

## ON PROBABILISTIC APPROACH TO THE EIGENVALUE PROBLEM FOR MAXIMAL ELLIPTIC OPERATOR

MASATOSHI FUJISAKI

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### 0. Introduction

Let  $D$  be a bounded domain of  $R^d$  with its smooth boundary  $\partial D$ . Let  $L^\alpha, \alpha \in A$  ( $A$  is a given set), be non-degenerate second-order linear differential operator with parameter  $\alpha$  of the following type:

$$(0.1) \quad L^\alpha = \sum_{i,j=1}^d a_{ij}(\alpha, x) \partial_i \partial_j + \sum_{i=1}^d b_i(\alpha, x) \partial_i - c(\alpha, x)$$

and let also  $L$  be a (nonlinear) operator, given by the formula  $L = \sup_{\alpha \in A} L^\alpha$ , which is called the maximal operator. Let consider a Dirichlet eigenvalue problem with respect to  $L$  on  $D$ ;

$$(0.2) \quad Lu + \lambda u = 0 \text{ on } D, \quad u > 0 \text{ on } D, \quad \text{and } u|_{\partial D} = 0.$$

In this paper, under the assumption that there exist an eigenvalue  $\lambda$  and corresponding (smooth) eigenfunction  $u$  satisfying (0.2), we will discuss various properties of  $\lambda$  and also obtain a probabilistic representation for  $\lambda$ .

We shall prove in §2 that  $\lambda$  is smaller than any  $\lambda^\alpha, \alpha \in A$ , where for each  $\alpha \in A$ ,  $\lambda^\alpha$  is the smallest eigenvalue of linear operator  $L^\alpha$ . In §3, we shall show that  $\lambda$  is the limit of a sequence of the principal eigenvalues, each of them corresponds to a linear operator in the class. Finally, in §4, we shall obtain a probabilistic representation for  $\lambda$ .

Our method of proof is similar to that of [3] and [4]. Namely, it proceeds firstly by considering the transformation such that  $v = -\log u$ , after then by applying Bellman equation's method to the equation with respect to  $v$  obtained by such logarithmic transformation of (0.2). Therefore our results mainly rely on the theory of stochastic control and Bellman equation developed by [1], [5] and [6] (See also [2]).

C.Pucci showed in [8] that  $\lambda = \min \lambda^\alpha$ , in the case where  $a(\alpha, x) = \alpha(x)$  and  $\alpha(\cdot)$  is a matrix-valued bounded measurable function. In [8], he also gave an interesting example in which  $D = \{x \in R^d; |x| < 1\}$ , and there exist an eigenvalue and corresponding (smooth)

eigenfunction relative to Eq.(0.2). His method of proof depends upon entirely the theory of differential equations (to the detail, see Remark 1.1 below).

On the otherhand, in [3] and [4], stochastic representation problems for the principal eigenvalue of second order linear differential equations are discussed. Result in [3] is concerned to a Dirichlet problem on a bounded domain, while one in [4] is concerned to the whole domain, namely finite-horizon problem.

### 1. Notations, definitions and preliminaries

Let  $D$  be an open set in  $R^d$  with compact closure  $\bar{D}$  and smooth boundary  $\partial D$ . Let  $A$  be a convex subset in a Euclidean space  $R^l$ . For any component  $\alpha$  in  $A$ , define the operator  $L^\alpha$  by (0.1), whose coefficients  $a(\alpha, x)$ ,  $b(\alpha, x)$  and  $c(\alpha, x)$ ,  $(\alpha, x) \in A \times R^d$ , are  $d \times d$ -matrix,  $d$ -vector and nonnegative valued functions, respectively. Assume that they satisfy the following conditions;

**(A.1)** for any  $\alpha \in A$ ,  $f(\alpha, x)$  is Lipschitz continuous on  $R^d$  and  $f$  is continuous in  $\alpha$  uniformly with respect to  $x$  ( $f = a, b$  and  $c$ ). Furthermore, the matrix  $a$  is symmetric and uniformly positive definite; there exists a positive constant  $\mu$  such that

$$(1.1) \quad \mu|\xi|^2 \leq (a(\alpha, x)\xi, \xi), \quad \text{for all } \alpha \in A, x \in R^d \text{ and } \xi \in R^d.$$

