

MEROMORPHIC FUNCTIONS ADMITTING AN ALGEBRAIC ADDITION THEOREM

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0. Introduction

Abelian functions are meromorphic functions of n complex variables having $2n$ independent periods. Today the theory of abelian functions is remarkably developed. One of the important properties of abelian functions is to admit an algebraic addition theorem. In the one dimensional case Weierstrass obtained the result that a meromorphic function admits an algebraic addition theorem if and only if it is an elliptic function or a rational function or a rational function of an exponential function. Weierstrass also noted an analogous statement in the case of n complex variables. But he did not publish his proof. Even its manuscript does not remain. This fact is seen in [10] and in [14]. The first attempt to prove Weierstrass' statement is due to Painlevé([10]). Painlevé's argument, however, is not acceptable at least for the author. Later, Severi called meromorphic functions of n complex variables with μ ($< 2n$) independent periods admitting an algebraic addition theorem quasi-abelian functions (or degenerate abelian functions), and published a thick book[13] about them(see also [11]). However his theory is not written in a clear language so that it is difficult for us to understand what he wrote. So it seems to the author that Weierstrass' statement is not yet established.

We study this problem in this paper. First we give examples which are simple but suggestive for our problem. Through these examples we can recognize the notion of algebraically non-degenerate fields, which we introduce in Section 2(Definition 2.9), is natural. Let $\mathfrak{M}(\mathbb{C}^n)$ be the meromorphic function field on \mathbb{C}^n , and let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$, whose transcendence degree $\text{Trans } K$ over \mathbb{C} is n . We prove that if K is non-degenerate and admits an algebraic addition theorem, then there exist a discrete subgroup Γ of \mathbb{C}^n and a projective algebraic variety Y such that K is considered as a subfield of the meromorphic function field $\mathfrak{M}(\mathbb{C}^n/\Gamma)$ on \mathbb{C}^n/Γ and K is isomorphic to the rational function field $\mathbb{C}(Y)$ on Y (Theorem 2.6). Applying this result to the one-dimensional case, we give a short proof of Weierstrass' theorem (Theorem 2.7).

Next we consider the meromorphic extension of $f \in K$ to a compactification \overline{G} of $G = \mathbb{C}^n / \Gamma$ in Section 3. The meromorphic extension of f to \overline{G} is equivalent to the holomorphic extension of the line bundle given by f to \overline{G} . In Section 5 we characterize extendable line bundles on G to \overline{G} (Theorem 5.10). We note that G has a fibre bundle structure $\sigma : G \rightarrow A$ on a k -dimensional abelian variety A with fibres $\mathbb{C}^{p+s} \times (\mathbb{C}^*)^{q+t}$. We take complex coordinates (x_1, \dots, x_n) of \mathbb{C}^n such that the projection to the space of the first k variables gives this fibration.

Using above results, we finally prove the following main theorem (Theorem 6.2).

Main Theorem *Let K be a non-degenerate subfield of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$ which admits (AAT). Let Γ be the discrete subgroup of \mathbb{C}^n in Theorem 2.6. We take complex coordinates (x_1, \dots, x_n) of \mathbb{C}^n as in Section 3. Suppose that K is algebraically non-degenerate with respect to (x_1, \dots, x_n) . Then we have*

$$\mathbb{C}^n / \Gamma = \mathbb{C}^p \times (\mathbb{C}^*)^q \times A,$$

where A is an r -dimensional abelian variety ($n = p + q + r$), and K is a subfield of $\mathbb{C}(z_1, \dots, z_p, w_1, \dots, w_q, g_0, g_1, \dots, g_r)$, where z_1, \dots, z_p and w_1, \dots, w_q are coordinate functions of \mathbb{C}^p and $(\mathbb{C}^*)^q$ respectively, and g_0, g_1, \dots, g_r are generators of the abelian function field $\mathfrak{M}(A)$ on A .

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1. Definitions and examples

Let $\mathfrak{M}(\mathbb{C}^n)$ be the meromorphic function field on \mathbb{C}^n . Consider a subfield $K \subset \mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$, where $\text{Trans } K$ is the transcendence degree of K over \mathbb{C} . Let f_0, f_1, \dots, f_n be generators of K .

DEFINITION 1.1. We say that f_0, f_1, \dots, f_n admit an algebraic addition theorem (in the sequel we abbreviate it (AAT)) if for any $j = 0, 1, \dots, n$ there exists a non-zero rational function R_j such that

$$(1.2) \quad f_j(x + y) = R_j(f_0(x), f_1(x), \dots, f_n(x), f_0(y), f_1(y), \dots, f_n(y)) \quad \text{for all } x, y \in \mathbb{C}^n.$$

The following lemma is elementary. So we omit its proof.

Lemma 1.3. *Let $K = \mathbb{C}(f_0, f_1, \dots, f_n)$. Suppose that generators f_0, f_1, \dots, f_n of K admit (AAT). Then other generators g_0, g_1, \dots, g_n also admit (AAT).*

DEFINITION 1.4. A subfield $K \subset \mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$ admits (AAT) if K has generators f_0, f_1, \dots, f_n of K which admit (AAT).

