; lim $a_i = \alpha$, $a_i \in V_{n_1, \ldots n_i}$*

$-V_{n_1 \dots n_{i+1}} \frown F$ *does not oscillate,* otherwise, the image of l converges on an arc A of $|z|=1$, then there is an arc $\underline{\Gamma}^*$ of the image of $\underline{\Gamma}$ in A, then there must be another fundamental domain D^* with $\underline{\Gamma}^*$, this contradicts that a_i are contained in the former fundamental domain D_0. Thus l converges to a certain point on $|z|=1$. $\underline{\Gamma}_{n_1, \dots n_i}$ cuts the fundamental domain into two parts, one of them containing $z=z_0$ and the other corresponds to $V_{n_1, \dots n_i}$ of α.

2) *If* $\alpha_1 \neq \alpha_2$, *then* $\underline{\alpha}_1 \neq \underline{\alpha}_2$. Since $\alpha_1 \neq \alpha_2$, there is a pair of neighbourhoods such as $V^i(\alpha_1) \frown V^j(\alpha_2) = 0$ on F, then there is a Jordan curve $\widehat{\alpha_1 \alpha_2}$ connecting α_1 and α_2 on the boundary of D_0, accordingly there is at least an analytic curve to which a substitution s corresponds, which must put $\underline{\Gamma}$ of D_0 on an image which is one of equivalent system of $\underline{\Gamma}$ on $|z|=1$ between $\underline{\alpha}_1$ and $\underline{\alpha}_2$, this follows that $\underline{\alpha}_1 \neq_2 . \underline{\alpha}$ Therefore the set of ideal boundary points of Riemann surface corresponds in one-to-one manner on the set $\underline{R}$ on the periphery of $|z|=1$ of a certain fundamental domain.

We denote by C the periphery $|z|=1$, and by $\bar{D}_0$, the closure of D_0 then.

Theorem 2. $\underline{R} = \bar{D}_0 \frown C - \Gamma$.

Proof. $\underline{R} \subset \bar{D}_0 \frown C - \Gamma$ is clear, if $\bar{D}_0 \frown C - \Gamma \ni p$, then there exists a sequence of a_i such as $\lim_i a_i = p$; $a_i \in D_0$ in the z-plane topology. In thinking the image $\bar{a}_i$ of a_i in F, which has no limit point in F, then they have a subsequence a_{n_i} such as $\lim_i \bar{a}_{n_i} = q \in R$. Let $z(q) = Q$, then $Q \in R$, $\lim_i a_{n_i} = Q \in \underline{R}$ but $\lim_i a_{n_i}, = p = \lim_i a_i$, therefore, we have

$$p = Q, \ Q \in \underline{R}$$

hence,

$$\bar{D}_0 \frown C - \underline{\Gamma} = \underline{R}$$

Thus $\underline{R}$ is closed in the z-plane topology and every point of R is accessible in F, then all boundary of D_0 are accessible in F, then all boundary of D_0 are accessible and the set of ideal point R is represented homeomorphically on $\bar{D}_0 \frown C - \underline{\Gamma}$ and the system of neighbourhood of α can be defined on D_0 as their image.

$\underline{R}$ lies on the periphery $|z|=1$, and moreover the number of all fundamental domains is enumerable, then we have

Corollary. *Let F be a relative zero-boundary Riemann surface and if we map F conformally onto $|z|<1$, then the set of image of the ideal boundary point set is F_σ on $|z|=1$.*

5. Smoothing process.

Theorem 3. (L. Sario.[8]) *If zero-boundary Riemann surface is divided into* F_0 *and other domains* $F_1, F_2, \dots F_n$, *where* $F_i \frown F_j = 0$, $i \neq j$, *and* F_0 *and* F_i *have common boundary* γ_i, *and in each* F_i *a harmmonic function* u_i *is defined and if*

$$\sum_i^n \int_{\gamma_i} \frac{\partial u_i}{\partial n} ds = 0$$

is satisfied, then there exists a uniform harmonic function f except at the singularity of u_i *in* F_i *such that*

$$D_{F_i}(f - u_i) < \infty .$$

The proof of this theorem is shown in C. R. Paris (1949), p 229.

Chapter II.

6. The Behaviour of the harmonic function and analytic function in the neighourhood of the harmonic measure zero-boundary.

Theorem 4. (R. Nevanlinna[1]).

Let F be a Riemann surface having the relative boundary Γ *and an ideal boundary set R of harmonic measure zero. Let us denote by* $dw = du + idv$ *an uniform differential on F with finite Dirichlet Integral over F. Then there exists a sequence of curves* $\gamma_i = \sum_j \gamma_{ij}$; $i = 1. 2. \dots$ *enclosing R on which*

i) $$\lim \int_{\gamma_i} |dw| = \lim \varepsilon_i = 0 .$$

ii) *if* $u(x)$ *is uniform and if we denote by* F_{γ_i} *the non compact domain bounded by* γ_i, *then*

$$\min_{x \in \gamma_i} u(x) \leq \underline{\lim}_{x \in F_{\gamma_i}} u(x) \leq \overline{\lim}_{x \in F_{\gamma_i}} u(x) \leq \max_{x \in \gamma_i} u(x).$$

iii) *if* $\overline{\lim}_{x \in F_{\gamma_i}} |u(x)| < \infty$, *then* $\underset{F_{\gamma_i}}{D} u(x) < \infty$.

7. Theorem. 5. *If* $\lim_{x \in F} |u(x)| < \infty$ *and uniform, then for the first kind of Stoilow's ideal boundary point p of R*

$$\lim_{x \to p} u(x)$$

exists, but if p is of the second kind, then $\lim u(x)$ *does not necessarily exist.*

Proof. If p is of the first kind,[3] then there exists a neighbourhood $V(p)$ of p which is planer, therefore every $\bar{\gamma}_i$ which is the part of γ_i of theorem 4 contained in $V(p)$ is made of only a proper cut ; $i_0 \geqq i_0(V(p))$.

If we denote by $V^i(p)$ the neighbourhoad of p cut by γ_i and contained in $V(p)$, then

$$\min_{x \in \bar{\gamma}_i} u(x) \leqq \underline{\lim}_{x \in V^i(p)} u(x) \leqq \overline{\lim}_{x \in V^i(p)} u(x) \geqq \max_{x \in \bar{\gamma}_i} u(x)$$

$$\text{but } |\max_{x \in \bar{\gamma}_i} u(x) - \min_{x \in \bar{\gamma}_i} u(x)| \leqq \int_{\gamma_i} |du| = \varepsilon_i : \lim \varepsilon_i = 0$$

hence, $\lim_{x \to P} u(x)$ exists.

8. Example 1.

$$y^2 = \prod_{n=1}^{\infty} \left(1 - \frac{x}{2^n}\right)\left(1 - \frac{x}{2^n + I_n}\right) \; ; \; n \geqq 1$$

I_n is defined afterward: $I_n =$ real.

The Riemann surface F^* of y spread on the x-plane, composed of two sheets and have first order branch points $a_n = 2^n$, $b_n = 2^n + I_n$ on the real axis, and $x = \infty$ is the only singular point of the second kind ideal boundary point. We connect cross-wise the upper and lower sheets at the intervals $S_n = [b_n \; a_n]$.

We denote by F the Riemann surface obtained after cutting two discs $|x| \leqq 1$, from F^*, then F has two boundaries, C_1, C_2 on $|x| = 1$, and zero boundary, Denoting by $U(x)$ the bounded harmonic function with the boundary values 1 on C_1 and O on C_2. F has Green function (next chapter) denoted by $g(x, x_0^1)$, $g(x, x_0^2)$, where x_0^i, $i = 1, 0$ means upper and lower sheets, and x_0 means the projection of x, then by Green's formula

$$u(x^*) = \int_{C_1} \frac{\partial g}{\partial n}(x, x^*) ds,$$

if $x \in S_n$, then

$$U(x_0^1) = \frac{1}{2\pi} \int_{C_1} \frac{\partial}{\partial n} g(x, x_0^1) ds = \frac{1}{2\pi} \int_{C_1} \frac{\partial}{\partial n} g(x, x_0^2) ds = U(x_0^2) = U(x).$$

$$\overline{\lim_{\substack{x \in S_n \\ x \to \infty}}} \; U(x_0^1) = \overline{\lim_{\substack{x \in S_n \\ x \to \infty}}} \; U(x_0^2).$$

$U(x^1)$ is harmonic bounded in the x-plane out of C_1 and $\sum S_n$ with the boundary value 1 on C_1, $U(x)$ on $\sum S_n$, $U(\overset{2}{x})$ is bounded harmonic in the x-plane out of C_2 and with the boundary value 0 on C_2, $U(x)$ on $\sum S_n$.

$\tilde{U}(x) = \underbrace{U(\overset{1}{x}) - U(\overset{2}{x})}$ is bounded harmonic in the x-plane except C and $\sum S_n$ with the boundary value 1 on C, 0 on $\sum S_n$.

By $z=\frac{1}{x}$, we inverse, then $\tilde{U}(x)=\tilde{U}(z)$, $\tilde{U}(z)$ is harmonich bounded in $|z|\leqq 1$.

$$U(z)=1: \quad |z|=1$$
$$U(z)=0: \quad z\in S_n',$$

where $\frac{1}{2^n}=\frac{1}{a_n}=a_n'$, $\frac{1}{b_n}=b_n'$; $S_n'=[a_n'\ b_n']$; where b_n' is defined by the next equation, where C is a constant such as $0<C<\frac{1}{2}$.

$$\frac{(1+a_n'b_n')-\sqrt{(1-a_n'^2)(1-b_n'^2)}}{a_n'+b_n'}$$
$$=C\left(\frac{1}{2}\right)^n\frac{\sqrt{1-a_n'^2}-\sqrt{1-b_n'^2}}{b_n'\sqrt{1-a_n'^2}+a_n'\sqrt{1-b_n'^2}}$$

Let $V(z)=1-\tilde{U}(z)$, then $0\leqq V(z)\leqq 1$.

$$V(z)=0: \quad |z|=1 \quad V(z)=1: z\in\sum S_n'.$$

We shall show that $\varlimsup_{x\to 0} V(0)<1$.

We denote by $\omega_n(z)$ the harmonic function, $0\leqq\omega^n(z)\leqq 1$, in $|z|<1-\sum S_n'$ such that

$$\omega_n(z)=0 \ ; \ z\in C$$
$$\omega_n(z)=1 \ ; \ z\in S_n',$$

then $V(z)\leqq\sum_n\omega_n(z)$, $V(0)\leqq\sum\omega_n(0)$. We map $|z|<1$ on to $|w|<1$ by

$$\frac{2-p_n}{1-p_nz}=w,$$

where $p_n=\frac{1+a_n'b_n'-\sqrt{(1-a_n'^2)(1-b_n'^2)}}{a_n'+b_n'}$,

then
$$b_n'\longrightarrow\beta_n \ : \ -1<\beta_n=\frac{\sqrt{1-b_n^{2\prime}}-\sqrt{1-a_n^{2\prime}}}{a\sqrt{1-a_n^{2\prime}}+a_n'\sqrt{1-b_n^{2\prime}}}<0$$
$$a_n'\longrightarrow a_n \ : \ 0<a_n=\frac{\sqrt{1-a_n'^2}-\sqrt{1-b_n'^2}}{b_n'\sqrt{1-a_n'^2}-a_n'\sqrt{1-b_n'^2}}<1$$
$$\longrightarrow 0'=0'=\frac{-1-a_n'b_n'+\sqrt{(1-a_n'^2)(1-b_n'^2)}}{a_n'+b_n'}.$$

$S_n'=[b_n'\ a_n']\to T_n=[\beta_n\ \alpha_n]$ on real axis. Denoting by $\bar{\omega}_n(w)$ the function which is harmonic and bounded in $|w|<1$. $0\leqq\bar{\omega}_n\leqq 1$; $\bar{\omega}_n(w)=1: w\in$ on the circle of which diameter is $|\alpha_n|=|\beta_n|$, and $\bar{\omega}_n(w)=0$; $|w|=1$, then $\bar{\omega}_n(w)=\frac{-\log|w|}{\log|m_n|}$: $m_n=|\alpha_n|=|\beta_n|$.

$$\bar{\omega}_n(w)>\omega_n(w), \text{ then } C\left(\frac{1}{2^n}\right)=\bar{\omega}_n(0')\geqq\omega_n(0'),$$
$$\sum\omega_n(0')>\sum\omega_n(0)>V(0),$$

but $\tilde{\omega}_n(p') = C\left(\frac{1}{2}\right)^n : C < \frac{1}{2}$, therefore $V(0) < 2C < 1$, finally $1 - V(0) = \tilde{U}(\infty) > 0$, then $\lim_{x \to \infty} |U(\overset{1}{x}) - U(\overset{2}{x})| > 0$. Accordingly $U(x)$ has no limit when x converges to ∞ on $F - \sum S_n$. In reality $x = \infty$ is irregular for Dirichlet problem,[9] and

$$I_n \doteqdot \frac{2 \times 2^n \left(\frac{1}{2}\right)^n}{1 - 3K_n} \doteqdot \frac{C}{2}, \quad \text{where} \quad K_n = \left(\frac{1}{2}\right)^n C.$$

9. **Theorem 6. Generalizaion of the identity theorem.** *Let F be a Riemann surface with relative boundary Γ and an ideal boundary R of a relative harmonic measure zero. If $f(x)$ is in F a regular bounded, and non constant function, then for every ideal boundary point $p \in R$*

1. $\lim_{x \to p} f(x)$ *exists.*
2. *$f(x)$ is continuous in $F + R$.*
3. *For every constant C, the number of roots of the equation $f(x) = C$ in $F + R$ is uniformly bounded.*

Proof. Let us denote by front A, int A and $\bar{A}$, the boundary, interior point and the closure of set A.

Lemma 1. *If F is the Riemann surface satisfying the conditions of Theorem 6, then front $(f(F)) - f(\Gamma)$ is a set of the logarithmic capacity zero.*

Since $f(F): x \in F$ is continuous and bounded, then

front $f(F) +$ int $f(F) = \overline{f(F)} \subset f(\bar{F}) = f(\bar{F} - R - \Gamma) + f(R \cap \bar{F}) + f(\Gamma)$.

Since $f(x)$ is regular, if $p \in F$, then $f(p) \in$ int $f(F)$, therefore,

front $f(F) \subset f(R \cap \bar{F}) + f(\Gamma) \subset f(\bar{F})$.

Let $E =$ front $f(F) - f(\Gamma)$ then, $E \cap f(F) = 0$, $E \cap f(\Gamma) = 0$, and $f(\Gamma)$. is closed.

We suppose that Cap $E > 0$, we denote by E_m the set of E having distance larger than $\frac{1}{m}$ from $f(\Gamma)$, then

$$E = \sum_{m=1}^{\infty} E_m,$$

therefore there is a certain m_0 such as Cap $(E_{m_0}) > 0$, then there is at least one point w_0 of E_{m_0}, such as for any small disc K of which is centre is w_0, Cap $(E_{m_0} \cap K) > 0$, therefore there exists a closed subset E'_{m_0} of E_m havingno common point with the periphery of K and Cap $(E'_{m_0} \cap K) > 0$, and dia $K < \frac{1}{2m_0}$.

Hence $E \in f(F) - f(\Gamma)$, we take a connected piece on K which is denoted by $\underset{w}{F}\underset{m_0}{}$, and hence $\text{Cap}(E'_{m_0}\frown K)>0$, there exists a bounded harmonic function such as $0\leqq U(w)\leqq 1$.

$\overline{\lim}\ U(w)=1 \quad w\in(E'_{m_0}\frown K): \quad U(w)=0: w\in$ boundary of K

To E_{m_0}, a part of Riemann surface F_{m_0} coerresponds which have positive distance from Γ.

$U(x)=U(f^{-1}(w))$; $U(x)$ is harmonic in F_{m_0}.

We denote by $\omega_n(x)$ the harmonic function such as $0\leqq\omega_n(x)\leqq 1$: $\omega_n(x)=0: x\in\Gamma$, $\omega(x)=1: x\in\Gamma^n$ boundary of F_n of exhaustion, then $\omega_n(x)\geqq U(x)$. But R is harmonic measure zero set therefore

$$0\equiv\lim\ \omega_n(x)\geqq U(x)=0, \qquad \text{this is absurd.}$$

10. Corollary. *Let us denote by $v(R)$ the non compact domain containing R in its interior and bounded by the relative boundary γ; $\gamma=\sum\gamma_i$; in the w-plane denote by D_γ the maximal compact domain bounded by $f(\gamma)$, Then*

$$\text{front } f(v)-f(\gamma)=E_v\subset\bar{D}_\gamma\,.$$

We suppose that E_v has at least one point in the exterior of D_γ, it will be denoted by p, as $f(\gamma)$ is closed,

$$\text{dist. } (p.\ f(\gamma))\geqq\delta_0>0.$$

On the other hand there is at least an inner point q of+ $f(V)$ such as dist $|p.\ q|<\frac{\delta_0}{4}$.