Put  $\mathcal{L} = \{L^\alpha, \alpha \in A\}$ . Let  $\mathcal{G}$  be the class of functions  $u$  such that  $u \in C(\bar{D}) \cap H^{2,d}(D)$ ,  $u(x) > 0$  for  $x \in D$  and  $u(x) = 0$  for  $x \in \partial D$ . Let  $\Lambda$  be the set of real numbers  $\lambda^\alpha$  for which there exist an  $L^\alpha \in \mathcal{L}$  and a function  $u^\alpha \in \mathcal{G}$  such that

$$(1.2) \quad L^\alpha u^\alpha(x) + \lambda^\alpha u^\alpha(x) = 0 \quad \text{a.e. in } x \in D.$$

Then it is shown from the maximum principle that all such  $\lambda^\alpha$ 's are positive. Let  $L$  be the maximizing operator relative to the class  $\mathcal{L}$ , given by the formula; for any  $u \in \mathcal{G}$  and  $x \in D$ ,

$$(1.3) \quad Lu(x) = \sup_{\alpha \in A} L^\alpha u(x).$$

Throughout this paper we assume the following hypothesis.

**(A.2)** There exist a function  $u \in \mathcal{G}$  (we assume that  $u(x) \leq 1$ , for convenience) and a real number  $\lambda$  such that

$$(1.4) \quad Lu(x) + \lambda u(x) = 0 \quad \text{a.e. in } D.$$

REMARK 1.1. In [8](Theorem II), C.Pucci gave an interesting example that (A.2) holds, in which he assumed that  $D = \{x \in R^d; |x| < 1\}$  and  $a(\alpha, x) = \alpha(x)$ , where  $\alpha(\cdot)$

is a measurable function of  $x$ . In this case, therefore,  $A$  is a set of symmetric uniformly positive definite  $d \times d$ -matrices (in addition, it was assumed that the trace of  $\alpha = 1$  and  $\mu \leq 1/d$ ). It is also shown in [8](Theorem I) that if (A.2) is true then  $\lambda = \min_{\alpha \in A} \Lambda$ .

**2. Lower bound for  $\Lambda$**

Let assume (A.1) and (A.2). Let  $v(x) = -\log u(x)$  for  $x \in D$ , then  $v \in C(\bar{D}) \cap H_{loc}^{2,d}(D)$  (i.e.  $v \in H^{2,d}(D')$  for any  $D'$  such that  $D' \subset \bar{D}' \subset D$ ),  $v > 0$  in  $D$  and  $v(x) \nearrow \infty$  as  $x \rightarrow \partial D$ . Moreover, it is easily seen from (1.3) and (1.4) that

$$(2.1) \quad \sup\{\nabla v(x)^* a(\alpha, x) \nabla v(x)/2 - \text{trace } a(\alpha, x) \nabla^2 v(x)/2 - b(\alpha, x)^* \nabla v(x) - c(\alpha, x)\} = -\lambda, \text{ a.e. in } D.$$

Here,  $b^*$  denotes the transposed vector of  $b$  and so on, moreover we used the abbreviated notations such as  $a(\alpha, x) = \{a_{i,j}(\alpha, x), 1 \leq i, j \leq d\}$ ,  $b(\alpha, x) = \{b_i(\alpha, x), 1 \leq i \leq d\}$ ,  $\nabla v(x) = \{\partial_i v(x), 1 \leq i \leq d\}$ ,  $\nabla^2 v(x) = \{\partial_i \partial_j v(x), 1 \leq i, j \leq d\}$ , etc. Eq.(2.1) implies that for all  $\alpha \in A$  and for a.a. $x$ ,

$$(2.2) \quad \text{trace } a(\alpha, x) \nabla^2 v(x)/2 + b(\alpha, x)^* \nabla v(x) + c(\alpha, x) - \nabla v(x)^* a(\alpha, x) \nabla v(x)/2 \geq \lambda,$$

which can be written in the form;

$$(2.3) \quad \text{trace } a(\alpha, x) \nabla^2 v(x)/2 + b(\alpha, x)^* \nabla v(x) + c(\alpha, x) + \inf_{\beta \in R^d} \{(\sigma(\alpha, x)^* \nabla v(x), \beta) + |\beta|^2/2\} \geq \lambda,$$

where  $\sigma(\alpha, x)$ ,  $\alpha \in A$ ,  $x \in R^d$ , is a  $d \times d$ -matrix such that  $\sigma \sigma^* = a$ . Note that in this case  $\sigma$  can be taken so as to be also Lipschitz continuous in  $x$ .