DEFINITION 1.5. A meromorphic function f on \mathbb{C}^n is *degenerate* if there exist an invertible linear transformation $L : \mathbb{C}^n \rightarrow \mathbb{C}^n, z = L(\zeta)$ and a natural number r with $r < n$ such that $f(L(\zeta))$ depends only on ζ_1, \dots, ζ_r . We say that f is *non-degenerate* if it is not degenerate.

DEFINITION 1.6. A subfield $K \subset \mathfrak{M}(\mathbb{C}^n)$ is said to be *non-degenerate* if there exists a non-degenerate meromorphic function $f \in K$.

We give some examples which are simple but suggestive in our investigation.

Example 1.7. Let $f_1(z, w) = z, f_2(z, w) = e^z$. We put $K_I := \mathbb{C}(1, f_1, f_2) \subset \mathfrak{M}(\mathbb{C}^2)$. Then $\text{Trans } K_I = 2$, and K_I is degenerate. The generators $1, f_1, f_2$ admit (AAT). The period group Γ_I of K_I is equal to $\{0\} \times \mathbb{C}$.

Example 1.8. Take $g_1(z, w) = z$ and $g_2(z, w) = e^w$. Let $K_{II} := \mathbb{C}(1, g_1, g_2) \subset \mathfrak{M}(\mathbb{C}^2)$. Then $\text{Trans } K_{II} = 2$, and K_{II} is non-degenerate. Generators $1, g_1, g_2$ admit (AAT). The period group Γ_{II} of K_{II} is equal to $\{0\} \times 2\pi\sqrt{-1}\mathbb{Z}$.

Example 1.9. Let $h_1(z, w) = z + w, h_2(z, w) = e^w$. We set $K_{III} := \mathbb{C}(1, h_1, h_2) \subset \mathfrak{M}(\mathbb{C}^2)$. Then $\text{Trans } K_{III} = 2$, and K_{III} is non-degenerate. Generators $1, h_1, h_2$ admit (AAT). The period group Γ_{III} of K_{III} is equal to $2\pi\sqrt{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbb{Z}$.

The subfield K_{III} is mapped to K_{II} isomorphically by the following invertible linear transformation (T) ,

$$(T) \begin{cases} \zeta = z + w \\ \eta = w \end{cases}$$

Moreover, if we restrict K_{III} to $\{z = 0\}$, then we obtain K_I , i.e. $\mathbb{C}(1, h_1(0, w), h_2(0, w)) = K_I$. Therefore K_I, K_{II} and K_{III} are isomorphic each other. So we set $K := K_I (\cong K_{II} \cong K_{III})$. Functions g_1, g_2 are extendable meromorphically to \mathbb{P}_2 as functions on $\mathbb{C}^2/\Gamma_{II} = \mathbb{C} \times \mathbb{C}^*$. Since $g_1(0, w) = 0, g_2(0, w) = e^w$, the transcendence degree of K_{II} restricted to $\{z = 0\}$ is 1. Similarly, the transcendence degree of K_{II} restricted to $\{w = 0\}$ is 1. Therefore, we can see after the following consideration that K_{II} is a better model of K (see Section 2).

2. Picard varieties

Throughout this section we assume that a subfield K of $\mathfrak{M}(\mathbb{C}^n)$ is non-degenerate, $\text{Trans } K = n$ and admits (AAT).

DEFINITION 2.1. Meromorphic functions $f_1, \dots, f_m \in \mathfrak{M}(\mathbb{C}^n)$ are *analytically independent* at $a \in \mathbb{C}^n$ if

- (a) f_1, \dots, f_m are holomorphic at a ,
- (b) the (m, n) -matrix $\left[\frac{\partial f_i}{\partial z_j}(a) \right]$ has rank m , where (z_1, \dots, z_n) are complex coordinates of \mathbb{C}^n .

Lemma 2.2. *For a non-degenerate meromorphic function $f \in K$, there exist $a^{(1)}, \dots, a^{(n)} \in \mathbb{C}^n$ such that $f(z+a^{(1)}), \dots, f(z+a^{(n)})$ are analytically independent at $z = 0$.*

Proof. Since f is non-degenerate, $\partial f / \partial z_1, \dots, \partial f / \partial z_n$ are linearly independent over \mathbb{C} . Then there exist $a^{(1)}, \dots, a^{(n)} \in \mathbb{C}^n$ such that f is holomorphic at $a^{(j)}$ ($j = 1, \dots, n$) and

$$\det \left[\frac{\partial f}{\partial z_i}(a^{(j)}) \right] \neq 0.$$

Thus we obtain the lemma. \square

Proposition 2.3. *There exist holomorphic functions $\varphi_0, \varphi_1, \dots, \varphi_N$ on \mathbb{C}^n such that these functions give a holomorphic immersion*

$$\Phi := (\varphi_0 : \varphi_1 : \dots : \varphi_N) : \mathbb{C}^n \longrightarrow \mathbb{P}_N$$

into the N -dimensional complex projective space \mathbb{P}_N and $h_1 := \varphi_1/\varphi_0, \dots, h_N := \varphi_N/\varphi_0$ generate K .