We can take a circular neighbourhood $v^*(q)$ of which the radius $<\frac{\delta_0}{4}$ and composed of only inner point of $f(v)$, let us take a non compact and simply connected domain G containing the point at infinity and $v^*(q)$ and denote its boundary by $\mathfrak{B}$ satisfying dist. $(\mathfrak{B}.\ f(\gamma))\geqq\frac{\delta_0}{8}$.

Hence $f(v)$ is compact, ∞ is exterior point of $f(v)$, dimension of $(G\frown f(v))=2$, but $(G\frown\text{front }(f(v)))=\text{front }(G\frown f(v))$.

$$\dim\ \text{front }(G\frown f(v))-\mathfrak{B})=1, \qquad E_v\ni(\text{front }(G\frown f(v))-\mathfrak{B})$$

but Cap $E_v=0$.

This is a contradiction.

Proof of 1. $\bar{D}_\gamma$ is a connected set, because γ is connected by a curve in $v(R)$. Take a point $p\in R$, then there exists a sequence of curves γ_n of theorem 4 on which

$$\lim_n\int_{\gamma_n}|dw|=0\,.$$

We denote by $V_n(p)$ the neighbourhood of p determined by γ_n and denote its boundary by $\bar{\gamma}_n \subset \gamma_n$.

If $x \in V_{n+1}(p)-R$, then $f(x)$ is regular, and D_{V_n} is a connected set We have $\overline{f(V_{n+1}(p)-R)}+E_{V_{n+1}} \subset \bar{D}_{V_n}$.

$$\text{diameter } D_{V_n} \leq \frac{1}{2}\int_{\gamma_n}|dw| = \frac{\varepsilon_n}{2};\ \lim_n \varepsilon_n = 0 .$$

$f(\bigcap_n^\infty V_n(x)) \subset \bigcap_n^\infty \bar{D}_{V_n}$ is only one point. Finally $\lim_{x\to p} f(x)$ exists. It is clear that $f(x)$ is continuous in $F+R$.

11. Proof of 2. Take n points $x_1, x_2 \dots x_n$ in the neighbourhood of Γ and denote by K_i the disc of which the centre is x_i and K_i contains x_{i-1}, x_{i+1} in its interior

We connect x_i and x_{i+1} by a curve so that the image of the curve in the w-plane may be a straight line ${}_il_{i+1}$, all straight lines ${}_1l_2, {}_2l_3 \dots {}_nl_1$ make up a polygon denoted by π. Because, let K_i be a disc with centre p in F denote by K_i' the maximal disc contained in $f(K)$, of which the centre is $f(p)$. Then we can connect p and the other point of $f^{-1}(K_i')$ with a curve of which the image in the w-plane is a straight line.

Let us denote by Π its outer polygon made of the outer side of π, and the neighbourhood of R determined by $f^{-1}(\pi)$ is denoted by V_{π_F} or F_{π_F}; $\pi_F = f^{-1}(\pi)$.

12. **Lemma 2.** $\qquad \Pi \cap f(R \cap F_{\pi_F}) = 0$.

Suppose $\Pi \cap f(R \cap F_{\pi_F}) \ni p$, we take a closed curves π_F' in the neighbourhood of π_F. As $F_{\pi_F} - V_{\pi_F'}$ is compact, therefore $f^{-1}(p)$ is finite number of points $x_1 \cdots x_S$ in $F_{\pi_F} - V_{\pi_F'}$, and take $v_i(x_i)$ of neighbourhood in $F - V_{\pi_F'}$, so that $|f(x)-p| \geq \delta_0 > 0$, if $x \in$ inte $(F_{\pi_F} - V_{\pi_F'}) - \sum^S v_i(x_i)$. In F we construct a non compact domain containing R bounded by relative boundary γ^* contained in $F_{\pi_F} - F_{\pi_F'} - \sum^S v_i(x_i)$ and denote by Π^* the maximal domain bounded by $f(\gamma^*)$ in the w-plane, then dist $(p\ f(\gamma^*)) \geq \delta_0$; $\Pi \supset \Pi^*$, then $p \in$ exterior of Π^*, but $p \in V_{\pi_F}$, this is a contradiction from corollary of Lemma 1.

We denote by $n(w)$ the number of times when w is convered by $f(x)$: $x \in F_{\pi_F}$ and by D_n the set $E[n(w) \geq n]$, this is clearly open relative to π.

Lemma 3. If E_n = *boundary of* $(D_n - (\pi))$ *is not zero* : $n < \infty$, *then* $\qquad$ Cap $E_n = 0$.

Proof. Suppose that Cap $E_n \neq 0$, then the boundary E_n is the set of point which is covered by $f(F_{\pi_F})$ at most $n-1$ times.

We denote by S_i the set which is covered i times by $f(F_{\pi_F})$ exactly and denote by S_{im} the set of S_i which has a distance larger than $\frac{1}{m}$ from π.

$$E_n = \sum_{i=1}^{n-1} S_i = \sum_{i=1}^{n-1} \sum_{m=1}^{\infty} S_{im} .$$

therefore there exists at leasts one point p_0, and a number m_0 and i_0 such as $\mathrm{Cap}(S_{i_0 m_0} \frown K) > 0$ for any small disc K of which the center is p_0, and the closed subset $S'_{i_0 m_0}$ of $S_{i_0 m_0}$ such as ($S'_{i_0 m_0} \frown$ Boundary of $K) = 0$, and Cap $(S_{i_0 m_0} \frown K) > 0$.

Then there exist discs $K_1, K_2, \ldots K_{m_0}$ on p, and there is another disc K_0 which does not cover a positive capacity set $S'_{i_0 m_0} \frown K =$ far from π. Therefore there exists a non constant bounded harmonic function $1 \geqq U(w) \geqq 0$, such as

$$\lim U(w) = 1 \quad : \quad w \in S'_{i_0 m_0} \frown K$$
$$U(w) = 0 \quad : \quad w \in \text{boundary of } K .$$

This is a contradiction (see Lemma 1).

Since from Lemma, $f(R \frown F_\pi) \frown \Pi = 0$, and $f(R)$ is closed; accordingly dist $(f(R), \Pi) > \varepsilon_0 > 0$. Take a point q on Π, and denote by $V_{\frac{\delta}{2}}(q)$ a circular neighbourhood of q of which radius is $\frac{\delta_0}{2}$.

If $(\text{inte } \Pi \frown V_{\frac{\delta_0}{2}}(q) - R) \ni w$, *then* $n(w) < \infty$.

Proof. Suppose $n(w_0) = \infty$, then there exists a sequenc of $x_1, x_2 \ldots$, such as $\lim_i x_i \in R$; $f(x) = w_0$, consequently we have, $f(R) \ni w_0 = \lim_i f(x_i) \in V_{\frac{\delta_0}{2}}(q)$. This is absurd.

13. Since D_n is compact, therefore the outer boundary Π_n of D_n is a continuum contained in π. Assuming that $\Pi_n \ni q$, we take a point p in the neighbourhood of q such as $V(q) \ni p \bar{\in} f(R)$, then $n(p) = n_0 < \infty$, accordingly D_{n_0+1} does not exist in the neighbourhood of q, because if it were not so, $q \in D_{n_0+1}$. As $q \in f(R)$, if we deform π_F into $\pi_{F'}$ in adding a small disc of which the centre is $f^{-1}(q)$, so that $q \in \text{int } \Pi'$. Thus $n(q) = n(p) = n_0$.

The complement of π in the w-plane is composed of a non compact domain and a finite number of compact domains which have no common point and their boundaries are made of π, we denote by $\mathfrak{D}(p)$ the compact domain of the complement of π devided by π and containing p, the boundary of $\mathfrak{D}(p)$ is a subset of π. *Then* $\mathfrak{D}(p) \frown D_{n_0+1} = 0$. Take another compact domain next to $\mathfrak{D}(p)$, and denote by S the common boundary of $\mathfrak{D}(p)$ and π, and take a point q on S. Then for any point t in the neighbourhood $v(q)$ such as $t \in v(q) \frown \mathfrak{D}(p)$, we deform F_π a little into $F_{\pi'}$ so that q and t may be contained in $\mathfrak{D}'(p)$; where $\mathfrak{D}'$ is $\mathfrak{D}(p)$ corresponding to π', then $n'(p) = n'(q)$, where n' means the times when

p, and q are covered by $f(F_{\pi'})$, consequently the difference of $n(q)$ and $n(p)$ by $f(\pi_F)$ is at most m which is the number of times when q is covered by π.

then $$\mathfrak{D} \bigwedge D_{n_{0 J k}} = 0 \quad : \quad k \gtreqless m_0 \ .$$

But the number of domains is finite, therefore $n(w) < M$; for every $w \in F_\pi$.

In reality $n(w)$ is equal to the order of w with respect to π.

Let $f(R) \ni p$, then p may be the image of only M different ideal boundary points.

Proof. Suppose that $p = f(Q_1) = f(Q_2) \ldots f(Q_{M+1}) \ldots . : Q_i \in R$.

Take neighbourhoods $v(Q_i)$ of Q_i, each of them has no common point and their boundaries may be denoted by γ_i, and by $\bar{D}_{\gamma_i}$ the maximal campact domain bounded by $f(\gamma_i)$, then the set of $\bar{D}_{\gamma_i}$ which is not covered by $f(v(Q_i))$ is capacity zero set.

$$\bigwedge_{M+1} f(v(Q_i)) = D,$$

D is covered at least $M+1$ times except at most capacity zero set of D. This is a contradiction.

Consequently the number of roots of the equation $f(x) = C$ is $\leq 2M$ in $\bar{F}_\pi$.

14. **Corollary.**

i. $|f(x)| < +\infty, x \in F+R$, *if the number of root of* $f(x) = C$ *is infinitely many, then* $f(x) \equiv C$.

ii. *Let $f(x)$ be non constant analytic function in F and the number of roots is infinite for at least a value, then $f(x)$ is not bounded and further $f(x): x \in F$ covers almost all point of the w-plane except at most a non dense set for ano small neighbourhood $V(p)$ of $p \in R$.*

15. *Definition. Generalized local parameter of which the center is an ideal point p.* If $|f(x)| < \infty : x \in F+R$, then we can take a small neighbourhood $v(p)$ of p so that $f^{-1}f(p)$ may have only p in $v(p)$, then $f(v(p))$ covers at most finitely many times and $f(v(p))$ is compact then we call $v(p)$ a generalized disc of p and $f(x)$ a generalized local parameter.

Chapter III.

16. Green function on the relative zero-boundary Riemann surface F and its generalization.

Let p be an inner point of F which has harmonic measure zero set ideal boundary R and has relative boundary Γ.

Definition. If $G(x, p)$ is harmonic positive in F except only p where $G(x, p)$ has logarithmic singularity and zero on Γ and its Dirichlet integral on $F - V(p)$ is finite, then $G(x, p)$ is called the *Green function* *of F with pole at p.*

Theorem 7.[7] *If we denote by $x=m(z)$ the mapping function of F onto $|z|<1$, then,*

$$G(x,p)=\sum g(z,s(p)):\ x=m(z),$$

where the summation is taken over all the substitutions of Fuchsoid group, and $g(z,s(p))$ is the Green function of $|z|<1$ with pole at $s(p)$.

Proof. From mes $\sum\limits_i s_i(\Gamma)=2\pi$,

i) $\sum\limits^n g(z,s(p))$ is harmonic in $|z|<1$ except at $\sum\limits_i^n s_i(p)$ hence

$$\int\limits_{\underline{\Gamma}} \frac{\partial}{\partial n}\sum_i^n g(z,s_i(p))ds=\sum^n\int\limits_{\underline{\Gamma}}\frac{\partial}{\partial n}g(z,s(p))ds=$$

$$=\int\limits_{\sum\limits_i^n S_i(\underline{\Gamma})}\frac{\partial}{\partial n}g(z,s(p))ds=2\pi-\underset{n}{\varepsilon}<\int\limits_{|z|=1}\frac{\partial}{\partial n}g(z,p)ds=2\pi\,.$$

$$m(\underline{\Gamma})=\Gamma$$

ii) Let us denote by z_0 a certain point in the circle $|z|<1$, except $\sum\limits_i s_i(p)$, then there exists at least a fundamental domain D_0 which has z_0 at its inner point D_0 or an its boundary.

Case 1. z_0 is in the interior of a fundamental domain denoted by D_0, which has the pole p and denote by α and β the two ends of $\underline{\Gamma}_0$, corresponding to D_0, and connect with curve C_1, z and a and with C_2, z and $s(a)$, where a and $s(a)$ are equivalent and situated on the two arcs of D_0 which are nearest to $\underline{\Gamma}_0$, so that the simply connected subdomain of D_0, bounded by, $\widehat{\alpha a}$, C_1, C_2 and $\widehat{s(a)\beta}$ does not contain p. Then $g(z,p)+(g(z,s(p))$ is invariant with respect to the substitution s. In denoting by G_n the sum $\sum\limits_i^n g(z,s_i(p))+g(z,ss_i(p))$, this is also invariant with respect to s and is harmonic in the circle except the poles $\sum\limits_i^n (s_i(p)+ss_i(p))$. All terms of this series are positive and zero on $\sum\limits_i^n s_i(\Gamma)+ss_i(\Gamma)$. $m(C_1+C_2)$ in F enclose with Γ a compact domain F_{D_0}, and indicate by $\omega(x)$ the harmonic function on F_{D_0} having the boundary values 1 at $m(C_1)+m(C_2)$ and 0 on Γ, $\omega(x)=\omega(m(z))=\omega(z)$ is automorphic in $|z|<1$, therefore by Harnack's theorem, there exists a constant q depending continuously on C_1+C_2 such as

$$\frac{1}{q}G_n(z_0,p)\leqq G_n(z,p)\leqq qG_n(z_0,p):\ z,z_0\in J_1+C_2\,,$$

$$G_n(z,p)=G_n(s(z),p).\qquad \frac{\partial}{\partial n}G_n(z)ds=\frac{\partial}{\partial n}G_n(s(z))ds(s)\,.$$

$$\frac{\partial}{\partial n}\omega(z)ds=\frac{\partial}{\partial n}\omega(s(z))ds(s)\,.$$

By Green's formula we have

$$\int G_n(z)\frac{\partial\omega}{\partial n}ds=\int\omega(z)\frac{\partial}{\partial n}G_n(z)ds\,,$$

where integration is taken on $\underline{\Gamma}_0+C_1+C_2+\widehat{s(a)\beta}+\widehat{a\alpha}$
but

$$\int_{\widehat{\alpha a}+\widehat{S(a)\cdot\beta}} G(z)\frac{\partial\omega}{\partial n}ds=0\,,\qquad \int_{\widehat{\alpha a}+\widehat{(Sa)\cdot\beta}}\omega\frac{\partial G_n}{\partial n}ds=0\,,$$

on the other hand

$$\int_{C_1+C_2}\frac{\partial\omega}{\partial n}ds\gtreqless 0\,,$$

then we have

$$\frac{1}{q}\frac{2\pi-\varepsilon_n}{\int_{C_1+C_2}\frac{\partial\omega}{\partial n}ds}\leq G_n(z_0)\leq q\frac{2\pi-\varepsilon_n}{\int_{C_1+C_2}\frac{\partial\omega}{\partial n}ds}\,,$$

the last inequality holds for every n, therefore $\sum_i^\infty g(z,s_i(a))$ is absolutely bounded except $\sum_{i=1} s_i(p)+p$.

Case 2. $z_0\in$ boundary of a certain fundamental domain D_0, let us denote by δ_0 the side containing z_0, then there exists at least a fundamental domain D_1 which has δ_0 in common with D_0, then we do in the same way in D_0+D_1 as in D_0, in taking D_0+D_1 in the place of D_0.