For any fixed  $x \in D$  and  $\alpha \in A$  such that  $\lambda^\alpha \in \Lambda$ , let consider the following stochastic differential equation;

$$(2.4) \quad \begin{cases} dX_t &= \sigma(\alpha, X_t) dB_t - a(\alpha, X_t) \nabla v^\alpha(X_t) dt + b(\alpha, X_t) dt \\ X_0 &= x, \end{cases}$$

where,  $(B_t), 0 \leq t < \infty$ , is a  $R^d$ -valued Brownian motion,  $v^\alpha = -\log u^\alpha$  and  $u^\alpha \in \mathcal{G}$ . Then we have a lemma.

**Lemma 2.1.** *For each  $\alpha \in A$  such that  $\lambda^\alpha \in \Lambda$  and  $x \in D$ , there exists a solution  $(X_t^{\alpha,x})$  of Eq.(2.4) on a probability space satisfying the standard conditions, and this process does not leave  $D$  at last. Moreover,  $\lambda^\alpha$  is represented as follows;*

$$(2.5) \quad \liminf_{t \rightarrow \infty} E \left[ \int_0^t \{c(\alpha, X_r^{\alpha,x}) + \nabla v^\alpha(X_r^{\alpha,x})^* a(\alpha, X_r^{\alpha,x}) \nabla v^\alpha(X_r^{\alpha,x})/2\} dr \right] / t = \lambda^\alpha.$$

(for the proof, see [3]).

Let  $\{D_n\}_{n \in \mathbb{N}}$  be an increasing sequence of bounded open sets in  $R^d$  such that for each  $n$ ,  $D_n \subset \bar{D}_n \subset D_{n+1}$  and  $\bigcup_{n \geq 1} D_n = D$ . For any fixed  $n, \alpha \in A$  such that  $\lambda^\alpha \in \Lambda$  and  $x \in D_n$ , applying the Ito formula to  $v$  and  $(X_t^{\alpha,x})$ , we obtain the following.

$$\begin{aligned}
 (2.6) \quad & v(X_{t \wedge \tau_n}^{\alpha,x}) - v(x) \\
 &= \int_0^{t \wedge \tau_n} \{ \text{trace } a^\alpha \nabla^2 v + (b^\alpha)^* \nabla v - (\nabla v)^* a^\alpha \nabla v^\alpha \} (X_r^{\alpha,x}) dr \\
 &+ (\text{square integrable martingale}),
 \end{aligned}$$

where  $\tau_n$  is the first exit time of  $(X_t^{\alpha,x})$  from  $D_n$ , and we used the abbreviations such as  $f^\alpha(x) = f(\alpha, x)$  ( $f = a, b$ ), etc. Adding to the both terms of (2.6) the quantity

$$\int_0^{t \wedge \tau_n} \{ (\nabla v^\alpha)^* a^\alpha \nabla v^\alpha / 2 + c^\alpha \} (X_r^{\alpha,x}) dr,$$

and taking the expectation, one sees that

$$\begin{aligned}
 (2.7) \quad & E[v(X_{t \wedge \tau_n}^{\alpha,x}) - v(x) + \int_0^{t \wedge \tau_n} \{ (\nabla v^\alpha)^* a^\alpha \nabla v^\alpha / 2 + c^\alpha \} (X_r^{\alpha,x}) dr] \\
 &= E[\int_0^{t \wedge \tau_n} \{ \text{trace } a^\alpha \nabla^2 v / 2 + (b^\alpha)^* \nabla v + c^\alpha \\
 &\quad - (\nabla v)^* a^\alpha \nabla v^\alpha + (\nabla v^\alpha)^* a^\alpha \nabla v^\alpha / 2 \} (X_r^{\alpha,x}) dr] \\
 &\geq \lambda E[t \wedge \tau_n].
 \end{aligned}$$