Proof. Take a non-degenerate function $f \in K$. Let $a^{(1)}, \dots, a^{(n)} \in \mathbb{C}^n$ be vectors in Lemma 2.2. We set $g_1(z) := f(z + a^{(1)}), \dots, g_n(z) := f(z + a^{(n)})$. Let

$$G(z) := \det \left[\frac{\partial g_i}{\partial z_j}(z) \right].$$

Then $G(z)$ is a meromorphic function on \mathbb{C}^n . We set

$$A := \{z \in \mathbb{C}^n; \text{some } g_i \text{ is not holomorphic at } z \text{ or } G(z) = 0\}.$$

Then we obtain a holomorphic immersion

$$(g_1, \dots, g_n) := \mathbb{C}^n \setminus A \longrightarrow \mathbb{C}^n.$$

Since A is an analytic set of codimension 1 in \mathbb{C}^n , there exist a finite number of vectors $b^{(1)}, \dots, b^{(r)} \in \mathbb{C}^n$ such that

$$A \cap (A - b^{(1)}) \cap \dots \cap (A - b^{(r)}) = \emptyset,$$

where $A - b^{(i)} := \{a - b^{(i)}; a \in A\}$. Letting $b^{(0)} = 0$, we define

$$g_{ij}(z) := g_j(z + b^{(i)}), \quad i = 0, \dots, r; j = 1, \dots, n.$$

Take a set of generators $\{f_0, f_1, \dots, f_n\}$ of K . We set

$$\{h_1, \dots, h_N\} := \{g_{ij}; i = 0, \dots, r \text{ and } j = 1, \dots, n\} \cup \{f_0, f_1, \dots, f_n\}.$$

There exist holomorphic functions $\varphi_0, \varphi_1, \dots, \varphi_N$ on \mathbb{C}^n such that $\varphi_0, \varphi_1, \dots, \varphi_N$ have no common divisor and $h_i = \varphi_i / \varphi_0$ ($i = 1, \dots, N$). Then these functions give the desired mapping. \square

Let $\varphi_0, \varphi_1, \dots, \varphi_N$ and h_1, \dots, h_N be holomorphic functions and meromorphic functions in Proposition 2.3 respectively. Let $(x_0 : x_1 : \dots : x_N)$ be homogeneous coordinates of \mathbb{P}_N . Consider all the algebraic relations among h_1, \dots, h_N . We denote by \mathcal{P} the set of all corresponding homogeneous polynomials, i.e.

$$\mathcal{P} = \{\text{homogeneous polynomial } P \text{ with } P(\varphi_0, \varphi_1, \dots, \varphi_N) = 0\}.$$

We set

$$Y := \{(x_0 : x_1 : \dots : x_N) \in \mathbb{P}_N; P(x_0, x_1, \dots, x_N) = 0 \text{ for all } P \in \mathcal{P}\}.$$

Then Y is an algebraic subvariety of \mathbb{P}_N . It follows from the definition of Y that $\Phi(\mathbb{C}^n) \subset Y$ and Y is the Zariski closure of $\Phi(\mathbb{C}^n)$. We set $\Omega := \Phi(\mathbb{C}^n)$.

Proposition 2.4. *Ω is a connected complex abelian Lie group whose group structure is induced from the abelian group $(\mathbb{C}^n, +)$.*

Proof. For $p, q \in \Omega$ we define

$$p \cdot q := \Phi(z + w), \quad z \in \Phi^{-1}(p), w \in \Phi^{-1}(q).$$

We now check this definition is well-defined.

Take $z, z' \in \Phi^{-1}(p)$. If $\varphi_i(z) \neq 0$, then $\varphi_i(z') \neq 0$ and

$$\frac{\varphi_j(z)}{\varphi_i(z)} = \frac{\varphi_j(z')}{\varphi_i(z')}, \quad j = 0, 1, \dots, N.$$

Similarly we have the same property for $w, w' \in \Phi^{-1}(q)$. If $\varphi_i(w) \neq 0$, then $\varphi_i(w') \neq 0$ and

$$\frac{\varphi_j(w)}{\varphi_i(w)} = \frac{\varphi_j(w')}{\varphi_i(w')}, \quad j = 0, 1, \dots, N.$$

Consider $(\varphi_0(z+w) : \varphi_1(z+w) : \cdots : \varphi_N(z+w))$. There exists i such that $\varphi_i(z+w) \neq 0$. Without loss of generality we may assume that $i = 0$. Since K admits (AAT), there exists a rational function R_j for any $j \neq 0$ such that

$$\begin{aligned} h_j(z+w) &= R_j\left(h_1(z), \dots, h_N(z), h_1(w), \dots, h_N(w)\right) \\ &= R_j\left(h_1(z'), \dots, h_N(z'), h_1(w'), \dots, h_N(w')\right) \\ &= h_j(z'+w'). \end{aligned}$$

The following properties are obvious;

- (a) $p \cdot q = q \cdot p$ for $p, q \in \Omega$
- (b) $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ for $p, q, r \in \Omega$.