Therefore $\sum_s g(z,s(p))$ is convergent in $|z|<1$ except at $\sum s(p)$.

iii) $G(z,p)=\sum_s g(z,s(p))$ is evidently automorphic

$$G(z,p)=\sum_s g(z,s(p))=0\,,\quad\text{if}\quad z\in\sum s(\underline{\Gamma})\,.$$

If $z\in\sum s(\Gamma)$, then there exists at least a $s^0(\underline{\Gamma})\ni z$, and $D^0\ni s^0(\Gamma)$ therefore $G_n(z,p)$ is regular in D_0 except only $s^0(p)$, and clearly

$$G_n(z,p)=0\quad\text{if}\quad z\in s^0(\Gamma),\quad\text{so}\quad 0=\lim_n G_n(z,p)=G(z,p)\,.$$

Finally we have that $G(x,p)=G(m(z),p)$ is harmonic positive in F and 0 on Γ, and has logarithmic pole at p.

17. **Lemma 1.** *Let $u(x)$ be harmonic and positive on the non compact part F of zero-boundary Riemann surface with a relative boundary Γ.*

If $$\int_{\Gamma} \frac{\partial u}{\partial n} ds = 0, \quad then \quad D_F(u) < +\infty.$$

Proof. If $D_F(u)=\infty$, by Nevanlinna's theorem u is not bound in F, therefore there exists a sequence of points $p_1, p_2, \ldots, \lim p_i=$ ideal boundary point such as $u(p_i)=M_i$; $\lim_i M_i=\infty$.

Take $M_0 > \max u(z): x \in \Gamma$, and trace a niveau curve C_{M_0} on which $u(x)=M_0$, C_{M_0} devides F into two parts $\overline{C}_{M_0}$ and $\underline{C}_{M_0}$ in which $u(x)$ is $\gtreqless M_0$ respectively.

Proof. Case 1. Every C_{M_i} encloses the compact part with Γ and does not intersect with Γ, hence

$$0=\int_{\Gamma} \frac{\partial u}{\partial n} ds = \int_{C_{M_i}} \frac{\partial u}{\partial n} ds, \quad 0 \geqq \frac{\partial u}{\partial n} \text{ on } C_{M_i},$$

then $$\frac{\partial u}{\partial n}=0 \quad \text{on} \quad C_{M_i},$$

$$\lim_i D_{\underline{C}_{M_i}}(u)=\int_C u\frac{\partial u}{\partial n} ds - \int_{C_{M_i}} u\frac{\partial u}{\partial n} ds = \int_C u\frac{\partial u}{\partial n} ds < \infty.$$

Case 2. Γ and C_{M_i} enclose non compact part.

Let be $\gamma_1, \gamma_2, \ldots$ a sequence of curves enclosing the ideal boundary point and denote by $\tilde{\gamma}_j^i$ the part of γ_j lying in $\underline{C}_{M_i}$, then it is evident that $\Gamma+\tilde{\gamma}_j^i+C_{M_i}$ enclose the compact part $\underline{C}_{M_i}^j$ of $\underline{C}_{M_i}$.

Denoting $\tilde{\omega}_j^i$ the harmonic function in $\underline{C}_{M_i}^j$ having the boundary values 0 and $\Gamma+C_{M_i}$ and 1 on $\tilde{\gamma}_j^i$, and denote by ω_j being harmonic and in $F-V(\gamma_j)$, having the boundary values 0 on Γ and 1 on γ_j, we see directly $0<\tilde{\omega}_j^i<\omega_j$ for every i, j.

Since F has zero-boundary,

$$0=\lim_{j=\infty} \omega_j \geqq \lim_{j=\infty} \tilde{\omega}_j^i = 0.$$

Denote by u_j^i the harmonic function in $C_{M_i}^j$ such as

$$u_j^i = u \quad \text{on} \quad \Gamma+C_{M_i},$$
$$u_j^i = M_i \quad \text{on} \quad \tilde{\gamma}_j^i,$$
$$0 \leqq u_j^i - u \leqq \tilde{\omega}_j^i M_i, \qquad \text{for every } i,$$

therefore u_j converges uniformly in C_{M_i}; $\lim_{j=\infty} u_j^i = u$, but from

$$\lim_{j=\infty} \frac{\partial u_j^i}{\partial n} = \frac{\partial}{\partial n} \lim_{j=\infty} u_j^i = \frac{\partial u}{\partial n},$$

$$\int_\Gamma u\frac{\partial u}{\partial n}ds=\lim_{j=\infty}\int_\Gamma u_j^i\frac{\partial u_j^i}{\partial n}ds=\lim_{j=\infty}D_{\underline{C}_{M_i}^j}(u_j^i)=\lim D_{\underline{C}_{M_i}}(u),$$

this inequality holds for every M_i, then

$$D_F(u)=\int_\Gamma u\frac{\partial u}{\partial n}ds<+\infty.$$

18. **Lemma 2.** *Let C be a proper cut dividing F into two parts; if the part of F bounded by R+C has no pole, then* $\int_C\frac{\partial G}{\partial n}ds=0$.

Proof. Let C' be the image in the fundamental domain in $|z|<1$ and is ending in two points a and b and $s(a)=b$; whene s is a substitution.

Case 1. s is parabolic.

$\overline{C'}=\sum_{n=1}^{\infty}s^{+n}(C')+s^{-n}(C')+C'$ ends in the fixed point of s on $|z|=1$ making a closed curve $\overline{C'}$ not enclosing p by hypothesis, then we have

$$\sum_s s(C')=\sum{}' s(\overline{C'}),$$

where the summation on the left is over all substitution and the right is all over except s. The right side is the sum of closed curves not containing (p) in their interior, finally

$$\int_{C'}\frac{\partial G}{\partial n}ds=\sum_s{}'\int_{s(\overline{C'})}\frac{\partial}{\partial n}g(z,p)ds=0\ .$$

Case 2. s is hyperbolic.

$$\sum_{n=1}^{\infty}s^n(C')+s^{-n}(C')+C'=\overline{C'},$$

is a Jordan curve ending in the two fixed points a and b. $\overline{C'}$ is transformed by other substitution into $s(\overline{C'})$, every $s(\overline{C'})$ has on its outerside the image of Γ, but mes $\sum_s s(\Gamma)=2\pi$, therefore $\sum_s s(C')=\sum_s{}' s(\overline{C'})$ are sum of closed Jordan curves not containing p in their interiors, then we have as in the case 1:

$$\int_{\sum_s s(C')}\frac{\partial G}{\partial n}ds=0\ .$$

From Lemma 1 and 2 $D_{F-V_{(p)}}(x,p)<+\infty$.

19.1. **Green function with its pole on an ideal boundary point.**

Definition. Generalized module. Let γ_i be a proper cut, we define a harmonic function ${}_i\omega_s$ in the surface bounded by γ_i, γ_s and Γ and has the boundary value 0 on Γ, and 1 on $\gamma_i+\gamma_s$, when γ_s converges to ideal boundary set R, ${}_i\omega_s$ is decreasing monotonously, therefore ${}_i\omega_s$ converges to ${}_i\omega$ being non constant harmonic function

$$\lim_{s=\infty}\int_\Gamma \frac{\partial_i\omega_s}{\partial n}ds=\int_\Gamma\frac{\partial}{\partial n}\lim_{s=\infty}{}_i\omega_s\ ds>0\,.$$

Then $\infty>\int\frac{\partial_i\omega}{\partial n}ds>0$ is called generalized module of the surface F bounded by Γ and γ_i,

$$D_{F-V_{(\gamma_i)}}(\omega_i)=\int_{\gamma_i}{}_i\omega\frac{\partial_i\omega}{\partial n}ds=\int_{\gamma_i}\frac{\partial_i\omega}{\partial n}ds=\int_\Gamma\frac{\partial_i\omega}{\partial n}ds\,,$$

and if $F_i\supset F_{i'}$ then $\omega_i>\omega_{i'}$, in putting $N=\dfrac{1}{\int_\Gamma\frac{\partial_i\omega}{\partial n}ds}$, we have $N_i<N_{i'}$.

Definition. Regular ideal boundary point. From Harnack's theorem, for positive harmonic function $u(x)$, there exists a constant q depending on the curve C in the defining domain such that

$$\frac{1}{q}u(x_0)\leqq u(x)\leqq qu(x_0)\ :\ \text{if } x,\ x_0\in C\,.$$

Let us denote q by $q(C)$.

If for an ideal boundary point α, there exists a sequence γ_i of proper cuts enclosing α, on which every positive and finite except α harmonic function must satisfy $q(\gamma_i)\leqq q$, then α is named a regular ideal boundary point.

19.2. **Theorem 7'.** *If α is an ideal boundary point, then we can define a Green function $G(x,\alpha)$, and further if α is a regular point, then $G(x,\alpha)$ is uniquely determined.*

Proof. Take a sequence of point p_i in F, such as ; $p_i\in V_i(\alpha)-V_{i+1}(\alpha)$: $\lim_{i=\infty}p_i=\alpha$ and sequence of Green function corresponding to p_i

$$C(x,p_1),\quad G(x,p_2)\ \ldots\ldots\ldots\ldots\,.$$

So long as p_i is contained in F

$$G(x,p)=\sum_s g(z,s(p))\,.$$

Therefore

$$\int_\Gamma\frac{\partial G}{\partial n}ds=2\pi$$

and from Lemma 2, for every proper cut γ of which the domain bounded does not contain p,

$$\int_{\gamma} \frac{\partial G}{\partial n} ds = 0 .$$

If $p \in V(\gamma)$ and $p \bar{\in} \sum V(\gamma_i)$, then $G(x, p)$ is regular harmonic in the domain bounded by $\Gamma+\gamma$ and $\sum \gamma_i$, accordingly

$$\int_{\Gamma+\gamma+\sum\gamma_i} \frac{\partial G}{\partial n} ds = 0 .$$

Finally

$$\int_{\gamma} \frac{\partial G}{\partial n} ds = 2\pi .$$

As $G(x, p) \geqq 0$, there exists q for such that

$$\frac{1}{q} G(x_0, p) \leqq G(x, p) \leqq qG(x_0, p) ; \ x, \ x_0 \in \gamma ,$$

and if we denote by F_γ the non compact domain not containing p bounded by γ, then

$$\int_{\gamma} \frac{\partial G}{\partial n} ds = 0 ,$$

accordingly

$$D_{F-V(\gamma_i)}(G(x, p)) < \infty .$$

Then

$$\max_{x \in \gamma} G(x, p) \geqq \overline{\lim} \, G(x, p) ; \ x \in F_\gamma ,$$

after all

$$G(x, p_i) \leqq M_{i_0} : x \in F - V(\gamma_{i_0}) , \quad V(\gamma^i) \ni p_i : i \geqq i_0 .$$

We can extract a sequence of $G(x, p_i)$ which converges uniformly in every compact domain contained in F,

$$\lim_{i=\infty} G(x, p_i) = G(x, a) .$$

Then the limit function $G(x, \alpha)$ is clearly non constant and $\int_{\gamma_i} \frac{\partial G(x, \alpha)}{\partial n} ds = \int_{\gamma} \frac{\partial G(x, \alpha)}{\partial n} ds = 2\pi$, because $\lim_i \frac{\partial G(x, p_i)}{\partial n} = \frac{\partial}{\partial n} \lim_i G(x, p_i)$.

We call $G(x, \alpha)$ a Green function also.

20. *The behaviour of $G(x, \alpha)$ in the neighbourhood S of ideal boundary points.*

Case 1. When x converges in the other boundary point α', there exists a number j_0 a proper cut γ_j ; $j \geqq j_0$, such that the non compact part F_{γ_j} of F cut by γ_j do not contain $p_i : i \geqq i_0$, hence by Lemma 1,

$\int_{\Gamma} \frac{\partial G}{\partial n} ds = 0$, then

$$\max_{x \in \gamma_j} G(x, \alpha) \geq \overline{\lim_{x \in F_{\gamma_j}}} G(x, \alpha) .$$

Case 2. We see directly that $G(x, \alpha)$ is not bounded in $V(\alpha)$, if α is a regular ideal point there exist a sequence of p_i such as

$$G(p_i, \alpha) = M_i , \quad p_i \in V(\alpha) : M_i = \infty .$$

We make the curve C_{M_i} on which $G(x, \alpha) = M_i$, these curves are composed of a finite number of closed curves or open curves tending to α and each of them divides F into two parts in which $G(x, \alpha) \gtreqless M_i$ respectively. But in this case C_{M_i} does not tend to the ideal point α, if it were so then there exists a certain γ such as γ intersects the curve C_{M_i} where

$$M_i < \frac{1}{q} N_\gamma 2\pi \ : \ \lim N_\gamma = \infty .$$

This is a contradiction.

Remark. When $p \in F$, $G(x, p)$ is expressed in a uniformly convergent series of Green functions $g(z, s(p)) : |x| < 1$. But when p converges to the ideal boundary set, this loses its meaning, because $|s(p)| \to 1$ as $p \to R$ and all $g(z, s(p)) \to 0$, but $G(x, p) \neq 0$, that is, an ideal boundary is singular point with respect to this series.

21. For the regular boundary point α, there exists a sequence of γ_i on which

$$\frac{\underset{x \in \gamma_i}{\text{Max}} \ G(x, \alpha)}{\underset{x \in \gamma_i}{\text{Min}} \ G(x, \alpha)} \leq q^2 \ : \ x \in \gamma_i .$$

If there were two $G_1(x, \alpha)$ and $G_2(x, \alpha)$, then $\frac{G_1}{G_2}$ is non constant. Let us denote by k_i

$$\text{Min} \ \frac{G_1(x, \alpha)}{G_2(x, \alpha)} = k_i : x \in \gamma_i$$

then, k_i is a constant and $G_1 - k_i G_2 \geqq 0$ in the domain bounded by Γ and γ_i, because other boundary is harmonic measure zero set, then

$$D_{F - V_i(\alpha)}(G_1 - k_i G_2) < +\infty .$$

From the maximum principle k_i is taken on γ_i and

$$k_1 > k_2 > \ldots\ldots \ k_n > \ldots\ldots \ k > \frac{1}{q^2} \geqq 0 .$$

$k \gtreqqless 0$ follows from that

$$\frac{1}{q} N_i \leqq G_1(x, \alpha) \leqq qN_i, \ \frac{1}{q} N_i \leqq G_2(x, \alpha) \leqq qN_i,$$

where N_i means the generalized module of the domain bounded by γ_i and Γ. Let $\varepsilon_n = k_n - k$, then $\lim_{n=\infty} \varepsilon_n = 0$, and there exist x_i on γ_i such as

$$G_1(x_i) - kG_2(x_i) = \varepsilon_i G_2(x_i)$$

$$0 < G_1(x) - kG_2(x) < q\varepsilon_i G_2(x) : \ x \in \gamma_i .$$

is true for every i. That is on γ_i,

$$G_1(x) - kG_2(x) = \circ (G_2(x)) : \text{ as } n \to \infty . \tag{1}$$

But $G(x_i) - k_1 G_2(x_i^*)$ for a certain x_i^* on γ_i, for every i,

$$G_1(x^*) - k_1 G_2(x^*) = \varepsilon_1 G_2(x^*) . \tag{2}$$

(1) and (2) contradict each other for $\varepsilon_1 \leqq \frac{\varepsilon_n}{q}$,

$$\frac{G_1(x)}{G_2(x)} = \text{const. but } \int_\Gamma \frac{\partial G_1}{\partial n} ds = \int_\Gamma \frac{\partial G_2}{\partial n} ds = 2\pi ,$$

finally $G_1(x, \alpha) = G_2(x, \alpha)$.

For closed Riemann surface α is a regular ideal boundary point.

$G(x, \alpha)$ generally depends on the sequence of $p_i : \lim_{i=\infty} p_i = \alpha$ and is not always uniquely determined.

22. **Example 2.** Let us consider the Riemann surface of Example 1. Let us denote by $G(x, 1.5)$ the Green function with pole at 1.5 on the upper sheet, then $G(x, 1.5)$ has no limit. When x converges to on the upper and lower sheets, it has no limit.

Take two sequences p_i, q_i on the upper or the lower sheet such as: $\lim_i p_i = \infty$, $\lim_i q_i = \infty$ and

$$\lim_i G(p_i, 1.5) = A + B = \lim_i G(q_i, 1.5)$$

$$G(1.5, p_i) = G(p_i, 1.5); \ G(1.5, q_i) = G(q_i, 1.5) .$$

Then we have two Green functions

$$\lim_i G_1(1.5, p_i) = G_1(1.5, \infty) + G_2(1.5, \infty) = \lim_i G(1.5, q_i) .$$

Property of Green function 1.