The last inequality is due to (2.3), in which we put  $\beta(x) = -\sigma^\alpha(x)^* \nabla v^\alpha(x)$  such that  $\sigma^\alpha(x) \sigma^\alpha(x)^* = a^\alpha(x)$ . From (A.2), it is not difficult to show (e.g.[8]) that there exist positive constants  $c_1$  and  $c_2$  such that for any  $x \in D$ ,

$$(2.8) \quad c_1 d(x) \leq u(x) \leq c_2 d(x),$$

where  $d(x)$  ( $x \in D$ ) denotes the distance between  $x$  and  $\partial D$ . It is also shown that

$$(2.9) \quad \sup_{t \geq 0} \left\{ \liminf_{n \rightarrow \infty} E[v(X_{t \wedge \tau_n}^{\alpha,x})] \right\} < \infty,$$

for  $\forall x \in D$  and  $\forall \alpha \in A$  such that  $\lambda^\alpha \in \Lambda$  (c.f.[3]). Dividing by  $t$  and then taking the limit in (2.7) as  $n \rightarrow \infty$  and  $t \rightarrow \infty$  (note that  $\tau_n \nearrow \infty$  as  $n \nearrow \infty$ ) we deduce that for any  $\alpha$ ,

$$(2.10) \quad \liminf_{t \rightarrow \infty} E[\int_0^t \{ (\nabla v^\alpha)^* a^\alpha \nabla v^\alpha / 2 + c^\alpha \} (X_r^{\alpha,x}) dr] / t \geq \lambda,$$

from which we obtain a theorem.

**Theorem 2.1.** *Let assume (A.2), then  $\lambda \leq \lambda^\alpha$  for any  $\lambda^\alpha \in \Lambda$ .*

REMARK 2.1. Let  $\alpha(\cdot)$  be a smooth function on  $R^d$  with its values in  $A$ , then it is well known that there exist an eigenvalue  $\lambda^{\alpha(\cdot)}$  and a corresponding eigenfunction belonging to  $\mathcal{G}$ . Moreover, we can show by the same way as Theorem 2.1 that  $\lambda \leq \lambda^{\alpha(\cdot)}$  for such  $\alpha$ 's.

Now let consider the minimizing operator  $L'$ , given by  $L' = \inf_{\alpha \in A} L^\alpha$ . Moreover, let assume that there exist a  $u' \in \mathcal{G}$  and a real number  $\lambda'$  such that

$$(2.11) \quad L'u' + \lambda'u' = 0 \text{ a.e. in } D.$$

Then we have the following.

**Theorem 2.2.** *Assume (2.11), then  $\lambda^\alpha \leq \lambda'$  for any  $\lambda^\alpha \in \Lambda$ .*

Proof. Since the proof is almost similar to that for  $\lambda$ , we will sketch only the different points. First, remark that if  $\lambda^\alpha \in \Lambda$  then it holds that

$$(2.12) \quad \lambda^\alpha \leq \liminf_{t \rightarrow \infty} E[\int_0^t \{(\nabla v)^* a^\alpha \nabla v/2 + c^\alpha\} (\bar{X}_r^{\alpha,x}) dr]/t,$$

( $a^\alpha(x) = a(\alpha, x)$ , etc.). Here, for each  $\alpha \in A$  and  $x \in D$ ,  $(\bar{X}_t^{\alpha,x})$ ,  $0 \leq t < \infty$ , is a process determined by the following stochastic differential equation;

$$(2.13) \quad \begin{cases} dX_t &= \sigma^\alpha(X_t)dB_t - a^\alpha(X_t)\nabla v(X_t)dt + b^\alpha(X_t)dt \\ X_0 &= x, \end{cases}$$

(c.f.Eq.(2.4)). In order to show (2.12) it must be important to note that the process  $(\bar{X}_t^{\alpha,x})$  satisfies the inequality (2.9)(to the detail, see [3], Th.3). On the otherhand, it can be deduced easily from Eq.(2.11) that for all  $\lambda^\alpha \in \Lambda$ ,

$$(2.14) \quad \liminf_{t \rightarrow \infty} E[\int_0^t \{(\nabla v)^* a^\alpha \nabla v/2 + c^\alpha\} (\bar{X}_r^{\alpha,x}) dr]/t \leq \lambda'.$$

The assertion of the theorem follows immediately from (2.12) and (2.14). □

From these results we obtain the following.

**Corollary 2.1.** *If  $\lambda^\alpha \in \Lambda$  then  $\lambda \leq \lambda^\alpha \leq \lambda'$ .*

REMARK 2.2. (a) C.Pucci also proved in his formulation that  $\lambda' = \max \Lambda$  by using differential equation's method ([8]).