We set $e := \Phi(0)$. Then $e \cdot p = p \cdot e = p$ for all $p \in \Omega$. For any $p \in \Omega$, $p' := \Phi(-z)$ ($z \in \Phi^{-1}(p)$) is the inverse element of p . Hence Ω has a group structure, and $\Phi : \mathbb{C}^n \rightarrow \Omega$ is an epimorphism.

We define $\Gamma := \text{Ker } \Phi = \Phi^{-1}(e)$. Since $\Phi : \mathbb{C}^n \rightarrow \Omega$ is a holomorphic immersion, Γ must be a discrete subgroup of \mathbb{C}^n . Then we obtain an isomorphism $\bar{\Phi} : \mathbb{C}^n/\Gamma \rightarrow \Omega$ between two abelian groups, which is a biholomorphic mapping. Since mappings $(z, w) \mapsto z + w$ and $z \mapsto -z$ are holomorphic, we can verify that mappings $\Omega \times \Omega \rightarrow \Omega$, $(p, q) \mapsto p \cdot q$ and $\Omega \rightarrow \Omega$, $p \mapsto p^{-1}$ are holomorphic. The connectedness of Ω is trivial. Then Ω is a connected complex abelian Lie group, and $\bar{\Phi} : \mathbb{C}^n/\Gamma \rightarrow \Omega$ is a Lie group isomorphism. \square

By the above proposition we can consider K as a subfield of $\mathfrak{M}(\mathbb{C}^n/\Gamma)$, where $\mathfrak{M}(\mathbb{C}^n/\Gamma)$ is the meromorphic function field on \mathbb{C}^n/Γ . Let $G := \mathbb{C}^n/\Gamma$. We denote by $\mathbb{C}(Y)$ the rational function field on Y .

Proposition 2.5. *The Lie group isomorphism $\bar{\Phi} : G \rightarrow \Omega$ induces the isomorphism $\bar{\Phi}^* : \mathbb{C}(Y) \rightarrow K$, $R \mapsto R \circ \bar{\Phi}$. Therefore $\dim Y = \dim G = n$.*

Proof. It is obvious that $\bar{\Phi}^* : \mathbb{C}(Y) \rightarrow K$ is a homomorphism between two fields. Then it suffices to show that $\bar{\Phi}^*$ is onto.

For any $f \in K$ there exists a rational function R such that $f = R(h_1, \dots, h_N)$. Then we can take homogeneous polynomials P and Q such that

$$f = R(h_1, \dots, h_N) = \frac{Q(\varphi_0, \dots, \varphi_N)}{P(\varphi_0, \dots, \varphi_N)}, \quad P(\varphi_0, \dots, \varphi_N) \neq 0.$$

We define

$$S(x_0, \dots, x_N) := \frac{Q(x_0, \dots, x_N)}{P(x_0, \dots, x_N)}.$$

Then $S \in \mathbb{C}(Y)$ and $\bar{\Phi}^* S = f$. \square

The following theorem summarizes the above results.

Theorem 2.6. *Let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$. Suppose that K is non-degenerate and admits (AAT). Then there exist holomorphic functions $\varphi_0, \varphi_1, \dots, \varphi_N$ on \mathbb{C}^n , a discrete subgroup Γ of \mathbb{C}^n , an algebraic subvariety Y of \mathbb{P}^N and a connected complex abelian Lie group Ω in Y such that*

(a) $\varphi_0, \varphi_1, \dots, \varphi_N$ give a Lie group isomorphism

$$\overline{\Phi} = (\varphi_0 : \varphi_1 : \dots : \varphi_N) : G := \mathbb{C}^n / \Gamma \longrightarrow \Omega,$$

(b) $\varphi_1/\varphi_0, \dots, \varphi_N/\varphi_0$ generate K and K is considered as a subfield of the meromorphic function field $\mathfrak{M}(G)$ on G ,

(c) Y is the Zariski closure of Ω and

$$\overline{\Phi}^* : \mathbb{C}(Y) \longrightarrow K$$

is an isomorphism, therefore $\dim Y = \dim G = n$.

Applying Theorem 2.6 in the case $n = 1$, we can give a short proof of the necessity in the following well-known theorem of Weierstrass.