23.1. Let F be a Riemann surface with relative boundary Γ and a harmonic measure zero ideal boundary point set R. We denote by $G(x, p_i)$ the Green function of F with pole p_i: where $p_i \in F + R - \Gamma$, then $G(x, p_i)$ are not always uniquely determined, and denote by G_M^i, $\overline{C}_M^i$, $\underline{C}_M^i$ the niveau curve of $G(x, p_i)$ on which $G(x, p_i) = M$, domain in which

$G(x, p) \geqq M$, and domain in which $g(x, p) < M$, then

$$\int_{C_M^1 *} (Gx, p_1) \frac{\partial G(x,p_2)}{\partial n} ds = \int_{C_M^2 *} G(x, p_2) \frac{\partial G(x,p_1)}{\partial n}$$

Proof. C_M may converge to the ideal point set, we denote them $q_1 \dots q_i \in C_M$, then we take a system of neighbourhood of $q_1 \dots$; $V_k(q_i) \ni q_i$, such as $\bigcap_k^\infty (\sum V_k(q_i)) = R \cap C_M$ and the boundary of $V_k(q_i)$ will be denoted by $\gamma_k(q^i)$, then $\underline{C}_M - \sum V_k(q_i)$ is a compact domain with the boundary Γ, the part $\sum \gamma_k'(q^i)$ or $\sum \gamma_k(q^i)$, and the part ${}_kC_M'$ of C_M.
Denoting by ${}_k\omega_M$ the harmonic measure of $\sum \gamma_M'$ with respect to $(\underline{C}_M - \sum V(q_i)$, then we have

$$ {}_k\omega_M = 1 \quad : \quad x \in \sum_k \gamma' $$
$$ {}_k\omega_M = 0 \quad : \quad x \in \Gamma + C_M' . $$

Let ${}_k\omega$ be the harmonic measure of $\sum \gamma_k$ with respect to $(F - \sum V_k)$, clearly

$$ {}_k\omega_M < {}_k\omega . $$

Hence harmonic measure of R is zero:

$$ \lim_k {}_k\omega_M = 0 , $$

but by Green's formula, we have

$$ \int_{\sum_k \gamma_M'} {}_k\omega_M \frac{\partial G(x,p_i)}{\partial n} ds = \int_{{}_kC_M + \sum \gamma_k'} G \frac{\partial_k \omega_M}{\partial n} ds $$

$$ \lim_{k=\infty} \int_{\sum_k \gamma_M'} \frac{\partial G(x,p_i)}{\partial n} ds = 0 . $$

From
$$ \frac{\partial G(x,p_i)}{\partial n} \geqq 0 : x \in {}_kC_M , $$

$$ \int_{\sum_k \gamma_M' + {}_kC_M'} \frac{\partial}{\partial n} G(x, p) ds = 2\pi , $$

then

$$ \lim_{k=\infty} \int_{\sum \gamma_k' + {}_kC_k'} \frac{\partial}{\partial n} G(x, p_i) ds = \int_{C_M} \frac{\partial}{\partial n} G(x, p_i) = 2\pi . $$

As $G(x, p_i)$ is bounded in the neighbourhood of p_j, $j \neq i$, we have

$$ \lim_{k=\infty} \int_{{}_kC_M{}^i} G(x,p_j) \frac{\partial G(x,p_i)}{\partial n} ds = \int_{C_M{}^i} G(x,p_j) \frac{\partial G(x,p^i)}{\partial n} ds . $$

and by Green's fonrmula we have the conclusion.

Since $G(x, p_i)$ is finite except for p_i,

$$\lim_{M=\infty} \bar{C}_{M^i} = p_i ,$$

$$\lim_{x \to P_j} G\ (x, p_i) = G(p_j, p) ,$$

$$\lim_{M=\infty} \int_{C_M{}^j} G(x,p_i) \frac{\partial G(x, p_j)}{\partial n} ds = 2\pi G(p_j, p_i) .$$

Especially if $p_i \in F-R-\Gamma$,

$$G(p_1, p_2) = G(p_2, p_1) .$$

23.2 **The properties of Green function.** *If $G(x, \alpha)$ is a Green function with pole $\alpha : \lim_i p_i = \alpha \in R$, and $V_2(\alpha) \subset V_1(\alpha)$ are two neighbourhoods and their bounday curves are denoted by C_2, and C_1, then*

$$m_1 = \min_{x \in C_1} G(x, \alpha) \leqq \min_{x \in C_2} G(x, \alpha) = m_2 .$$

Proof. If $m_1 > m_2$, let us take δ such as $m_1 - m_2 > \delta > 0$, then for any small number $\varepsilon < \frac{\delta}{4}$ there exists a number $i_0 = i_0(\varepsilon)$ such as

$$|\min_{x \in C_1} G(x, p) - m_1| < \varepsilon , \quad |\min_{x \in C_2} G(x, p_i) - m_2| < \varepsilon \quad \text{for every } p_i : i \geqq i_0 ,$$

hence $p_i \in F$, $\lim_{x \to p_i} G(x, p_i) = \infty$, taking a small neighbourhood $v(p_i)$ of p_i, then $\underset{F-v(P_i)}{D}(G(x, p_i)) < +\infty$

$$\min_{x \in C_1} G(x, p_i) \leqq \varlimsup_{x \in V_2 - v} G(x, p_i) \leqq \varlimsup_{x \in V_1 - v} G(x, p_i) = \min_{x \in C_2} G(x, p_i)$$

$$m_1 - \varepsilon < m_2 + \varepsilon , \quad \delta < m_2 - m_1 < 2\varepsilon .$$

This is absurd.

23.3 *If $G^*(x, \alpha)$ is the function satisfying the following conditions*

a° $G^*(x, \alpha) \geqq 0 : x \in F$, $G^*(x, \alpha) = 0 : x \in \Gamma$

b° $\min_{x \in C_1} G^*(x, \alpha) \leqq \min_{x \in C_2} G^*(x, \alpha)$: if $V_2(\alpha) \subset V_1(\alpha)$

c° $\int_\Gamma \frac{\partial G^*}{\partial n} ds = 2\pi$, $G^*(x, \alpha) < +\infty : x \in F-R$.

Then for any point $x_0 \in F$, we can choose a sequence of p_i, $\lim_i p_i = \alpha$ such that

$$\lim_{i=\infty} G(x_0, p_i) = G^*(x_0, \alpha) .$$

Proof. Let us denote by C_M the niveau curve on which $G^*(x, \alpha) = M$ and $\bar{C}_M$ such as $G^*(x, \alpha) \geqq M$, then $M_1 < M_2$ it follows that $\bar{C}_{M_1} \supset \bar{C}_{M_2}$

from c°, $$\int_{C_M} \frac{\partial G^*}{\partial n} ds = 2\pi . \quad \frac{\partial G^*}{\partial n} \geqq 0 \quad \text{on} \quad C_M .$$

Since $$\lim_{M=\infty} \overline{C}_M = \alpha : \lim M = \infty ,$$

σ_i^1 is a niveau curve of $G(x, x_0)$, σ_i^2 is a niveau curve of $G^*(x, \alpha)$, then

$$G^*(x_0, \alpha) = \frac{1}{2\pi} \int_{\sigma_i^1} G^*(x, \alpha) \frac{\partial}{\partial n} G(x, x_0)\, ds = \frac{1}{2\pi} \int_{\sigma_i^2} G(x, x_0) \frac{\partial}{\partial n} G^*(x, \alpha) ds .$$

therefore there exists a point p_i on σ_i^2, such as $G^*(x_0, \alpha) = G(p_i, x_0)$, but $\lim_{i=\infty} G(p_i, x_0) = \lim_{i=\infty} G(x_0, p_i) = G^*(x_0, \alpha), = \lim_{i=\infty} G(x_0, p_i)$ and it is clear that

$$\int_{C_M} \frac{\partial}{\partial n} G(x, \alpha)\, ds = 2\pi \neq 0 .$$

24. It is clear that $G(x, \alpha)$ is not bounded in the neighbourhood of α, but not always

$$\lim G(x, \alpha) = \infty ,$$

neverthless we see directly that if $\underline{\lim}_{x \to \alpha} G(x, \alpha) < +\infty$, then from 23, there is a sequence of C_M which converges into α.

Definition. If $\lim_{x \to \alpha} G(x, \alpha) = \infty$, we call it a regular Green function. It is easily seen that there is only regular function on the regular ideal point or inner point.

When an ideal point α has at least one regular Green function, we call α a *regular ideal point for Evans' problem.* This notion is a clearly local property.

Theorem 8. *It is necessary and sufficient for α to be regular for Evans' Problem, that there is a certain neighourhood $V(\alpha)$ and a harmonic function $U(x)$ satisfying the following conditions:*

1. *$U(x)$ is lower bounded in $\overline{V}(\alpha)$* ($\overline{V}$ is V's closure),
2. $U(x) < +\infty \quad x \in \overline{V}(\alpha) - \alpha$,
3. *$U(x)$ is harmonic in $\overline{V}(\alpha) - R$* ,
4. $\lim_{x \to \alpha} U(x) = +\infty$.

The necessity is clear, if $U(x)$ exists we can make a regular Green function. Let $W(x) \equiv 0$; $x \in F - \overline{V}(\alpha)$, by the smoothing process we gain a harmonic function $H(x)$ such as

$$0 < d = \int_\gamma \frac{\partial H}{\partial n} ds = \int_\gamma \frac{\partial U}{\partial n} ds ,$$

where γ is the boundary of $V(\alpha)$,

$$|H-U|<M : \quad x \in \overline{V}(\alpha) , \quad H=0 : x \in \Gamma ,$$

accardingly $\frac{2\pi H(x)}{d}$ is the Green function which we reguire.

25. **Theorem 9.** *If there is a certain neighbourhood $V(\alpha)$ of α in which a one-valued bounded analytic function $f(x) : x \in F-V$ exists, then all points of $V(\alpha) \cap R$ is regular for Evans' problem.*

Proof. By Theorem 7 there is a certain neighbourhood $V'(\alpha) \subset V(\alpha)$ in which $f(x)=f(\alpha)$ has no root except only α. $-\log|f(x)-f(\alpha)|$ is the function of theorem 9, accordingly α is regular for Evans' Problem.

Corollary. If $f(x) : x \in V(\alpha) \cap F$ *remains a second Category set in the w-plane in which $n(w) < +\infty$ then α is regular for Evans' Problem.*

If there is no generalized local parameter for any small $V(\alpha)$ then by theorem 7, $f(x)$ covers in $V(\alpha)$ the w-plane except at most non-dense set, therefore

$$V_1 \supset V_2 \ldots ; \; \bigcap_i V_i = \alpha , \; f(x) \in V_i(\alpha) \cap F$$

covers the w-plane infinitely many times except at most first category set. This is a contradiction.

Corollary. *When the Riemann surface F is given as the covering surface of an other abstract Riemann surface F^*, if F covers finitely many times and all points of F^* have generalized local parameter, then all points of F have generalized local parameter.*

We denote by x^* the projection of $x : x \in F$ on F^*. We define $f(x)=f(x^*)$, this is clearly the generalized local parameter. Especially if we take the w-plane as ground surface, if $V(\alpha)$ has finitely many times covers the w-plane, then all points of $V(\alpha) \in R$ is regular for Evans' problem.

Especially, let F^* be the covering surface, being finitely many sheeted on F, if F has finite numbers of genus then F is representable conformally as a sub-Riemann surface of closed Riemann surface F_c, since all point of F_c are regular for Evans's problem it follows that all point of F^* are regular, however infinitely many times sheeted covering surface on the w-plane F^* may be represented.

The problem whether all ideal harmonic measure zero points are regular for Evans' Problem is quite difficult but it seems very true and admissible.

Extension of Cauchy's integral formula.

26. When a curve C on F converges to the ideal point set R, we call C non compact. If F is bounded by a compact or non compact

curve Γ, and ideal point set, we take a system of neighbourhood V_k such as

$$\sum V_k \supset \bar{F} \cap R \ : \lim_{k=\infty} \sum V_k = \bar{F} \cap R \ ,$$

then $F_k = F - \sum V_k$ is compact domain with the boundary Γ'_k which is a part of Γ and γ'_k which is a part of the boundary $\sum V_k$, we denote by $\omega_k(x)$ a positive bounded harmonic function in F with the boundary value 1 on γ'_k, 0 on Γ'_k, if $\lim_{k=\infty} \omega_k \equiv 0$, then we say that F has zero ideal boundary point.

On the other hand we denote a neighbourhood systems

$V'_k(p) : p \in R \cap \Gamma$, such as $\sum V'_k(p) \supset R \cap \Gamma$, $\bigcap_k \sum V'_k = R \cap \Gamma$,

$\Gamma_k = \Gamma - \sum V_k$, $F'_k = F - \sum V'_k$; $\gamma'_k =$ boundary of $(\sum V'_k \cap F)$

then F'_k satisfies the conditions of theorem 7 then F'_k has Green function $G'_k(x, x_0)$.

Theorem 10. *If $f(x)$ is a one-valued bounded analytic function in* in F, *and* $\left|\frac{\partial f}{\partial n}\right| < +\infty$ *on* $\Gamma - R$, *then*

$$f(x_0) = \lim_{k=\infty} \int_{\Gamma'_k} f(x) \frac{\partial G_k(x, x_0)}{\partial n} ds \ : \ x_0 \bar{\in} \Gamma \ .$$

Proof. Hence F'_k is compact then $G'_k = 0$; $x \in \Gamma'_k + \gamma'_k$. In denoting by C_M the niveau curve of $G'_k(x, x_0)$ then

$$\int_{C_M} \frac{\partial G'_k}{\partial n} ds = \int_{\Gamma'_k + \gamma'_k} \frac{\partial G_k}{\partial n} ds = 2\pi \quad \text{(see Lemma 1 of Nr. 17)}$$

$$\lim_{M=\infty} \bar{C}_M = x_0 \ , \quad \text{by theorem 7} \quad \lim_{x \to x_0} f(x) = f(x_0)$$

and in using Green's formula, we have

$$\int_{\Gamma'_k + \gamma'_k + C_M} f(x) \frac{\partial G'}{\partial n} ds = \int_{\Gamma'_k + \gamma'_k + C_M} G'_k \frac{\partial f}{\partial n} ds$$

Let $\omega'_k(x)$ be the bounded harmonic function in F'_k such as then we have

$$f(x_0) = \frac{1}{2\pi} \int_{\Gamma'_k + \gamma'_k} f \frac{\partial G'_k}{\partial n} ds$$

$$0 \leq \omega'_k(x) \leq 1 \ , \quad \omega'_k = 0 \ : \ x \in \Gamma'_k \ , \quad \omega_k = 1 \ : \ x \in \gamma'_k \ ,$$

then $\omega'_k \leq \omega_k \ : \lim \omega'_k \equiv 0 \ .$

$$\int_{\gamma'_k} \frac{Gg'_k}{\partial n} ds = \int_{\gamma'_k} G'_k \frac{\partial \omega_k}{\partial n} ds \ : \ |G_k| \leq M \ ; \ x \bar{\in} V(x_0)$$

$$\lim_{k=\infty}\int_{\gamma_k'}\frac{\partial G_k'}{\partial n}ds=0 \quad \text{but} \quad \frac{\partial G_k'}{\partial n}\geqq 0 \quad \text{on} \quad \Gamma_k ,$$

then

$$\lim_{k}\int_{\gamma_k'}\frac{\partial G_k'}{\partial n}ds=\int_{\Gamma}\frac{\partial G_\infty'}{\partial n}ds$$

finally

$$\frac{1}{2\pi}\lim_{k}\int_{\Gamma_k'} f\frac{\partial G_k'}{\partial n}ds \text{ exists and equal is to } f(x_0) .$$

Remark. $G_k(x, x_0)$ is not always uniquely determined.

Chapter IV.

Potential theory on the abstract zero-boundary Riemann surface.

27. If we would like to establish the potential theory on the zero-boundary Riemann surface, we must construct the function $X(p, 0, q)$ which has the same role as $\phi\left(\frac{1}{r}\right)$ in $u(p)=\int\phi\left(\frac{1}{r_{pq}}\right)d(q)$ in the general potential theory, and study its fundamental properties which are very useful.

Distance function X. Let H be the disc, that is simply connected, and compact domain of zero-boundary Riemann surface F which is mapped conformally on to the circle $|z|\leqq 1$ of the z-plane, and its centre is denoted by 0. In the preceding we recognized that Green function $G_1(x, p)$ of $F-H$; exists where p is an inner point of $F-H$ or the boundary point R.