(b) The results, obtained in the following sections of this paper with respect to  $(L, \lambda, v)$ , can be also extended easily to  $(L', \lambda', v')$  without any change of the proof.

**3. Successive approximation for  $\lambda$**

We deduced in Theorem 2.1 that  $\lambda \leq \inf \Lambda$ . In this section, we shall prove that  $\lambda$  could be approximated by a sequence of the principal eigenvalues, each of them corresponds to a linear operator with smooth coefficients, by using the analogous method as [8]. Assume (A.2). Let  $\alpha = \alpha(x)$  be a measurable function with values in  $A$ . Put

$$(3.1) \quad \begin{aligned} L^{\alpha(\cdot)}u(x) + \lambda u(x) &\equiv \sum_{i,j=1}^d a_{ij}(\alpha(x), x)\partial_i\partial_j u(x)/2 \\ &+ \sum_{i=1}^d b_i(\alpha(x), x)\partial_i u(x) - c(\alpha(x), x)u(x) + \lambda u(x) \end{aligned}$$

Since  $A$  is a convex subset in a Euclidian space, for each  $n$  there exist a function  $\alpha_n(x)$  given on  $R^d$  which is infinitely differentiable, has values in  $A$  and a constant  $k_n$  such that

$$(3.2) \quad \begin{aligned} \|h_n\|_{L^\infty(D)} &\leq 1/n, \quad \sup_x \sup_{l \in R^d} |\alpha_{n(l)}(x)| < \infty, \\ \|\sigma(\alpha_n(x), x) - \sigma(\alpha_n(y), y)\| + \|b(\alpha_n(x), x) - b(\alpha_n(y), y)\| &\leq k_n|x - y|, \\ &\text{for all } x \in R^d, \end{aligned}$$

where for each  $n, x$  and  $l, \alpha_{n(l)}$  means the derivative of  $\alpha_n$  in the  $l$ -direction, and we put

$$(3.3) \quad h_n(x) = L^{\alpha_n(\cdot)}u(x) + \lambda u(x).$$

$L^{\alpha_n(\cdot)}$  is given in (3.1), in which  $\alpha(\cdot)$  is replaced by  $\alpha_n(\cdot)$ (see [5] p.215 or [2]). For any  $n$ , let  $u_n \in \mathcal{G}$  and  $\lambda_n$  be the solution of the eigenvalue problem (1.2) in which the parameter  $\alpha$  is replaced with  $\alpha_n(\cdot)$  (see Remark 2.1).

For each  $n$ , let  $t_n = \sup_{x \in D} u(x)/u_n(x)$  then  $0 < t_n < \infty$  (c.f.(2.8)). For any  $n$ , if we put  $s_n = t_n \lambda / \lambda_n$  then

$$\lambda_n s_n u_n(x) - \lambda u(x) \geq 0 \text{ in } D.$$

From these results one sees that

$$(3.4) \quad L^{\alpha_n(\cdot)}(s_n u_n - u)(x) + h_n(x) \leq 0 \text{ a.e. in } D.$$

For each  $n$ , let  $w_n$  be a solution of the following equation

$$(3.5) \quad \begin{cases} L^{\alpha_n(\cdot)} w_n(x) = h_n(x) \text{ a.e. } x \in D \\ w_n(x) = 0 \text{ for } x \in \partial D. \end{cases}$$

Now suppose that for  $\forall x \in D$  and  $\forall n$ ,  $w_n$  satisfies the condition;

$$(3.6) \quad |w_n(x)| \leq M d(x)/n,$$

where  $M$  is a constant independent of  $n$  and  $x$  (see Remark 3.1 below). We deduce from (3.2)~(3.6) that

$$(3.7) \quad \begin{cases} L^{\alpha_n(\cdot)}(s_n u_n - u + w_n) \leq 0 \text{ a.e. in } D \\ s_n u_n - u + w_n = 0 \text{ on } \partial D. \end{cases}$$

In virtue of the maximum principle, it follows by (3.7) that for any  $x$  and  $n$ ,

$$s_n u_n(x) - u(x) + w_n(x) \geq 0.$$

From the definitions of  $s_n$  and  $t_n$ , one deduce that for  $\forall n, \forall x \in D$ ,

$$(3.8) \quad \begin{aligned} \lambda/\lambda_n &\geq u(x)/t_n u_n(x) - w_n(x)/u(x) \\ &\geq u(x)/t_n u_n(x) - M/cn \geq 1 - M/cn \end{aligned}$$

where  $c$  is a constant independent of  $n$  and  $x$  (see (2.8) and (3.6)). As  $n \rightarrow \infty$  in (3.8) we obtain a theorem.