Theorem 2.7 (Weierstrass). *Consider a subfield K of the meromorphic function field $\mathfrak{M}(\mathbb{C})$ on \mathbb{C} with $\text{Trans } K = 1$. In this case, K is a maximal subfield admitting (AAT) if and only if K coincides with an elliptic function field or $\mathbb{C}(\exp(a\zeta))$ ($a \in \mathbb{C}^*$) or $\mathbb{C}(\zeta)$. Here we say that K is a maximal subfield admitting (AAT) if K has no algebraic extension L/K in $\mathfrak{M}(\mathbb{C})$ such that L admits (AAT) and $(L : K) \geq 2$.*

Proof. We prove the necessity. Suppose that K is a maximal subfield admitting (AAT). By Theorem 2.6 there exist holomorphic functions $\varphi_0, \varphi_1, \dots, \varphi_N$ on \mathbb{C} , a discrete subgroup Γ of \mathbb{C} , an algebraic subvariety Y of \mathbb{P}_N and a connected complex abelian Lie group Ω such that

- (a) $\overline{\Phi} = (\varphi_0 : \varphi_1 : \dots : \varphi_N) : G := \mathbb{C}/\Gamma \longrightarrow \Omega$ is a Lie group isomorphism,
- (b) $\varphi_1/\varphi_0, \dots, \varphi_N/\varphi_0$ generate K and K is considered as a subfield of $\mathfrak{M}(G)$,
- (c) Y is the Zariski closure of Ω and $\dim Y = 1$.

When $\text{rank } \Gamma = 2$, K is an elliptic function field.

If $\text{rank } \Gamma = 0$ or 1 , then $G = \mathbb{C}$ or \mathbb{C}^* . Assume that some $\varphi_i/\varphi_0 \in K$ is not extendable meromorphically to the one-dimensional complex projective space \mathbb{P}_1 . Changing indices if necessarily, we may assume that $h := \varphi_1/\varphi_0$ is so. By Picard's theorem we can take $c \in \mathbb{C}$ such that the number of the elements of the set $\{\zeta \in \mathbb{C}; h(\zeta) = c\}$ is infinite. We define a hyperplane H of \mathbb{P}_N by

$$H := \{(x_0 : x_1 : \dots : x_N) \in \mathbb{P}_N; x_1 - cx_0 = 0\}.$$

Then the number of the points of the set $H \cap \Omega$ is infinite. Since $\dim Y = 1$, $H \cap Y = Y$ or a finite set. On the other hand $H \cap \Omega \subset H \cap Y$. Then $H \cap Y = Y$. This means that $h \equiv c$ on $G = \mathbb{C}/\Gamma$. This is a contradiction. \square

DEFINITION 2.8. Let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ satisfying the assumptions in Theorem 2.6. The n -dimensional projective algebraic variety Y is called a *Picard variety* of K .

Let K' be another subfield of $\mathfrak{M}(\mathbb{C}^n)$ satisfying the assumptions in Theorem 2.6, and let Y' be a Picard variety of K' . Then K and K' are isomorphic if and only if Y and Y' are birationally equivalent.

In the examples of the previous section, both K_{II} and K_{III} have the same Picard variety \mathbb{P}_2 . The invertible linear transformation (T) gives an isomorphism between K_{II} and K_{III} . The linear transformation (T) is extendable to an automorphism of \mathbb{P}_2 . The restriction of K_{III} to $\{z = 0\}$ is K_I . However K_{II} does not have such a property. Then K_{II} is a better model than K_{III} .

Let L be a complex linear subspace of \mathbb{C}^n . For $f \in \mathfrak{M}(\mathbb{C}^n)$ we denote by $I(f)$ the set of indeterminacy of f . We define

$$f|_L := \begin{cases} \text{the restriction of } f \text{ to } L, & \text{if } L \not\subset I(f), \\ 0, & \text{if } L \subset I(f). \end{cases}$$

We set $K|_L := \{f|_L; f \in K\}$.

The following definition reflects the above phenomenon.

DEFINITION 2.9. A subfield K of $\mathfrak{M}(\mathbb{C}^n)$ is *algebraically degenerate with respect to complex coordinates* (x_1, \dots, x_n) if there exists i ($1 \leq i \leq n$) such that for the $(n-1)$ -dimensional complex linear subspace $L := \{(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)\}$ of \mathbb{C}^n

$$\text{Trans } K = \text{Trans } K|_L.$$

If there does not exist such a complex linear subspace L , then K is said to be *algebraically non-degenerate with respect to coordinates* (x_1, \dots, x_n) .

3. Extension to compactifications of G

Throughout this section we assume that a subfield $K \subset \mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$ is non-degenerate and admits (AAT). We have just proved in the previous section that K is considered as a subfield of $\mathfrak{M}(\mathbb{C}^n/\Gamma)$. By Remmert-Morimoto's theorem([7] and [9]), we have

$$G := \mathbb{C}^n/\Gamma \cong \mathbb{C}^p \times (\mathbb{C}^*)^q \times X,$$

where $X = \mathbb{C}^r/\Gamma^*$ is a toroidal group of rank $\Gamma^* = r + m$ ($1 \leq m \leq r$) and $p + q + r = n$. Here we say that a connected complex Lie group G_0 is a toroidal

group if $H^0(G_0, \mathcal{O}) = \mathbb{C}$. It is well-known that a toroidal group is abelian (Morimoto [8]). Since there exists a non-degenerate meromorphic function on X , it is a quasi-abelian variety (cf. [3] and [6]). For the definition of quasi-abelian varieties, see Definition 4.1 in the next section. By Andreotti-Gherardelli fibration theorem ([5], see also [2]), there exists a fibre bundle structure $\sigma : X \rightarrow A$ on a k -dimensional abelian variety A with fibres $\mathbb{C}^s \times (\mathbb{C}^*)^t$, where $0 \leq 2s \leq r - m$, $t = r - m - 2s$ and $k = m + s$.