And we make the Green function of H with its pole at 0, which is $-\log|z-0|$: $z\in H$

$$\int_{|z|=1}\frac{\partial}{\partial n}G_1(x)ds=2\pi , \quad \text{and} \quad G_2=\log|x| .$$

By smoothing process, we can construct the function $X(x, 0, p)$ such as

$$D_{F-H}(-G_1+X)<\infty$$
$$D_H(-G_2(x, 0)+X)<\infty ,$$

in this process we used an assistant curve Γ which is on the outer side of the boundary of H. Then, in the neighbourhood of $x=0$

$$X(x, 0, p)=-\log|x-0|+U(x) ,$$

where $U(x)$ is harmonic in the neighbourhood of 0, and determine an adequate constant so that

$$\chi(x, 0, p) = |x| + \mathcal{E}(x) : \lim_{x \to 0} \mathcal{E}(x) = 0 .$$

Theorem 11. *After the normalization in the above process, $\chi(x, 0, p)$ is uniquely determined depending only on $G_1(x, p)$ and $G_2(x, 0)$, but not on the assistant curve Γ.*

Proof. If there exist two functions χ_1 and χ_2 which have the same singularty, then

$$D_H(\chi_1 - G_2(x, 0)) + D_{F-H}(x_1 - G_1(x, p)) < +\infty$$
$$D_H(\chi_2 - G_2(x, 0)) + D_{F-H}(\chi_2 - G_1(x, p)) < +\infty$$

$$D_F(\chi_1 - \chi_2) = D_H(\chi_1 - \chi_2) + D_{F-H}(\chi_1 - \chi_2) = \sum_{i=1}^{2} D_F((\chi_1 - G_i) - (\chi_2 - G_i)$$
$$= \sum_{i=1}^{2} D(\chi_1 - G_i) + D(\chi_2 - G_i) - D(\chi_1 - G_i, \chi_2 - G_i) ,$$

on the other hand $D(\chi_1 - G_i, \chi_2 - G_i) \leqq \sqrt{D(\chi_1 - G_i) D(\chi_2 - G_i)}$

finally $D_F(\chi_1 - \chi_2) < +\infty$,

but $\chi_1 - \chi_2$ is uniform, then $\chi_1 - \chi_2 = \text{const}$, but this constant is zero by the normalization at 0.

Remark. we can prove easily that

$$\chi(x, 0, p) = \chi(p, 0, x) \quad : \quad x, p \in F$$
$$\chi(x, 0, p) = \int_\gamma \chi(p, 0, x) \frac{\partial}{\partial n} \chi(x, 0, p)\, ds \quad : \quad p \in R ,$$

where the integration is on the curve γ which is the niueav curve of $\chi(p, 0, x)$.

28.1. **Property** A. *Let us denote by $V(x)$ and $V(0)$ the neighbourhood of x and 0 respectively. Then $\chi|(x, 0, p)| \leqq M(x)$ wherever the parameter point p may be situated, including ideal boundary points, so long as $p \bar{\in} V(x) + V(0)$, where M depends on only x but not on p.*

Case 1. from $\chi(x, 0, p) = \chi(p, 0, x) : p \in F,\ x \in F$

$$|\chi(x, 0, p)| \leqq M(x) \quad \text{if} \quad p \bar{\in} V(x) + V(0) : \quad p \in F ,$$

because $\chi(x, 0, p)$ is harmonic in $F - V(p) - V(0)$, $|\chi(x, 0, p)| \leqq \max |\chi(x, 0, p)|$ on the boundaries of $V(x)$ and $V(0)$, (see Nevanlinna's theorem) Case 2. when p converges in the set of ideal boundary point α, let us fix x at present.

If $p \in V(\alpha) : \alpha \in R$, then there exist two neighbourhoods $V(x)$ and $V(0)$ such as if $p \bar{\in} V(0) + V(x)$ then, $\underset{F-V(0)-V(x)}{D}(x, 0, p) < +\infty$, it follows that

$$\overline{\lim_{p \in F - V(0) - V(x)}} |\chi(x, 0, p)| \leqq \max |\chi(x, 0, p)| \text{ on the boundary of } V(0) \text{ and } V(x).$$

Property A′. If $x \bar{\in} V(p) + V(0)$, then

$$|\chi(x, 0, p)| \leqq M(p) < +\infty .$$

This can be proved in the same way.

28.2. **Property** B. *In the neighbourhood of a positive pole p,*

$$\chi(\lambda - p, 0, p) = \log \frac{1}{|\lambda - p|} + U_p(\lambda),$$

where the distance is in the local parameter, and $U_p(\lambda)$ is a harmonic function depending on p, and $U_p(\lambda)$ is continuous with respect to p, that is to say for every positive number ε, there exists a circle ${}_pC_d$ with a diameter d and the centre at p and δ such as if $|p-p'|<\delta$, then $|U_p(\lambda) - U_{p'}(\lambda)| < \varepsilon : \lambda \in {}_pC_d \cap {}_{p'}C_d : d$ is larger than δ.

Proof. It can be considered that $\chi(x, 0, p)$ is made by smoothing process from two Green functions G_1 and G_2

$$G_1(x, 0, p) = \log \frac{1}{|x-p|} + \tilde{U}_p(x) : x \in {}_pC_d .$$

First we prove that $\tilde{U}_p(x)$ is a continuous function with respect to p. $G(x, p)$ seems that which is made by smoothing process from

$U_0 = \log \frac{1}{|x-p|}$ in ${}_pC_d$, and $\omega_0=0$ in $F-H-{}_pC_d$ with assistant curve Γ traced in ${}_pC_d$.

ω_1 is a harmonic function in the domain bounded by Γ_0 and Γ, and $\omega_1 = 0$ on Γ_0, $\omega_1 = U_0$ on Γ; where Γ_0 is the boundary of H.

$D_{{}_pC_d}(U_1 - U_0) < +\infty$, $U_1 = \omega_1$ on ${}_pC_d$'s periphery.

Let $S_1 = U_1 - U_0 : S_1$ being harmonic in ${}_pC_d$

....

$S_n = U_n - U_{n-1}$; $\omega_n = U_n$ on C's periphery ; $\omega_n = 0$ on Γ_0

$$T_n = \omega_n - \omega_{n-1} .$$

If $|p-p'|$ is so small that $||U_0 - U_0'| < \varepsilon$, on Γ, then $|\omega_1 - \omega_1'| \leqq \varepsilon$

$$|S_1 - S_1'| \leqq L|\omega_1 - \omega_1'|$$

$$|\omega_2 - \omega_2'| \leqq K|S_1 - S_1''| \leqq KL|\omega_1 - \omega_1'|$$

$$|S_2 - S_2'| \leqq L|\omega_2 - \omega_1| \leqq L^2K|\omega_1 - \omega_1'| \qquad K, L, < 1$$

in general

$$|S_n - S_n'| \leqq L^n K^{n-1}|\omega_1 - \omega_n'|, \quad |\omega_n - \omega_n'| \leqq L^{n\ 1} K^{n-1}|\omega_1 - \omega_1'|,$$

after all
$$G(x, p) = U_0 + \sum S_n = U^\infty : \quad x \in {}_pC_d ,$$
$$G(x, p) = U_0' + \sum S_n' = U'^\infty : \quad x \in {}_{p'}C_d ,$$

therefore
$$\tilde{U} = \sum S_n , \quad \tilde{U}' = \sum S_n' .$$

$$\sum S_n - \sum S'_n \leqq \sum |S_n - S'_n| \leqq M\varepsilon : M = M(K, L,) < +\infty \text{ in } {}_pC_a \cap {}_{p'}C_a$$

$$|\tilde{U} - \tilde{U}'| \leqq M\varepsilon \quad \text{in} \quad {}_pC_a \cap {}_{p'}C_a,$$

in the same way we have $|{}_pU(x) - {}_{p'}U(x)| \leqq M\varepsilon$.

29. The function $X(x, 0, p)$ is harmonic when $x \in F$ except at 0, where it is negatively infinite and p, where it is positively infinite ($p \in F$) or a regular ideal point, but if p is not a regular boundary point the function has no limit but in all cases $\frac{1}{2\pi}\int \frac{\partial X}{\partial n} ds = 1$. This fact means that a positive mass one is distributed on p in the sense of the potential theory, then we can define ideal mass on the ideal boundary point.

Definition. Mass distribution μ. Mass distribution is so defined as in the general potential theory, μ is defined for the set in the Riemann surface and its regular boundary (or all ideal boundary point set). The family of which μ is defined must be additive class and μ is completely additive. The corn of mass distribution is defined in the same manner (of course μ is invariant with respect to conformal mapping).

Then the potential will be defined as the Lebesgue-Stieltjes-Radon integral

$$u(x) = \int X(x, 0, p) d\mu(p).$$

The value of the function $X(x, 0, p)$ is not determined only by the distance $|x-p|$ as $\phi\left(\frac{1}{r}\right)$, but it depends on the location of x, and further distance is not defined in the Riemanian surface in general except locally, so the potential defined with X is not homogenous.

We must verify to what extent the properties of general potential will hold. We see directly

1° at $x = 0$ $\quad u(x) = \int X(x, 0, p) d\mu(p) = \log|x| \int d\mu(p)$,

and from the properties A and A′, $u(p)$ is harmonic, continuous and finite wherever no mass is scattered, for instance at inner point of F or at the boundary, if only x is not situated in the corn of mass distribution.

2° Hence $u(p) = \lim_{N=\infty} \int X^N(p, 0, Q) d\mu(Q)$, $u(p)$ is lower semi-continuous,

where

$$X^N \begin{cases} = X, & \text{if } X \leqq N \\ = N; & \text{if } X > N. \end{cases}$$

3° From the definition of integral which expresses the potential, it is necessary and sufficient for the potential to be bounded and continuous on the closed set T not containing $x=0$, that there exists a circles of radius δ, so as the potential engendered by the mass containded in this circle C_δ is $<\varepsilon$, for any positive number ε.

30. **Theorem 12.** **(G. C. Evans. F. Vasilesco)** *Let μ is zero out side of the closed set T not containing $x=0$. If the potential is continuous as the function defined on T where the mass is distributed, then it is also continuous in F.*

This theorem will be proved by means of the property B.

Proof. In the sequel the distance is assumed that it is defined by the local parameter.

From B $\chi(x, 0, p)=\log \dfrac{1}{|x-p|}+U_p(x) : x\in {}_pC_d$, where ${}_pC_d$ is the center at p of radius d and moreover $U_p(x)$ is continuous with respect to p in this circle. We denote by $u_\delta(p)$ the potential engendered by the mass in the circle with its center at p and radius δ. As $u(p)$ is contInous as the function on the corn of mass distridution, we can choose 2δ and d for any positive number ε so that the following condition may be satisfied, at any point Q of the corn.

i. $|U_p(x)-U_Q(x)|\leqq\varepsilon<\frac{1}{2}\log 2 : \text{if } x\in C_d(Q): d>2\delta\,;\ |p-Q|\leqq 2\delta,$

ii. $|U_Q(Q+h)-U_Q(Q)|\leqq\varepsilon : 0\leqq h\leqq 2\delta,$

iii. $\log\left|\frac{1}{2\delta}\right|\geqq 2U_Q(Q) \quad : \quad \text{if } |U_Q(R)|>5,$

$\qquad\geqq 10 \quad : \quad \text{if } |U_Q(Q)|\leqq 5,$

iv. $$|U_{2\delta}(Q)|\leqq\frac{\varepsilon}{1+\dfrac{\frac{3}{2}\log 2}{U_Q(Q)-\frac{1}{2}\log 2}} \quad : \quad \text{if } |U_Q(Q)|>5,$$

$$|U_{2\delta}(Q)|\leqq\frac{\varepsilon}{1+\frac{3}{2}\log 2} \quad : \quad \text{if } |U_Q(Q)|\leqq 5.$$

Take a point p of F, and denote by u_δ the potential engendered by the mass in the circle with the centre at p of radius δ.

If the distance between p and the set of corn K of the mass distribution is larger than δ, then $u_\delta=0$, and if this distance is smaller than δ, then $C_\delta(p)$ is contained completely in a circle $C_{2\delta}(Q): Q\in K$.

Let Q be a certain point of K which is nearest to p, then

$$|u_\delta(p)| \leqq \int_{K \cap C(p)} \chi(p, 0, M) d\mu(M) \leqq \int_{K \cap_2 C(p)} |\chi(p, 0, M)| d\mu(M)$$
$$= \int \frac{|\chi(p, 0, M)||\chi(Q, 0, M)|}{|\chi(Q, 0, M)|} d\mu(M).$$

$$\frac{\chi(p, 0, M)}{\chi(Q, 0, M)} = \frac{\log \frac{1}{|pM|} + U_p(M)}{\log \frac{1}{|QM|} + U_Q(M)} = 1 + \frac{\log\left|\frac{QM}{pM}\right| + U_p(M) - U_Q(M)}{\log\left|\frac{1}{QM}\right| + U_Q(M)}$$

$$m \begin{cases} m = 1 + \dfrac{\frac{3}{2}\log 2}{U_Q(Q) - \frac{1}{2}\log 2} & : \text{ if } U_Q(Q) > 5, \\ m = 1 + \dfrac{\frac{3}{2}\log 2}{4} & : \text{ if } U_Q(Q) \leqq 5, \end{cases}$$

therefore $|u_\delta(p)| \leqq m \left|\int |\chi(Q, 0, M)| |d\mu(M)|\right| < \varepsilon$, consequently $u(p)$ is is continuous.

Remark. $\left|\frac{MQ}{Mp}\right| < 2$, because their proportion is greatest when M is situated at the point where C_δ and the extension of QP intersect each other.

From the property of $\chi(x, 0, p)$, $u(p)$ is sub-harmonic except at 0, and satisfies all conditions of logarithmic potential, accordingly energy integral, problem of equilibrium, sweeping out process, capacity and the transfinite diameter will be defined in the same manner and all theorems of general potential theory will hold so long as we consider only the set (point, curve, domain) of inner point of F being different from 0.

But it is neccessary and interesting to consider the problem regarding the ideal boundary point set R, where $\chi(x, 0, p)$ is not always determined uniquely.

It is well known that the equilibrium problem becomes easy in the case of logarithmic potential by making use of Green function, so called Robin's problem.

31. **Robin's problem.** *Let D be a domain compact or not (i.e. bounded by ideal boundary point set) composed of a finite number of domains D_i satisfying the following conditions: 1°. The boundary of D_i are analytic curves Γ_i or more generally regular curves for Dirichlet problem, 2°. Every D_i does not contain the point 0. 3°. Every curve Γ_i does never converge to any ideal boundary point.*

Theorem 13. *The Equilibrium Problem is soluble with respect to D.*

Let us denote by $g_{F-D}(\zeta, 0): \zeta \in F$ the Green function of $F-D$ with pole at 0 and call γ_D *Robin's constant of D*. Defined by the next formula

$$\gamma_D = \lim_{x\to 0} g_{F-D}(x, 0) - \log \frac{1}{|x-0|},$$

where $g_{F-D}(\zeta, 0) =$ Green function of $F-D$ with pole at 0. Hence

$$u(\zeta, 0, x) = -X(\zeta, 0, x) + g_{F-D}(\zeta, x) - g_{F-D}(\zeta, 0)$$

is regular harmonic in $F-D$ and finite in the neighbourhood of ideal boundary point contained in $F-D$, and $u = X$ on $\sum_i = \Gamma$.

By Green's formula

$$u(\zeta, 0, x) = \frac{-1}{2\pi}\int_\Gamma X(x, 0, \zeta^*) \frac{\partial}{\partial n} g_{E-D}(\zeta^*, \zeta)\, ds,$$

$$\frac{\partial g_{F-D}}{\partial n} \geqq 0, \quad -\frac{1}{2\pi}\int \frac{\partial}{\partial n} g_{F-D} ds = 1,$$

and hence $\qquad \lim_{\zeta\to 0}(g_{F-D}(\zeta, x) + X(\zeta, 0, x)) = \gamma_D$.

Case 1. If $x \in F-D-\gamma_D + g_{F-D}(\zeta, 0) = \dfrac{-1}{2\pi}\displaystyle\int_\Gamma X(\zeta, 0, \zeta^*) \frac{\partial}{\partial n} g_{F-D}(\zeta^*, 0) ds.$

Case 2. if $x \in \Gamma$, then $g_{F-D}(\zeta, x) = 0$

$$\gamma_D = \frac{1}{2\pi}\int_\Gamma X(\zeta, 0, \zeta^*) \frac{\partial}{\partial n} g_{F-D}(\zeta^*, 0)\, ds.$$

Case 3. If $x \in D$, $g_{F-D}(\zeta, x)$ is cancelled.

$$\gamma_D = \frac{1}{2\pi}\int_\Gamma X(\zeta, 0, \zeta^*) \frac{\partial}{\partial n} g_{F-D}(\zeta^*, 0)\, ds.$$

Here the potential $u(x)$ engendered by the positive mass distribution $\frac{\partial}{\partial n} g_{F-D}(\zeta^*, 0)$ is continuous $(= \gamma_D)$ on Γ, where the mass is distributed, on account of the theorem 14, $u(x)$ is continuous in F except 0.