**Theorem 3.1.** *Let  $\{\alpha_n\}$  be a sequence of smooth functions on  $R^d$  such that  $\alpha_n(x) \in A$  for all  $x \in D$ , satisfying the conditions such as (3.2). Let also  $\lambda_n$  be the first eigenvalue of Eq.(1.2) associated with  $\alpha_n$ . If the condition (3.6) is true, then  $\lambda \geq \limsup_{n \rightarrow \infty} \lambda_n$ .*

Combine this theorem with Theorem 2.1 and Remark 2.1, we deduce

**Corollary 3.1.** *Let assume the condition (3.6). Then  $\lambda = \lim_{n \rightarrow \infty} \lambda_n$ .*

REMARK 3.1. The condition (3.6) would be satisfied if  $D$  is convex. In fact, using the results in [7] or [6](p.207) it can be shown that if  $h_n$  is given by (3.3), then the (unique) solution  $w_n$  of Eq.(3.5) has the property such that  $\sup_{x \in D} |\nabla w_n(x)| \leq M/n$ , where  $M$  is a constant independent of  $n$ . For any  $x \in D$ , let  $\xi$  be the nearest point to  $x$  on  $\partial D$ . By the mean value theorem,

$$w_n(x) - w_n(\xi) = \int_0^1 \nabla w_n(\eta x + (1 - \eta)\xi) \cdot (x - \xi) d\eta.$$

Now, if we suppose that  $D$  is convex, then  $\eta x + (1 - \eta)\xi \in D$  for  $\forall \eta \in (0, 1)$ . Since  $w_n(\xi) = 0$  for  $\forall \xi \in \partial D$ , we get

$$|w_n(x)| \leq \sup_{x \in D} |\nabla w_n(x)| |x - \xi| \leq M \cdot d(x)/n,$$

where  $M$  does not depend on  $n$  and  $x$ .

#### 4. Variational formula for $\lambda$

Let  $\lambda$  and  $u$  be given in (A.2) and in this section we want to obtain a probabilistic representation for  $\lambda$  like as one for  $\lambda^\alpha$  (c.f.(2.5)). For that purpose, in addition to the assumptions made in Section 1 ~ 3, we impose to assume that  $A$  is compact. Then it is well known (see [1], p.168, e.g.) that there exists a measurable function  $\alpha(x)$  with values in  $A$ , which satisfies the following equation;

$$(4.1) \quad L^{\alpha(\cdot)} u(x) + \lambda u(x) = 0 \quad a.e. \text{ in } D$$

Then we have a theorem.

**Theorem 4.1.** Assume (A.2) then  $\lambda$  can be represented as follows; for all  $x \in D$ ,

$$(4.2) \quad \lambda = \liminf_{t \rightarrow \infty} E[\int_0^t \{ \nabla v(X_r^{\alpha(\cdot)}) * a(\alpha(X_r^{\alpha(\cdot)}), X_r^{\alpha(\cdot)}) \nabla v(X_r^{\alpha(\cdot)}) / 2 + c(\alpha(X_r^{\alpha(\cdot)}), X_r^{\alpha(\cdot)}) \} dr] / t$$

where  $v(x) = -\log u(x)$  and  $(X_t^{\alpha(\cdot)}), 0 \leq t < \infty$ , is a solution of the following stochastic differential equation;

$$(4.3) \quad \begin{cases} dX_t = \sigma(X_t) dB_t - a(X_t) \nabla v(X_t) dt + b(X_t) dt \\ X_0 = x. \end{cases}$$

Here  $x \in D, (B_t), 0 \leq t < \infty$ , is a  $R^d$ -valued Brownian motion,  $b(x) \equiv b(\alpha(x), x)$ , and for any  $x, \sigma(x)$  is a  $d \times d$ -matrix such that  $\sigma(x)\sigma(x)^* = a(x) (\equiv a(\alpha(x), x))$ .