REMARK. Such a fibration is not always unique for X (see an example in [4]).

Consider one of such fibrations $\sigma : X \rightarrow A$. Replacing fibres $\mathbb{C}^s \times (\mathbb{C}^*)^t$ with \mathbb{P}_1^{s+t} , we obtain the associated \mathbb{P}_1^{s+t} -bundle $\bar{\sigma} : \bar{X} \rightarrow A$. We note that G has the structure of $\mathbb{C}^{p+s} \times (\mathbb{C}^*)^{q+t}$ -bundle $\tau : G \rightarrow A$ on A by $\sigma : X \rightarrow A$. Let

$$\bar{G} := \mathbb{P}_1^p \times \mathbb{P}_1^q \times \bar{X}.$$

Then the structure of \mathbb{P}_1^{s+t} -bundle $\bar{\sigma} : \bar{X} \rightarrow A$ gives a \mathbb{P}_1^ℓ -bundle $\bar{\tau} : \bar{G} \rightarrow A$, where $\ell = p + q + s + t$. Take complex coordinates (x_1, \dots, x_n) of \mathbb{C}^n such that the projection to the space of the first k variables gives the $\mathbb{C}^{p+s} \times (\mathbb{C}^*)^{q+t}$ -bundle $\tau : G \rightarrow A$, and fix them.

Proposition 3.1. *Let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$. Suppose that K is non-degenerate and algebraically non-degenerate with respect to coordinates (x_1, \dots, x_n) , and admits (AAT). We set $H := \{(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)\}$ for any i . Then $\text{Trans } K|_H = n - 1$ and $K|_H$ is non-degenerate and algebraically non-degenerate with respect to coordinates $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and admits (AAT).*

Proof. It is obvious that $K|_H$ is non-degenerate and admits (AAT). By Lemma 2.2 and a well-known fact that the analytical independence induces the algebraical independence, we have

$$\text{Trans } K|_H \geq n - 1.$$

It follows from Proposition 2.10 and the assumption of the proposition that

$$\text{Trans } K|_H \leq n - 1.$$

The rest is to show that $K|_H$ is algebraically non-degenerate with respect to coordinates $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Suppose that $K|_H$ is algebraically degenerate with respect to coordinates $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Then there exists an $(n - 2)$ -dimensional complex linear subspace $H_0 = \{x_i = x_j = 0 \ (i \neq j)\}$ such that

$$\text{Trans } K|_H = \text{Trans } K|_{H_0}.$$

Let $L := \{(0, \dots, 0, x_i, 0, \dots, 0)\}$. Then

$$\mathbb{C}^n = L \oplus H.$$

We set $H' := L \oplus H_0 = \{(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)\}$. We take a non-zero function $f \in K$ with $f|_H = 0$. Then $f|_{H'} \neq 0$. Since $\text{Trans } K|_{H_0} = n - 1$, there exist $g_1, \dots, g_{n-1} \in K$ such that $g_1|_{H_0}, \dots, g_{n-1}|_{H_0}$ are algebraically independent. In this case, $f|_{H'}, g_1|_{H'}, \dots, g_{n-1}|_{H'}$ are algebraically independent. In fact, if these functions are algebraically dependent, then there exists a non-zero irreducible polynomial $P(X, Y_1, \dots, Y_{n-1})$ such that

$$P(f|_{H'}, g_1|_{H'}, \dots, g_{n-1}|_{H'}) = 0.$$

Since $f|_{H_0} = 0$, we obtain

$$P(0, g_1|_{H_0}, \dots, g_{n-1}|_{H_0}) = 0.$$

This contradicts that $g_1|_{H_0}, \dots, g_{n-1}|_{H_0}$ are algebraically independent. Therefore we obtain

$$\text{Trans } K|_{H'} = n.$$

Hence K is algebraically degenerate with respect to coordinates (x_1, \dots, x_n) . This is a contradiction. \square

By an inductive argument we obtain the following corollary.

Corollary 3.2. *Let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ as in Proposition 3.1. Then, for any k -dimensional complex linear subspace $L = \{x_{i_1} = 0, \dots, x_{i_{n-k}} = 0\}$ of \mathbb{C}^n we have*

$$\text{Trans } K|_L = k.$$

Theorem 3.3. *Let K be a subfield of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$. We assume that K is non-degenerate and admits (AAT). Consider a \mathbb{P}_1^ℓ -bundle $\bar{\tau} : \bar{G} \rightarrow A$ and coordinates (x_1, \dots, x_n) as above. We further assume that K is algebraically non-degenerate with respect to coordinates (x_1, \dots, x_n) . Then any function $f \in K$ is extendable meromorphically to \bar{G} .*

Proof. For $a \in A$ we set

$$F_a := \tau^{-1}(a) \cong \mathbb{C}^{p+s} \times (\mathbb{C}^*)^{q+t},$$

$$\bar{F}_a := \mathbb{P}_1^\ell, \quad \ell = p + s + q + t.$$

Let $\pi : \mathbb{C}^n \rightarrow G$ be the projection. Any $f \in K$ is considered as a meromorphic function on \mathbb{C}^n with period Γ .