The behaviour of $u(x)$ in the neighbourhood of the ideal boundary. We see directly that $u(x)$ is bounded in absolute value depending on D, on acconnt of property A, especially in side of C with respect to D, by Nevanlinna's theorem $u(x) = \gamma_D$, becase $u(x)$ is harmonic except 0 and Γ and R where $u(x)$ is finite, therefore $u(x)$ is the potential being the solution of Robin's problem.

Or more precisely we take Γ_i' near Γ_i surrounding E which is the part of R contained in D, since $x \dot{\in} V(0)$, $u(x) \geqq M(D_i) > -\infty$ in D and $\displaystyle\int_{\Sigma\Gamma_i'} \frac{\partial U}{\partial n} ds = 0$, then by Lemma 1 of Nr. 7, $D_{D_i}(u) < +\infty$, so we have

$u(x)=\gamma_D : x \in D.$

From Case 1, we have

$$\min -\chi(x,0,\zeta) \leqq g_{F-D}(0,x)-\gamma_D \leqq \max -\chi(x,0,\zeta) : \zeta \in \Gamma$$

this follows that the theorems about the z-plane set in R. Nevanlinna's[1] (pp. 111-129) will be proved in the zero-boundary Riemanian surface.

Green's function of F,

$H \subset F_1 \subset F_2 \ldots \lim_i F_i = F$ is the exhaustion of F, and denote by g_n the Green function of F_n with pole at 0, then

$$g_1 < g_2 < g \ldots\ldots .$$

If F is a zero boundary Riemanian surface, then $\lim_n g_n \equiv \infty$. This equivalency with $\lim_n \omega_n \equiv 0$, is proved by P. J. Myrberg[4]. Let us denote by γ_n the Robin's constant of $F-F_n$, then capacity of $F-F_n$ is defined by $C=e^{-\gamma_n}$: where $F-F_n$ is non compact.

Clearly $F_i \subset F_j$ follows $g_i < g_j$ and $\text{Cap}(F-F_i) > \text{Cap}(F-F_j)$.

Finally $\lim_n \text{Cap}(F-F_n)=0$ is equivalent with $\lim \omega_n \equiv 0$ and $\lim_n g_n \equiv \infty$.

32. Let F be the Riemann surface with relative boundary Γ_0 and relative harmonic measure zero ideal boundary point set R.

Denote by $G(x,p)$ the Green function of F with pole at p, then we can discuss the potential defined with $G(x,p)$ as in the case of $\chi(x,0,p)$

$$u(x)=\frac{1}{2\pi}\int G(x,p)d\mu(p),$$

$u(x)$ has property A, A', B and its lower semi-continuity and theorem of Evans-Vasilesco is valid.

If the mass distribution μ is zero on Γ_0 and on R, then we call

$$I(\mu)=\frac{1}{2\pi}\int G(p,q)d\mu(p)d\mu(q),$$

the *energy integral*[10] with respect to μ.

33.1. Let D be a compact domain of F not containing Γ_0 on $\bar{D}, F$, and the boundary of D is regular for Dirichlet Problem, then there exists a positive mass distribution μ_D^* on D, of which total is 1 and the energy integral is minimum, so called the equilibrium distribution, in this case μ is zero out of the boundary of D and the potential engendered by this distribution is constant on D which is equal to $I_D(\mu^*)$.

This is proved by using the following properties in the same way as in general potential theory[6]:

1. If $\mu_D = \lim_n \mu_{nD}$, then $I_D(\mu) \leq \underline{\lim} I_D(\mu_n)$.
2. $U(x)$ is harmonic in F except the corn of mass distribution μ.
3. $U(x)$ is lower semi-continuous in $F-R$.

If D is not compact, denote by $\bar{D} \bigwedge R$ the subset of R which is contained in $\bar{D}$. Let be $G_1, G_2, \ldots$ a sequence of closed domains enclosing $\bar{D} \bigwedge R$, such as

$$\bigwedge_i G_i = \bar{D} \bigwedge R ,$$

and every boundary of G_i never converges to R, then we define

$$I_D(\mu^*) = \lim_i I_{D-G_i}(\mu^*)$$

for $I_D(\mu^*)$ decreasing function of set.

33.2. **Lemma 1.**

$$I_D(\mu^*) = \lim_i I_{D-G_i}(\mu^*) = I_{D-G_1}(\mu^*) = I_{D-G_2}(\mu^*) \ldots$$

If we denote by Γ_D the outer boundary of D, then

$$U_i(x) = \int_{D-G_i} G(x \cdot p) d\, \mu^*_{D-G_i}(p) = I_{D-G_i}(\mu^*) ,$$

$U_i(x)$ is harmonic in $F-D$, but $\displaystyle\int_{\Gamma_D} \frac{\partial U_i}{\partial n} ds = 1$.

Since

$$D_{F-D}(U_1(x) - U_i(x)) = (I_{D-G_1}(\mu^*) - I_{D-G_i}(\mu^*)) \int_{\Gamma_D} \frac{\partial}{\partial n}(U_1(x) - U_2(x)) ds = 0 ,$$

$$U_1(x) \equiv U_i(x) \ : \ x \in F-D \ :$$

therefore

$$I_{D-G_1}(\mu^*) = \lim_i \ I_{D-G_i}(\mu^*) .$$

33.3. **Transfinite diameter**[9].

We denote by ${}_D D_n$, the transfinite diameter of order n of non-compact set D.

$$\frac{1}{{}_D D_n} = \frac{1}{2\pi} \lim_j \frac{1}{{}_n C_2} (\min_{p_s; p_t \in D - G_j} \sum_{s<t}^{n} G(p_s \cdot p_t)) .$$

For ${}_{D-G_j} D_n$ is monotonously decreasing with respect to j.

Lemma 2.

$$\frac{1}{{}_D D_n} = \frac{1}{{}_{D-G_1} D_n} = \frac{1}{{}_{D-G_2} D_n} = \ldots\ldots$$

If we denote by C and C_i the boundary of D and G_i respectively then all $p_i (i=1, 2, 3, \dots)$ lie on C.

If it were not so, there is at least one point p_0 on C_j such as

$$\frac{1}{{}_{D-G_i}D_n} = \frac{1}{2\pi}\frac{1}{{}_nC_2}\Big(\sum G(p_0 \cdot p_s) + \sum_{\substack{s<t \\ s\neq 0}} G(p_s \cdot p_t)\Big) \quad \text{is minimum.}$$

Hence $\lim\limits_{p\to p_s} G(p \cdot p_s)$, and $U(p_0) = \sum\limits_s G(p_0 \cdot p_s) < +\infty$, $p_0 \in F - \sum\limits_s v(p_s)$: where $v(p_s)$ is a neighbourhood of p_s, and R is harmonic measure zero therefore $U(p_0)$ takes its minimum on C, this is a contradiction, accordingly we have the conclusion.

${}_DD_n$ is monotonously decreasing with n.

Then $\lim {}_DD_n = {}_DD$ is called the *transfinite diameter of D.*

Lemma 3. *From general potential theory*[3]

$$\frac{1}{{}_{D-G_i}D} = I_{D-G_i}(\mu^*): \qquad \textit{for each } i,$$

then

$$\frac{1}{{}_DD} = I_D(\mu^*)\,.$$

Lemma 4. *We denote by $\omega_D(x)$ the bounded harmonic function of $F-D$; such that $0 \leqq {}_D\omega(x) \leqq 1$, $\omega_D(x) = 0: x \in \Gamma_D$, $\omega_D(x) = 1: x \in C$, and let*

$$w_D(x) = \frac{\omega_D(x)}{\int\limits_{\Gamma_0} \frac{\partial \omega_D(x)}{\partial n} ds}, \qquad W_D = \frac{1}{\int\limits_{\Gamma_0} \frac{\partial \omega_D(x)}{\partial n} ds},$$

Then $w_D(x)$ is constant for $x \in C$ and $\int\limits_{\Gamma_0} \frac{\partial \omega_D(x)}{\partial n} ds = 1$.

We easily see that

$$I_D(\mu^*) = W_D\,.$$

Let
$$V_n(M) = \frac{\sum\limits_{i=1}^{n} G(M \cdot p_i)}{2\pi n} : M \in C\,.$$

Since C is closed $V_n(M)$ is lower semi-continuous, $V(M)$ attains its minium M_n on C which is denoted by $V(p_1, p_2, \dots p_n)$. We take $p_1, p_2, \dots p_n$ on C so that $V(p_1, \dots p_n)$ may be its upper bound ${}_DV_n$ and p_i; $i = 1, 2, \dots, n$ and M converge

$$p_i \to p_i^\infty\,; \quad i = 1, 2, \dots, n\,.$$
$$M \to M^\infty.$$

$$V^* = \frac{1}{2\pi n}\left(\sum_{i=1}^{n} G(M \cdot p_i^\infty)\right),$$

then

$$\underset{M \in C}{V^*}(M) \geqq {}_D V_n .$$

Lemma 5. $${}_D V_n \geqq \frac{1}{{}_D D_{n+1}}{}^{6)}$$

$$\frac{1}{{}_D D_{n+1}} = \frac{1}{2\pi}\frac{1}{{}_{n+1}C_2} \min\left(\sum_{\substack{s<t \\ p_s \in C}} G(p_s \cdot p_t)\right)$$

$$= \frac{1}{4\pi\, {}_{n+1}C_2} \sum_{=1}^{n+1} \sum_{i \neq}^{n+1} G(p_i \cdot p_s),$$

then $${}_D V_n \geqq \frac{1}{2\pi} \min_{p \in C}\left(\sum_{i=1}^{n+1} G(p \cdot p_i)\right).$$

p in the right hand term is p_1, otherweise $\frac{1}{{}_D D_{n+1}}$ cannot be minimum, because $G(p_j \cdot p_s) = G(p_s \cdot p_j)$.

We have $\underline{\lim}_{M \in D} V^*(M) \geqq \min_{M \in C} V^*(M) \geqq {}_D V_n \geqq \frac{1}{{}_D D_{n+1}}$.

Let A be closed set contained in $F + R - \Gamma_0$, and denoted by D_i domains such as

$$\bigcap_i D_i = A .$$

We define ${}_A D_n$, $I_A(\mu^*)$, and W_A in the following manner:

$$ {}_A D_n = \lim_i {}_{D_i} D_n ,$$
$$ I_A = \lim_i I_{D_i} ,$$
$$ W_A = \lim_i W_{D_i} .$$

If A is harmonic measure zero, then $W_A = \infty$.

Theorem. *If A is harmonic measure zero set, then*

$$I_A = \infty,\ D_A = 0,\ W_A = \infty \quad \textit{and vice versa.}$$

34. **Theorem 14. (G. C. Evans)**[6] *Let A be a closed and a set of relative harmonic measure zero of $F+R$ with respect to F, and let every point of A be regular for Evans Problem. Then there exists a positive harmonic function satisfying the following conditions,*

1°. $U(x) \geqq 0 : x \in F$. 2°. $U(x) = 0 : x \in \Gamma_0$. 3°. $\int_{\Gamma_0} \frac{\partial U}{\partial n} ds = 1$.

4°. $\lim_{x \to A} U(x) = \infty$. 5°. $\lim_{x \to R-A} U(x) < +\infty$.

Proof.

$$\frac{1}{D_A}=\frac{1}{2\pi}\lim_j\lim_i\lim_n\min_{\substack{p_s,\,p_t\in D_j-G_i\\ s\neq t\\ s<t}}\sum_{s<t}\frac{G(p_s\,.\,p_t)}{{}_nC_2}$$

$$=\lim_j I_D(\mu^*)=\lim_j W_{D_j}=\infty\,.\quad \lim_j\frac{1}{{}_{D_j}D}=\infty\,.$$

Therefore, for every number N, there exists $j_0(N)$ such as

$$\frac{1}{{}_{D_j}D}=\lim_n\ \underline{\min}\ \frac{1}{{}_nC_2}\sum^n G(p_s\,.\,p_t)\geqq 2N:\qquad j\geqq j_0(N)$$

therefore there exists $n_0(D_j)=n_0(N)$ such as

$$\frac{1}{{}_{n+1}C_2}\min\sum^{n+1}_{p_s\,.\,p_t\in\ boundary\ of\ D_j}G(p_s\,.\,p_t)\geqq N:\quad n\geqq n_0(N)\,.$$

But since $D'\supset D''$, it follows that $\frac{1}{{}_{D'}D_{n+1}}\leqq\frac{1}{{}_{D''}D_{n+1}}$ for every $n+1$.

We can choose adequately $n+1$ points $p_1^j,\dots,p_{n+1}^j$ on the boundary $D_j: j\geqq j_0(N)$ so that

$$\xi_n^j(x)=\frac{1}{2n_n}\sum_{i=1}^n G(x\,.\,p^j)\geqq N: x\in\overline{D}_j\,,$$

because harmonic measure of $D_j\frown R$ is zero.

Since $D_j,\ D_{j+1},\dots,\quad \lim D_n=A$, and since A is closed, we can choose from the sequence of systems $((p_1^j, p_2^j,\dots,p_n^j),\ j=1,2,3\dots$ a subsequence of $i>i_0$

such as $\qquad (p_1^{nj}\,.\,p_2^{nj}\dots p_n^{nj})$

$$\lim_j p_k^{nj}=p_k: p_k\in A: k=1,2,\dots,n\,.$$

Since from the hypothesis regular Green functions exist at p_k ($k=1,2,\dots,n$), we denote them by $G^*(x\,.\,p_k)$.

Let
$$\eta_n(x)=\frac{1}{2n}\sum_{k=1}^n G^*(x\,.\,p_k)\,,$$

Since $\lim_{x\to\sum p_k}\eta_n(x)=\infty$, there exists a system of neighbourhoods $v(p_k)$ whose boundary is γ_k curve in F, satisfying the following condition:

$$\eta_n(x)\geqq N;\qquad \text{if}\qquad x\in\sum_k^n v(p_k)\,.$$

We choose a subsequence $\{\xi_n^{nj'}(x)\}$ from $\{\xi_n^{nj}(x)\}$ such as

$$p_k^{nj'}\in v(p_k): k=1,2,\dots,n\,.$$

Then $\xi^{n'_j}(x)$ has the properties,

1°. $\int_{\Gamma_0} \frac{\partial \cdot \xi_n^{n'_j}}{\partial n} ds = \frac{1}{2}$. $\xi_n^{n'_j}(x) \leqq 0$. $\xi_n^{n'_j}(x) = 0 : x \in \Gamma_0$.

2°. $\xi_n^{n'_j}(x) \leqq m(x)$ if $x \bar{\in} \sum(p_k) + (R) \bigwedge A$, for every n'_j .

3°. $\underline{\lim}_{x \to \sum v(p_i)} \xi_n^{n'_j}(x) \geqq N$.

From 2° $\xi_n^{n'_j}(x)$ constitute a normal family in F, we can choose a subsequence which converges uniformly in F to the limit function $\xi_n(x)$

$$\lim_j \xi_n^{n''_j}(x) = \xi_n(x).$$

The boundary of $(D_{j_0} - \sum_{k=1}^{n} v(p_k)) = C_{j_0} + \sum^{n} \gamma_k$

Since $D_{j_0} - \sum^{n} v(p_k)$ has no mass for $\xi_n^{n''_j}(x)$, then

$$\overline{\lim}_{x \in D^j_0 - \sum v(p_n)} | \xi_n^{n''_j}(x) - \xi_n(x) | \leqq \max_{x \in C^j_0 + \sum \gamma_k} | \xi_n^{n''_j}(x) - \xi_n(x) |,$$

therefore $\xi_n^{n''_j}(x)$ uniformly converges to $\xi_n(x)$ in $D_{j_0} - \sum^{n} v(p_k)$.

Then, $\underline{\lim}_{x \to A \cap D_{j_0} - \sum v(p_n)} \xi_n(x) = \lim_{j''} \underline{\lim}_{x \to A \cap \bar{D}_{0j} - \sum v(p_n)} \xi_n^{n''_j}(x) \geqq N$.

Let $\zeta_n(x) = \xi(x) + \eta_n(x)$, then $\zeta_n(x)$ has next properties,

1. harmonic positive when $x \in F - \sum^{n} p_k$.