The previous results in Section 1 ~ Section 3 indicate us that in order to show the theorem it is sufficient to prove the existence of a (weak) solution  $(X_t)$  of Eq.(4.3)(c.f. Lemma 2.1). Since  $\alpha(\cdot)$  is not continuous in general (see Remark 4.1 below), the proof of the theorem will be divided into several stages. First of all, for any fixed  $x \in D$ , let us consider stochastic differential equation of the form;

$$(4.4) \quad \begin{cases} dX_t = \sigma(X_t)d\tilde{B}_t + b(X_t)dt \\ X_0 = x, \end{cases}$$

where  $(\tilde{B}_t)$  is a d-dimensional Brownian motion. Since the matrix  $\sigma$  is uniformly positive definite and, moreover, the coefficients  $\sigma$  and  $b$  are bounded measurable function on  $R^d$ , there exists a solution  $(X_t)$  of Eq.(4.4) on a suitable probability space  $(\Omega, \mathcal{F}, Q; \mathcal{F}_t)$  satisfying the standard conditions (see [5], Th.2.6.1, p.87).

Let  $\{D_n\}, n \in \mathbf{N}$ , be an increasing sequence of sets given in §2. Let also  $\{\psi_n\}, n \in \mathbf{N}$ , be a sequence of smooth functions on  $R^d$  such that  $\psi_n \in C^\infty(R^d), 0 \leq \psi_n(x) \leq 1, \psi_n(x) = 1$ (if  $x \in \bar{D}_n$ ),  $\psi_n(x) = 0$ (if  $x$  does not belong to  $D_{n+1}$ ). Put  $g_n(x) \equiv \psi_n(x)\nabla v(x)$  for any  $n \in \mathbf{N}$  and  $x \in R^d$ , then for each  $n, g_n(x)$  is bounded measurable function of  $x$ . For any  $n, t \in [0, \infty)$  and  $x \in D_n$ , let define a function  $\rho_n(t)$  by the formula

$$(4.5) \quad \rho_n(t) = \exp\left\{-\int_0^t \sigma(X_r)^* g_n(X_r) d\tilde{B}_r - \int_0^t |\sigma(X_r)^* g_n(X_r)|^2 dr/2\right\}$$

where  $(\tilde{B}_t, X_t)$  is the solution of Eq.(4.4). Since  $\sigma$  and  $g_n$  are bounded measurable functions, for each  $n, \rho_n(\cdot)$  is a square integrable martingale with respect to  $(\mathcal{F}_t, Q)$ . Given any fixed  $x \in D$ , let define  $P_n$  and  $(B_t^n), n \in \mathbf{N}, 0 \leq t < \infty$ , by the following formulas;i.e.,

$$(4.6) \quad P_n(\cdot) = \rho_n(t)Q(\cdot) \text{ on } \mathcal{F}_t,$$

$$(4.7) \quad dB_t^n = d\tilde{B}_t + \sigma(X_t)^* g_n(X_t)dt, \quad B_0^n = 0,$$

then the following result is due to I.V.Girsanov.

**Lemma 4.1.** *For any  $n, P_n$  is a probability measure on  $\mathcal{F}_\infty \equiv \bigvee_{t>0} \mathcal{F}_t$  and also the triple  $(B_t^n, \mathcal{F}_t, P_n)$  is an  $R^d$ -valued Brownian motion. This is equivalent to say that for each  $n$  and  $x \in D_n$ ,  $(\Omega, \mathcal{F}, P_n; \mathcal{F}_t, X_t, B_t^n)$  is a (weak) solution of the following stochastic differential equation;*

$$(4.8) \quad \begin{cases} dX_t = \sigma(X_t)dB_t^n - a(X_t)g_n(X_t)dt + b(X_t)dt \\ X_0 = x. \end{cases}$$

Applying the Ito formula to  $v$  and  $(X_t)$ , the solution of Eq.(4.8), we deduce for any  $n$  and  $x \in D_n$

(4.9)

$$\begin{aligned} v(X_{t \wedge \tau_n}) &= v(x) + \int_0^{t \wedge \tau_n} \left\{ \text{trace } a \cdot \nabla^2 v(X_r) / 2 - (\nabla v)^* \cdot a \cdot g_n(X_r) + (\nabla v)^* \cdot b(X_r) \right\} dr \\ &\quad + \{\text{square integrable martingale}\}, \end{aligned}$$

where  $\tau_n$  means the first exit time of  $(X_t)$  from  $D_n$ . To the both parts of (4.9), adding the quantity

$$\int_0^{t \wedge \tau_n} \{(\nabla v)^* \cdot a \cdot g_n / 2 + c\} (X_r) dr$$

and then taking the mathematical expectation with respect to  $P_n$  (denote by  $E_n[\cdot]$ ), one obtains