Take any 1-dimensional complex linear subspace $L = \{(0, \dots, 0, x_{k+i}, 0, \dots, 0)\}$ of \mathbb{C}^n with

$$\pi(L) \subset F_0 = \tau^{-1}(0).$$

Then $\pi(L) = \mathbb{C}$ or \mathbb{C}^* . By Corollary 3.2 $\text{Trans } K|_L = 1$ and $K|_L$ admits (AAT). Let $\bar{\Phi} : G \rightarrow \Omega \subset \mathbb{P}_N$ be the Lie group isomorphism in Theorem 2.6. Since $\bar{\Phi}|_{\pi(L)}$ is one-to-one, $K|_L$ is not doubly periodic. Therefore any $\psi \in K|_L$ is extendable meromorphically to \mathbb{P}_1 by Theorem 2.7. It follows from (AAT) that $f_a(x) := f(x + a) \in K$ for all $a \in \mathbb{C}^n$. Then any $f \in K$ is meromorphically extendable to \bar{F}_a by virtue of Hartogs' theorem for meromorphic functions([12]). Using again the same theorem, we can conclude that $f \in K$ is extendable meromorphically to \bar{G} . \square

4. Normal form of ample generalized theta factors

We collect here a part of results in [4], which is needed in our arguments.

Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group of rank $\Gamma = n + m$ ($1 \leq m < n$). We denote by \mathbb{R}_Γ^{n+m} the real linear subspace of \mathbb{C}^n spanned by Γ . Let $\mathbb{C}_\Gamma^m = \mathbb{R}_\Gamma^{n+m} \cap \sqrt{-1}\mathbb{R}_\Gamma^{n+m}$ be the maximal complex linear subspace contained in \mathbb{R}_Γ^{n+m} .

DEFINITION 4.1. A toroidal group $X = \mathbb{C}^n / \Gamma$ is said to be a *quasi-abelian variety* if there exists a hermitian form \mathcal{H} on \mathbb{C}^n such that

- (a) \mathcal{H} is positive definite on \mathbb{C}_Γ^m ,
- (b) the imaginary part $\mathcal{A} := \text{Im } \mathcal{H}$ of \mathcal{H} is \mathbb{Z} -valued on $\Gamma \times \Gamma$.

We call such a hermitian form \mathcal{H} an *ample generalized Riemann form* of X .

DEFINITION 4.2. An ample generalized Riemann form \mathcal{H} of a quasi-abelian variety $X = \mathbb{C}^n / \Gamma$ is of *kind k* if $\text{rank } \mathcal{A}_\Gamma = 2(m + k)$, where $\mathcal{A}_\Gamma := \mathcal{A}|_{\mathbb{R}_\Gamma^{n+m} \times \mathbb{R}_\Gamma^{n+m}}$.

Let $\rho : \Gamma \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ be a factor of automorphy(see [16] for the definition).

DEFINITION 4.3. A factor of automorphy ρ is called a *generalized theta factor* if it is represented as

$$\rho(\gamma, x) = \mathbf{e}(\mathcal{L}_\gamma(x) + c(\gamma))$$

for all $\gamma \in \Gamma$ and $x \in \mathbb{C}^n$, where \mathcal{L}_γ is a linear polynomial, $c(\gamma)$ is a constant and $\mathbf{e}(x) := \exp(2\pi\sqrt{-1}x)$.

A generalized theta factor $\rho : \Gamma \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ has the following expression

$$\rho(\gamma, x) = \psi(\gamma)\mathbf{e}\left[\frac{1}{2\sqrt{-1}}(\mathcal{H} + Q)(\gamma, x) + \frac{1}{4\sqrt{-1}}(\mathcal{H} + Q)(\gamma, \gamma) + \mathcal{L}(\gamma)\right]$$

for $(\gamma, x) \in \Gamma \times \mathbb{C}^n$, where \mathcal{H} is a hermitian form on \mathbb{C}^n with $\mathcal{A} := \text{Im } \mathcal{H}$ \mathbb{Z} -valued on $\Gamma \times \Gamma$, Q is a \mathbb{C} -bilinear symmetric form on \mathbb{C}^n , \mathcal{L} is a \mathbb{C} -linear form on \mathbb{C}^n and

ψ is a semi-character of Γ associated with \mathcal{A} ([7]). In this case we say that ρ is of type $(\mathcal{H}, \psi, Q, \mathcal{L})$. As usual theta factors, ρ is equivalent to a generalized theta factor of type $(\mathcal{H}, \psi, 0, 0)$, which we call a reduced generalized theta factor of type (\mathcal{H}, ψ) .