2°. $\int_{\Gamma_0} \frac{\partial \zeta_n}{\partial n} ds = 1$ $\zeta_n(x) = 0 : x \in \Gamma_0$

3°. $\overline{\lim}_{x \to R - A} \zeta_n(x) < +\infty$

4°. $\underline{\lim}_{x \to A} \zeta_n(x) \geqq N$.

We denote this function by $\zeta^N(x)$.

Take $N \geqq 3$, $N^1, N^2, \dots, N^n$: $\lim_n N^n = \infty$, and corresponding to N^i,

$$\frac{1}{2}\zeta^N(x), \quad \frac{1}{2^2}\zeta^{N^2}(x), \dots, \quad \frac{1}{2^i}\zeta^{N^i}(x), \dots$$

$$U^n(x) = \sum_i^n \frac{\zeta^{N^i}(x)}{2^i}, \quad U(x) = \lim_n U^n(x).$$

$U(x)$ has the properties mentioned in the theorem.

1° and 2° are clear, because if $F - R \ni p$, $U(x)$ is harmonic and

$U(p) < \infty$.

If $p \in R$, $p \bar{\in} A$, there exists j_0 such as $p \in D_j$; $j \geq j_0$ therefore there is a neighbourhood $v(p)$, $v(p) \wedge D_{j_0} = 0$, but $v(p)$ has no mass, then

$$\lim_{x \in v(p)} U(x) \leq \max_{x \in \text{ boundary of } v(p)} U(x) < +\infty.$$

$\lim U(x) \geq U^n(x)$, for every n.

$$\varliminf_{x \to A} U(x) \geq \varliminf_{x \to A} \lim_n U_n(x) \geq \lim_n \sum_i^n \varliminf_{x \to A} \frac{1}{2^i} \zeta^{N^i}(x) \geq \lim \frac{\left(\frac{N}{2}\right)^n}{\frac{N}{2}-1} = \infty .$$

If F is a zero boundary Riemann surface, let us denote by 0 an inner point of F and take a dise of centre 0 then by the smoothing process and normalization of constant we easily have the harmonic function $U(x)$ satisfying the next conditions:

1°. $U(x) = \log |x|$ *in the nighbourhood of* 0 *and x is the local parameter.*

2°. $\lim_{x \to A} U(x) = \infty$.

3°. $|\lim_{x \to R-A} U(x)| < \infty$, $\int_\gamma \frac{\partial U}{\partial n} ds = 1$: γ curve enclosing 0 or A.

We can discuss with χ in the same manner as G.

35. *Let F be a Riemann surface with the relative boundary Γ and the ideal boundary point set R, If there exists a harmonic function $U(x)$ such that*

$$U(x) \geq 0 , \qquad \lim_{x \to R} U(x) = \infty .$$

Then R is of a set of relative harmonic measure zero.

Proof. Let us denote by C_M the niveau curve on which $U(x) = M$, and by $\bar{C}_M$ the domain in which $U(x) \gtreqless M$ respectively. If $\infty > M \geq \max_{x \in \Gamma} U(x)$, then C_M is compact curve and surrounds R, and $\lim_{M=\infty} \bar{C}_M = R$.

On the other hand we dedote by $\omega_M(x)$ the bounded positive harmonic function such as

$$\omega_M(x) = 1 \quad : \quad x \in C_M ,$$
$$\omega_M(x) = 0 \quad : \quad x \in \Gamma ,$$

then $\frac{\partial \omega_M}{\partial n} \leq 0$ on C_M, where normal derivative is inner direction with respect to C_M,

$$0 \leq \frac{\partial \omega_{M+\delta}}{\partial n} \leq \frac{\partial}{\partial n} \omega_M : x \in \Gamma ; \text{ if } \quad \delta \geq 0 .$$

By Green's formula,

$$\int_{\Gamma + C_M} \frac{\partial U}{\partial n} \omega_M \, ds = \int_{\Gamma + C_M} U \frac{\partial \omega_M}{\partial n} ds$$

then

$$\int_{\Gamma} \frac{\partial U}{\partial n} ds = \int_{C_M} \frac{\partial U}{\partial n} ds = \int_{\Gamma} U \frac{\partial \omega_M}{\partial n} ds = M \int_{C_M} \frac{\partial \omega_M}{\partial n} ds ,$$

but

$$\left| \int_{\Gamma} \frac{\partial U}{\partial n} ds - \int_{C_M} M \frac{\partial \omega_M}{\partial n} ds \right|$$

is bounded for every M, on the other hand $M \to \infty$ and $\frac{\partial \omega_M}{\partial n} \int \geqq 0$ then $\lim_{M=\infty} \int_{C_M} \frac{\partial \omega_M}{\partial n} ds = 0$, this follows that $\lim_{M=\infty} \int_{\Gamma} \frac{\partial \omega_M}{\partial n} ds = 0$,

then

$$\lim \omega_M = 0 .$$

Thus R is a set of relative harmonic measure zero.

Corollary. *Let F^* be a Riemann surface of which every ideal boundary point is regular for Evans's problem of zero boundary, and let F be a covering surface of F^*. We denote by $n(p)$ the number of times when p is covered by F and by $D_n(F^*)$ the set*

$$D_n = E[n(p) \geqq n] .$$

If $\sup_{p \in F^*} n(p) \leqq N$, then it is necessary and sufficient for F to be of zero boundary, that $D_N(F^*)$ is a zero boundary Rienann surface.

Proof. The necessity is clear. We denote by F' the sub-Riemann surface which has its projection on $D_N(F^*)$, and denote by 0 an inner point of $D_N(F^*)$. We can construct a harmonic function $U(x^*) : x \in F^*$ such as negatively infinite at 0 and positively infinite at every point of the boundary of $D_N(F^*)$ and let

$$U(x) = U^*(x^*) \ ; \ x \in F', \ x \in D_N(F^*) .$$

Then from 32, F' has zero boundary, accordingly, $F (F \supseteqq F')$ is of zero boundary Riemann surface.

Chapter IV.

Function theory on an abstract Riemman surface.

36. The function theory of the z-plane has made much progress but in the Riemann surface, it is in infancy, this owing to the fact that the z-plane has very adequate metric when $z = \infty$ is the only essential singularity and even when the set of singularity is not one point but of capacity zero set, the same metric in a sense can be constructed by the benevolance of Evans' theorem of the potential theory. On the contrary in the Riemann surface, there is no adequate metric except conformal

distance from the automorphic function theory, that is an invariant metric with respect to Fuchsoid group named hyperbolic metric.

It is clear that the metric defined by the Evans' theorem is the best to study the function theory of Riemann surface, as in the z-plane.

We must begin with the notion of regularity of the function at x: $x \in F$, we call that $f(x)$ is regular at $x = x_0$, when $f(x)$ is regular with respect to the local parameter defined in the neighbourhood of x_0, then the notion of regularity will be defined at every inner point of F, nevertheless it loses its meaning at an ideal boundary point α, and when we can prolong the Riemann surface F so that F may be contained as an inner point in the prolonged surface, we can define regularity as the preceding, for instance, if F has a finite number of genus then F is contained in a closed Riemann surface[2], in this case the essential difference about the notion of the regularity or the singularity of the function between the z-plane and in the Riemann surface does not occur.

But the fatal distinction between the z-plane and the Riemann surface is that there can exist the genuin ideal boundary point, which cannot be inner point in the other surface by no means as the second kind boundary point of Stoilow, at this point the notion of regularity or the singularity loses its meaning completely. Therefore in the theory of function on the Riemann surface, there is two cases when $f(x)$ is not regular, one of them is the case when $f(x)$ is not regular with the local parameter defined in the neighbourhood of $x = x_0$, and the other case is when $f(x)$ has no local parameter. When there is no local parameter at $x = x_0$, we call that $f(x)$ has a *genuine* singular point at $x = x_0$. The behaviour of the function in the neighbourhood of a genuine singular point is most complicated, it can take any value without condition perfectly. But the theorem 6 shows the behaviour of the function to some extent. Thus even at a genuine singular point, we can define the regularity of $f(x)$ in the following manner, Let us denote by $V(p)$ a neighbourhood of p, 1°. We say that $f(x)$ is regular at p in extended meaning, if $f(x)$ is regular in $V(p) \cap F$ and finite in $V(p)$, and $f(x)$ is meromorphic in $V(p)$ extended meaning in the case when $f(x)$ is a rational function of regular function 2°. If $f(x)$ is not meromorphic at p, then we say that $f(x)$ has an essential singular point in extended meaning. We see directly that $f(x)$ is meromorphic at p then the number of roots of the equation $f(x)$ C is finite and if $f(x)$ is essential singular, then $f(x)$ covers almost all the w-plane excepe at most a non dense set infinitely many times for any small neighbourhood of p.

Hypothesesis

In the sequel we presuppose that all boundary ideal point are regular for Evans' problem.

The First fundamental theorem of Nevanlinna.

37. Let $f(x)$ be one-valued and meromorphic function on the zero-boundary Riemann surface and denote by E the set of genuise or essential singular point set, then we see directly that $E \subset F+R$ and closed. If $F-E$ is zero-boundary Riemann surface, we say that E is capacity zero set, If E is capacity zero set, and all point of E is regular for Evans' probrem, take an inner point denoted by 0 of $F-E$ and call it the origin.

By Theorem 14 there exists a harmonic function $U(x)$ which is negatively infinite at 0 and positively infinite at E and only there, take a conjugate function $h(x)$ of $U(x)$

$$z = e^{U(x)+ih(x)} = re^{i\theta} : 0 \leq r < \infty.$$

This parameter corresponds to z, $0 < |z| < \infty$, in the z-plane. Let C_r be the niveau curue $r(x) = \text{const } r$, then C_r consists of a finite number of Jordan curves surrounding E, we remark that

$$\int_{C_r} d\theta(x) = \int_{C_r} \frac{\partial U}{\partial n} ds = 2\pi ,$$

where ds is the arc length on C_r and n is the inner normal of C_r, we use the same notation in R. Nevanlinna's book.

As $w(x)$ is meromorphic, it is expressed in a power series with respect to the local parameter t in the neighbourhood of $x = x_0$ and the function z is one valued in the neighbourhood.

$$w(x) = c_{-k} t^{-k} + c_{-k+1} t^{-k+1}, \dots , c_0 + c_{k'} t^{k'} + c_{k'+1} t^{k+1} + \dots \cdot$$
$$z(x) = d_0 + d_\lambda t^\lambda + d_{\lambda+1} t^{\lambda+1}, \dots \qquad : x \in F-E$$

$w(x)$ can have a finite number of negative power terms but $z(x)$ is finite in $F-E$ accordingly has no negative power terms exact at 0.

The differential $\frac{dw}{dt}dt$, $\frac{dz}{dt}dt$ have the next transformation in the change of localparameter from t to τ.

$$\frac{dw}{d\tau} = \frac{dw}{dt} \frac{dt}{d\tau} , \qquad \frac{dz}{d\tau} = \frac{dz}{dt} \frac{d\tau}{dt} .$$

then $\left|\dfrac{\frac{dw}{dt}}{\frac{dz}{dt}}\right| = \left|\dfrac{dw}{dz}\right|$ is one-valued and meromorphic function we denote it by $|w'(x)|$. In the neighbourhood of 0,

$$x = e^{U(x)+ih(x)} \quad = e^{\log t^n} \quad + \varepsilon = t^* + \varepsilon ,$$

where t^* is the local parameter used in the normalization of constant of $U(x)$ or $X(x, 0, p)$ in the neighbourhood of 0,

then $$\frac{dz}{dt} = 1 \quad \frac{dw}{dt} = \frac{dw}{dz} \quad \text{at} \quad x = 0.$$

Theorem 1. *Let the domain bounded by C_r be denoted by Δ_r,*

$$n(r, a) = \text{the number of zero point of } w-a \text{ in } \Delta_r .$$

$$m(r, a) = \frac{1}{2\pi}\int_{C_r} \overset{+}{\log}\frac{1}{|w-a|} \quad d\theta(x) .$$

$$N(r, a) = \int_0^r \frac{n(r. a)}{r} dr .$$

Then

$$m(r, a) + N(r, a) = T(r,) + \varphi(r),$$

where $\varphi(r) \leq \log |a| + |\log |c|| + \log 2$ *and* c *is the first non-vanishing coefficient of the Taylor's expansion of w-a in the neighbourhood of* 0 *with respect to* t^*.

Proof. $U(x)$ is is one-valued, C_r does never intersect other $C_{r'}$, and C_r is composed of a finite number of analytic, compact and closed curve. They enclose a compact domain which is denoted by Δ_r therefore Δ_r has only finite number of zero or pole of $w(x)$, we denote by a , b_ν and by k_ν, h_ν their multiplicity, we assume that C_r has no a, b on it.

If $g_r(x, b)$ is the Green function of Δ_r with pole at b, then $\log |w(x)| -\sum h_\nu g_r(x, b_\nu) - \sum k_\nu g_r(x, a_\nu)$ is regular harmonic in Δ_r and $\log |w(x)|$ on C_r, accordingly by Green's formula

$$\log |w(x_0)| = \int_{C_r} \log |w(x)| \frac{\partial g_r(x, x_0)}{\partial n} ds + \sum h_\nu g_r(x_0, b_r)$$
$$- \sum k\, g_r(x_0, a\ ).$$

We put $x = 0$,

$$\log |w(0)| = \int_{C_r} \log|w(x)| d\theta + \sum h_\nu g_r(0, b_\nu) - \sum k_\nu g_r(0, a_\nu).$$

If $w(0) = 0$ or ∞ ,

$w(x) = c_\lambda t^{*\lambda} + c_{\lambda+1}\ t^{*\lambda+1} \ldots$ in the neighbourhood of 0. Since $\log |w(x)| - \lambda \log |z| = \lambda \log |t^*| + \log c_\lambda - \lambda \log |t^*| + \varepsilon$, and $|\log z|$ is $\log r$ on C_r

$$\log |c_\lambda| = \frac{1}{2\pi}\int_{C_r} \log |w(x)| d\theta + \sum h_\nu g_r(0, b_\nu) - \sum k_\nu g_r(0, a_\nu) + \lambda \log r\,.$$

On the other hand $g(0, a) = g(a, 0) = -U(a) + \log r = -\log r_a + \log r$, where $U(a) = \log r_a$ then we have the theorem.

We denote by K the Riemann of diameter 1 contacting the w-plane at $w = 0$, we put the cordal distance,

$$[a, b] = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}\,.\quad m^*(r, a) = \frac{1}{2\pi}\int_{C_r} \log \frac{1}{[w, a]} d\theta\,.$$

If we denote by $A(r)$ the area on K, which is covered by $w(x)$ when x varies in Δ_r then

$$m^*(r, a) + N(r, a) = \frac{1}{\pi}\int_0^r \frac{A(t)}{t} dt + \log \frac{1}{[w(0), a]}\,.$$

For if we denote by $|d\sigma|$ the line element on the w-sphere

$$|d\sigma| = \frac{|dw|}{1+|w|^2}\,.$$

$$U(w) = \log \frac{|dw|}{|d\sigma|} = \log |1+|w^2|| = 2\log |w| + \varepsilon\left(\frac{1}{w}\right) : \lim_{W=\infty} \varepsilon\left(\frac{1}{w}\right) = 0\,.$$

$$\Delta_2 v(z) = \Delta_2 U(w(x)) = \Delta_w \left|\frac{dw}{dz}\right|^2 \quad \text{in an invariant,}$$

$$ds\frac{\partial v}{\partial r} = \frac{\partial v}{\partial t}\frac{\partial t}{\partial r} ds \quad \text{is an invariant also,}$$

$df_z =$ area element with respect to z.

In the same manner as R. Nevanlinna,

$$r\frac{d}{dr}\int_{C_r} V(z) d\theta + 4\pi\, n(r, \infty) = 4\int \frac{|w'(x)|^2}{(1+|w|^2)} df_z$$

where $A(r) = \int_{\Delta_r} \frac{|w'(x)|^2}{(1+|w|^2)^2} df_z$ is the area on K when x varies in Δ_r, we integrate between r_0 and $r (0 < r_0 < r < \infty)$ and make r_0 converge to 0.