(4.10)

$$\begin{aligned} E_n[v(X_{t \wedge \tau_n}) + \int_0^{t \wedge \tau_n} \{(\nabla v)^* \cdot a \cdot g_n / 2 + c\} (X_r) dr] \\ = v(x) + E_n \left[ \int_0^{t \wedge \tau_n} \left\{ \text{trace } a \cdot \nabla^2 v / 2 - (\nabla v)^* \cdot a \cdot g_n / 2 + (\nabla v)^* \cdot b + c \right\} (X_r) dr \right]. \end{aligned}$$

On the otherhand, since  $u(x) = -\log v(x)$ ,  $\forall x \in D$  and  $u$  satisfies (4.1) a.e. in  $D$ , it turns out that  $v$  satisfies the following equation for a.e. in  $D$ ;

$$\begin{aligned} (4.11) \quad & \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j v(x) / 2 + \sum_{i=1}^d b_i(x) \partial_i v(x) + c(x) - \\ & \sum_{i,j=1}^d a_{ij}(x) \partial_i v(x) \partial_j v(x) / 2 = \lambda. \end{aligned}$$

Using (4.10) and (4.11) we deduce

(4.12)

$$\begin{aligned} E_n[v(X_{t \wedge \tau_n}) + \int_0^{t \wedge \tau_n} \{(\nabla v)^* \cdot a \cdot g_n / 2 + c\} (X_r) dr] \\ = v(x) + \lambda E_n[t \wedge \tau_n]. \end{aligned}$$

But if we note that

$$(\nabla v)^* \cdot a \cdot g_n(x) + c(x) \equiv (\nabla v)^* \cdot a \cdot \nabla v \cdot \psi_n(x) + c(x) \geq 0 \text{ on } D_n,$$

then for any  $t, n$  and  $x \in D_n$ ,

$$E_n[v(X_{t \wedge \tau_n})] \leq v(x) + \lambda t.$$

Since  $v(x) \geq 0$  on  $D$ , it follows that

$$E_n[v(X_{\tau_n}); \tau_n \leq t] \leq v(x) + \lambda t.$$

On the otherhand, remember that if  $x \in D$  then  $v(x) \geq -\log c_2 d(x)$  (because of (2.8)). This implies that  $v(x) \geq d_n$  for  $\forall x \in D_n$  where  $\{d_n\}$  is a sequence of numbers such that  $d_n \nearrow \infty$  as  $n \rightarrow \infty$ . Therefore for any  $t, n$  and  $x \in D_n$ ,

$$P_n(\tau_n \leq t) \leq \{v(x) + \lambda t\} / d_n.$$

Summarizing these results we have a lemma.

**Lemma 4.2.** *For all  $t > 0$  and  $x \in D$ ,*

$$(4.13) \quad \lim_{n \rightarrow \infty} P_n(\tau_n \leq t) = 0.$$

Using this lemma we deduce

**Lemma 4.3.** *There exists a solution (in weak sense) of Eq.(4.3) on a probability space  $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$  satisfying the standard conditions on which is given a  $R^d$ -valued Brownian motion  $(B_t)$ .*

(For the proof, see Stroock-Varadhan [9], Th.1.3.5 and Cor.10.1.2, etc.) Then we can show Theorem 4.1 in the same way as [3](Theorem 3) using Lemma 4.1~ Lemma 4.3 .

REMARK 4.1. In general, we could not expect that there exists a continuous function  $\alpha(x)$  satisfying Eq.(4.1) for almost all  $x \in D$  (see [8],Theorem II). But in the case where we can choose  $\alpha(x)$  to be continuous, the martingale problem for  $(a, ag_n + b)$  is well posed. Therefore the proof of the theorem is followed immediately from [3] and [9](see also [1],p.170).

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Kobe University of Commerce  
8-2-1, Gakuen-Nishi-mati, Nishi-ku,  
Kobe, 651-21, Japan  
e-mail: fujisaki@kobe.kobeuc.ac.jp