Then we have

$$\log \frac{1}{2\pi}\int_{C_r} \log\sqrt{1+|w^2(re^{i\theta})|^2} d\theta + N(r, \infty) = \frac{1}{\pi}\int_0^r \frac{A(t)}{t} dt + \log\sqrt{1+|w(0)|^2}$$

If we transform w into w_1, by $w_1 = \frac{1+\bar{a}w}{w-a}$, which this is the rotation of the Riemann sphere, then

$$\frac{1}{2\pi}\int_{C_r}\log\frac{1}{[w(re^{i\theta}),a]}+N(r,a)=\frac{1}{\pi}\int_0^r\frac{A(t)}{t}dt+\log\frac{1}{[w(0),a]}.$$

If $w(0)=a$ the right hand term is infinite but we cannot now replace other term as in the case of the z-plane because z is not uniform generally and has infinite number of periods corresponding to genus. Accordingly we assume that $w(0)\neq\infty$, which is always possible. We denote $T(r,\infty)=\int_{C_r}|w|d\theta+N(r,\infty)$. $T^*(r,\infty)\frac{1}{\pi}\int_0^r\frac{A(t)}{t}dr$. It is clear that

$$|T(r,\infty)-T^*(r,\infty)|\leqq\log 2+\log\frac{1}{[w(0),\infty]}.$$

Theorem 2. *$T^*(r)$ is a monotone and convex function of log r.*

Theorem 3. $$\underline{\lim}\frac{T(r)}{\log r}>0.$$

If $\lim_{r=\infty} T(r)=0$, $w(x)$ must reduce to a constant.

Proof. $$\int_0^r\frac{A(t)}{t}dt>\int_{r_0}^r\frac{A(t)}{t}dt>A(r_0)\log\frac{r}{r_0}.$$

Theorem 3'. *If $f(x)$ has an essential singularity (classical or extended meaning), then*

$$\underline{\lim}\frac{T(r)}{\log r}=\infty.$$

Let p be an essential singularity and $V_1\supset V_2\ldots\supset V_n,\ldots\to p$ be a sequence of neighbourhoods coverging to p and e_n be the set of values ommitted by $f(x)$ in V_n, then e_n is non dense, so that

$$e=\sum e_n \qquad \text{is of first category.}$$

Hence there exists a point w_0, which does not belong to e, then w_0 is convered by $f(x)$ infinitely many times about p so that

$$\underline{\lim}\frac{T(r)}{\log r}=\infty.$$

38. **Meromorphic functions defined in a compact domain in the zero-boundary Riemann surface.** Let D be a compact domain in the zero boundary surface bouded by Jordan curve C, then by theorem 11 we can construct the domain function $U(x)$ which is negatively infinite at an inner point 0 of D. Accordingly we can discuss the function theory as in the case $0\leqq|z|\leqq1$ in the z-plane.

39. **Meromorphic functions in a neighbourhood of a closed harmonic measure zero ideal bounbary point**[3].

Let D be a domain in the Riemann surface (positive or zero boundary) boundes by Jordan curve C and closed set E of $F+R$ of relative harmonic measur zero, with respect to domain D.

We easily have all theorems studies about the behaviour of function in the z-plane. Since harmonic measure $E=0$, by theorem 15 we can construct a harmonic function $U(x)$ satisfying all conditions of $U(x)$.

Then we write results without proof because it is the same in the z-plane.

Theorem 1′. *First fundamental theorem of Nevanlinna.*

$$T(r, a) = T(r) + 0(\log r), \text{ where } T(r) = \frac{1}{\pi}\int_{r_0}^{r} \frac{A(t)}{t} \quad dt$$

Theorem 4′. *Second fundamental theorem of Nevanilnna*[8]. *Let e be a bounded closed set of positive capacity on K. Then we can distribute a positive mass* $d\nu(a)$ *on e, such that*

$$\int_e \log \frac{1}{(w.\, a)} \quad d\nu(a): \quad \int_e d\nu(a) = 1\,,$$

is bounded on K, hence by Theorem 1′,

$$T(r) = \int N(r, a) d\nu(a) + 0(\log r),$$

and the order is defined by the formula $\overline{\lim} \dfrac{\log T(r)}{\log r} = \rho$,

Theoerm 3′. $\qquad \underline{\lim} \dfrac{T(r)}{\log r} > 0\,.$

E is singular point set, but in the Riemann surface, E can consist of only genuine singular points where $w(x)$ may have its behaviour as if it were regular point therefore we cannot expect that $w(x)$ is not bounded in the neighbourhood of E. But further if we suppose that E has at least an essential singular (classical or extended meaning) point, we can conclude that

$$\lim_{r=\infty} \frac{T(r)}{\log r} = \infty\,,$$

as in the same way used in Theorem 3.

40. **Some consequences of Fundamental theorems.**

Theorem[9] **5′.** *Let D be a part domain which is bounded by Jordan curves C and by a closed set of F+R of relative harmonic measure zero* *lying inside of C, and let w(x) be one-valued and meromorphic in D and* $\lim_{r=\infty}\frac{T(r)}{\log r}=\infty$ *(or E has at least an essential singular (classical or extended meaning) point. Then,*

1°, *w(x) takes any value infinitely many times, except a set of capacity zero. More precisely,* $\overline{\lim}\frac{N(r,a)}{T(r,a)}=1$, *except values of capacity zero.*

2°. *If further w(x) is of finte order* $\rho>0$ *and* $x_n=x(a_n)$ *is the zero point of* $w-a$ *and* $r_n(a)=r(z_n,)$ *then*

$$\sum_{n=1}^{\infty}\frac{1}{[r_n(a)^{\rho+\varepsilon}]}<\infty: \quad \textit{for all } a.$$

$$\sum_{n=1}^{\infty}\frac{1}{[r_n(a)^{\rho-\varepsilon}]}=\infty,$$

except values of capacity zero, where ε *is any positive number.*

41. **Theoorem 6.** (**W. Gross**)[7]. *Let* $w=w(x)$ *be one valued and meromorphic in F−E have at least one essential singular point of E, and let* $x=x(w)$ *be its inverse function.*

i. *If* $x(w)$ *is regular at w, then we can continuate* $x(w)$ *analytically on half lines;* $w=w_0+re^{i}\theta(0\leq r<\infty)$ *indefinitely, except for values or measure zero.*

ii. *If* $w(x)$ *is regular on a segment;* $w=w_0+r^{i\theta}(\leq r_0<r<r_1)$, then starting from $w=w_0+r$, *we can continuate* $x(w)$ *analytically along circles;* $w=w_0re^{i\theta}(-\infty<\theta<\infty)$ *indefinitely, except for r values of measre zero.*

42. **Theorem 7.** (**Cartwright-Noshiro**)[12]. *From Theorem 6 under the same condition as Theorem 5′ without* $\underline{\lim}\frac{T(r)}{\log r}=\infty$. *Let* $x=x(w)$ *be the inverse function of* $w=w(x)$ *and F be its Riemann surface spread over the w-plane and* (w_0) *be its boundary point, whose projection on the w-plane is w. Then* (w_0) *is an accessible boundary point and* w_0 *is asymptoic values of* $w(x)$, *i.e., there exists a curve L in D ending at a point on E, such that* $w(x)\rightarrow w_0$, *when* $x\rightarrow x_0$ *along L.*

Theorem 8. *Under the same condition as 5′ and if E has further at least one essential singularity, and if* $f(x)\neq a$ *in D, then there exists a curve L in D ending at a point x on E such that* $w(x)\rightarrow a$, *when* $x\rightarrow x_0$ *along L.*

43. **Direct transcendental singularity.**

Let $x(w)$ *be defined on a Riemann surface* F_w *spread over the w-plane and* (w_0) *be a boundary point of* F_w, *whose projection on the w-plane* *is w. F. Iverseu called (w_0) a direct transcendental singularity of $x(w)$* if w_0 is lacurary for a connecten piece F_0 of F which has (w_0) as its b oundary and lies above a disc K; $|w-w_0|\leqq\rho$ about w_0.

Theorem 9. *Under the same condition as Theorem 5′, the set of projection of direct transcendental singularity of the iverse function $x(w)$ of $w(x)$ on the w-plane is of capacity zero.*[13]

44. **Behaviour of the inverse function $x(w)$ of $w(x)$ at its transcendental singularity.**

Let $w=w(x)$ satisfy the same condition as theorem 5′ and $x=x(w)$ be its inverse function and F be its Riemannian surface spread over the w-plane. Let (w_0) be its boundary point, whose projection on the w-plane is w_0.

A *δ-neighbourhood* U of (w_0) is defined by a connected piece of F which lies above a disc $(w-w_0)<\delta$ and has (w_0) as its boundary point, let U correspond to a domain Δ on the Riemanian surface, then $[w, w_0]<\delta$ in Δ and $[w(x)-w_0]=\delta$ on the boundary $\wedge$ of Δ, except the point on E. Since (w_0) is an accessible boundary point of F, there exists a curve on F ending at (w_0), which corresponds to a curve L in Δ ending at a point x, on E. Let $z=r(x)e^{i\theta(x)}$ be defined as theorem 14 and the part of Δ, such that $r(x)\leqq r$ and $r(x)=r$ be denoted by Δ_r and θ_r respectively, let K be the Riemanian sphere of diameter 1, which touches the w-plane at $w=0$ we put $n(r, a)=$ the number of zero point of $w(x)-a$ in Δ, where $[a-w_0]<\delta$,

$$N(r, a, \Delta)=\int_{r^0}^{r}\frac{n(r, a ; \Delta)}{r}dr ,$$

$$m(r, a\, \Delta)=\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{[w(x), a]}d^{\theta}(x) ,$$

$T(r, a ; \Delta)=m(r, a ; \Delta)+N(r, a ; \Delta)$,

$A(r ; \Delta)$, $S(r ; \Delta)$ are the same for Δ.

$L(r)$ the sum of length of the curves on K, which corresponds to **Theorem 1′**.

Theorem 1″.

$$T(r, a ; \Delta)=T(r, \Delta)+0(\int_{r^0}^{r}\frac{L(r)}{r}dr)+0(\log r) ,$$

$$T(r, \Delta)=\int_{r^0}^{r}\frac{S(r ; \Delta)}{r}dr ,$$

where

$$L(r)=0(\sqrt{T(2r,\Delta)\log r}) \quad : \quad for\ all\ r\geqq r_0.$$
$$L(r)=0(\sqrt{T(r:\Delta)\log T(r:\Delta)},$$

except certain intervals I_n *such that*

$$\sum_{n=1}^{\infty} d\int_{I_n}\log\log r<+\infty,$$

the order being
$$\rho=\overline{\lim_{r=\infty}}\frac{T(r:\Delta)}{\log r}.$$

Theorem 3''.
$$\underline{\lim}\frac{T(r:\Delta)}{\log r}<0,$$

and if Δ *is bounded by* E *which containg at last one essential singular (classical or extended meaning) point, then*

$$\underline{\text{ilm}}_{r=\infty}\frac{T(r;\Delta)}{\log r}=\infty.$$

More generally, if

$$\underline{\lim}_{r=\infty}\frac{T(r;\Delta)}{\log r}=\infty,$$

then

i. $w(x)$ *takes only values in* $[w,w_0]<\delta$ *infinitely many times in* Δ, *except a set of values in* $[w,w_0]<\delta$ *of capacity zero.*

ii. *If further* $w(x)$ *is of finite order* $\rho(>0)$ *in* Δ, *then*

$$\left.\begin{aligned}&\sum\frac{1}{[r_i(a)]^{\rho+\delta}}<\infty.\\&\sum\frac{1}{[r_i(a)]^{\rho-\delta}}=\infty,\end{aligned}\right\} \quad : for\ all\ a\ in\ [w,w_0]<\delta.$$

except values in $[w-w_0]<\delta$ *of capacity zero, where* ε *is any small positive number and* $r(a_n)=r(x_n)$, x_n *being the zero point of* $w-a$ *in* Δ.

45. **Applications to the theory of the cluster set.**

Let F be an abstract Riemann surface with a relative boundary Γ_0.

In the sense of Stoilow, we call an ideal point α defined by the system of the neighbourhood of α such as $\bigcap_i V_i(\alpha)=\alpha$.

$\overline{F-V_i(\alpha)}$ has another set of boundary point R^i defined by the system of the neighbourhood $W_j(R^i)$ such as $\bigcap_j W_j(R^i)=R^i$.

Let us denote by $\omega_j(x)$ the positive harmonic function in $F-V_i(\alpha)-W_j(R^i)$ with the boundary values 1 on the boundary of $V_i(\alpha)$ and 0 on Γ_0 and the boundary of $W_j(R^i)$.

If $\lim_i\lim_j\omega_{ij}(x)=0$, then we call α a *point-wise* boundary point.

Let D be an arbitrary connected domain of Riemnna surface F and

C be its boundary point set of D included in $F+R$, and E be closed relative harmonic measure zero boundary point defined as the preceding, being contained in C and further suppose that α is a point-wise and not isolated from C.

$f(x)$ is one valued meromorphic function in D. We denote by $[\overline{f(x)}]_{x \in N}$ the closure of the set attained by $f(x)$ in N.

Let us associate two cluster sets,

$$S_{\alpha}^{(D)} = \bigcap_i [\overline{\underset{x \in D \cap V_i(x)}{f(x)}}], \quad S_{\alpha}^{(C)} = \bigcap_i (\bigcup_{p'} \underset{p' \in v_i(\alpha) - \alpha - E}{S_{p'}^{(D)}}),$$

then we easily have the following theorems as in the case of D being planer,

Theorem[11]. **(F. Iversen, A. Beuring, K. Kunugi, M. Tsuji)**

$$B(S_{\alpha}^{(D)}) \subset S_{\alpha}^{(C)}, \quad \textit{that is} \quad \Omega = S_{\alpha}^{(D)} - S_{\alpha}^{(C)}$$

is an open set, where $B(S_{\alpha}^{(C)})$ is the bounday of $S_{\alpha}^{(D)}$.

Theorem[12]. **(S. Kametani, M. Tsujii).**

Let F have a boundary point set R of at most relative harmonic measure zero and let all points of E are regular for Evans' problem. If $(\Omega) = S_{\alpha}^{(D)} - S_{\alpha}^{(C)}$ is not empty, then $f(x)$ tskes any value, except a set of at most capacity zero, belonging to Ω infinitely many times in any neighbourhood of α.

Reference

1) R. Nevanlinna: Quardratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit. Ann. Acad. Fenn. (1940).

2) S. Bochner: Fortsetzung Riemannschen Flächen. Math. Ann. 98 (1928).

3) S. Stoilow: Leçons sur les princepes topologiques de la théorie des fonctions analytiques. Paris. Gauther Villars (1938).

4) P. J. Myrberg: Ueber die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Flächen. Acta. Math. 61 (1933).

5) L. Ahlfors: Ueber die Anwendung differentialgeometrischer Methoden zur Unterschung von Ueberlagerungsflächen. Acta. Fenn (1939).

R. Bader: La théorie du potential sur une surface de Riemann. C. R. Paris. pp. 2001-2002 228 (1949).

M. Parreau: Comportemant à la frontiere de la fonction de Green d'une surface de Riemann. C. R. Paris (1950) 230. pp. 704-711.

M. Parreau: La théorie du potential sur les surface de Riemann a frontière positive. ibid. pp. 914-916.

M. Parreau: Sur la moyennes des fonctions harmoniques et le classification C. R. Paris. 231 (1950) pp. 679-681.

S. Johansson: Herstellung der automorphen Funktion. Acta. Fenn. 61. 2 (1912).

6) G. C. Evans: Potential and positively infinite singularities of harmonic functions. Monatschefte für Math. U. Phy. Bd. 43 (1936), pp. 419-423.

7) R. Nevanlinna: Eindeutige analytische. Fankrionen. Springer. (1936).

A. Pfluger: Ueber das Anwachsen eindeutiger analytischer Funktionen auf offener Riemannschen Flächen. Ann. Acad. Fenn. (1949).

8) L. Sario: Existence des fonction d'allure donné sur une surface de Riemann arbitraire. C R. Paris 229 (1949).

9) M. Tsuji: Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero. Jap. Journal. (1944).

O. Frostmann: Potential d'equilibré et capacité des ensemble avec queques application á la théorie des fonctions, Meddel. Lund Uni. Mat. Sem 3 (1935).

10) P. J. Myrberg: Ueber die analytische Fortsezung von beschränkter Funktionen 1949. Acta. Fenn. A. 58 (1949).

11) K. Kunugui: Sur un théorème de M. M. Seidel-Beurling. Proc. Acad. Jap. 15 (1939). pp. 27-32.

M. Tsuji: On the cluster set of a meromorphic function. Proc. Acd. Jap. 19 (1947). pp. 60-63.

12) K. Noshiro: On the theory of the cluster set of analytic function Jonrn of Math. Hokkaido Imp. Uni. vol. V. no. 4 (1938).

S. Kametani: The exceptional values of function with set of capacity zero of essential singularity, Proc. Acad Jap. 17 (1941).

(Received April 20, 1951)