Suppose that ρ is a reduced generalized theta factor of type (\mathcal{H}, ψ) . If \mathcal{H} is an ample generalized Riemann form, then ρ is said to be ample. We may assume that \mathcal{H} is positive definite on \mathbb{C}^n if \mathcal{H} is ample([5]). In this case there exists a theta factor $\tilde{\rho} : \tilde{\Gamma} \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ such that ρ is the restriction of $\tilde{\rho}$ to $\Gamma \times \mathbb{C}^n$, where $\tilde{\Gamma}$ is a discrete subgroup of rank $2n$ with $\Gamma \subset \tilde{\Gamma}$ and $A = \mathbb{C}^n / \tilde{\Gamma}$ is an abelian variety.

Now we assume that \mathcal{H} is of kind k ($0 \leq 2k \leq n - m$). We denote by H_n the Siegel upper half space of degree n . Let \tilde{P} and P be period matrices of $\tilde{\Gamma}$ and Γ respectively. After a suitable change of period matrices using invertible matrices and unimodular matrices, we obtain normal forms of \tilde{P} and P as follows

$$\tilde{P} = (W \ D), \quad W = (w_{ij}) \in H_n$$

$$D = \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & d_n & \end{bmatrix},$$

where d_i ($i = 1, \dots, n$) are positive integers with $d_1 | d_2 | \dots | d_n$, and

$$(4.4) \quad P = \begin{bmatrix} & d_1 & & & & \\ W' & & \ddots & & & \\ & & & d_{m+k} & & \\ & & & & d_{m+k+1} & \\ W'' & & & & & \ddots & \\ & & & & & & d_{n-k} \\ & & & & & & & 0 \end{bmatrix} = (P' \ P''),$$

where we put

$$W' := \begin{bmatrix} w_{11} & \dots & w_{1,m+k} \\ \vdots & & \vdots \\ w_{m+k,1} & \dots & w_{m+k,m+k} \end{bmatrix} \in H_{m+k},$$

$$W'' := \begin{bmatrix} w_{m+k+1,1} & \dots & w_{m+k+1,m+k} \\ \vdots & & \vdots \\ w_{n,1} & \dots & w_{n,m+k} \end{bmatrix}.$$

Any $x \in \mathbb{C}^n$ can be represented as

$$x = Wx' + Dx'', \quad x', x'' \in \mathbb{R}^n.$$

We assign

$$\mathbf{x} = \begin{bmatrix} x' \\ x'' \end{bmatrix} \in \mathbb{R}^{2n} \quad \text{for } x \in \mathbb{C}^n.$$

It is well-known that $\tilde{\rho}$ is equivalent to the following theta factor $\tilde{\rho}_0 : \tilde{\Gamma} \times \mathbb{C}^n \rightarrow \mathbb{C}^*$,

$$\tilde{\rho}_0(\tilde{\gamma}, x) = \mathbf{e} \left[- {}^t\gamma'x - \frac{1}{2} {}^t\gamma'W\gamma' \right]$$

for $\tilde{\gamma} \in \tilde{\Gamma}$ with $\tilde{\gamma} = W\gamma' + D\gamma''$ and $x \in \mathbb{C}^n$. Then ρ is equivalent to $\rho_0 := \tilde{\rho}_0|_{\Gamma \times \mathbb{C}^n}$. Take complex coordinates $(z, w) = (z_1, \dots, z_{m+k}; w_1, \dots, w_{n-m-k})$ of \mathbb{C}^n , where $z = (z_1, \dots, z_{m+k})$ and $w = (w_1, \dots, w_{n-m-k})$ represent the first $m+k$ rows and the last $n-m-k$ rows in (4.4) respectively. Every $\gamma \in \Gamma$ has the unique representation

$$\gamma = W \begin{bmatrix} a \\ 0 \end{bmatrix} + D \begin{bmatrix} b \\ 0 \end{bmatrix} = P'a + P''b,$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_{m+k} \end{bmatrix} \in \mathbb{Z}^{m+k}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-k} \end{bmatrix} \in \mathbb{Z}^{n-k}.$$

Then we have the explicit representation of ρ_0 as follows

$$\rho_0(\gamma, x) = \mathbf{e} \left[- {}^taz - \frac{1}{2} {}^t_aW'a \right]$$

for $\gamma = P'a + P''b$ and $x = (z, w) \in \mathbb{C}^n$. Thus we obtain the following proposition.

Proposition 4.5. *Let $\rho : \Gamma \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ be a reduced generalized theta factor of type (\mathcal{H}, ψ) . Assume that \mathcal{H} is ample and of kind k . Then we can take a period matrix P of Γ as in (4.4) and ρ is represented as*

$$(4.6) \quad \rho(\gamma, x) = \mathbf{e} \left[- {}^taz - \frac{1}{2} {}^t_aW'a \right]$$

for $\gamma = P'a + P''b \in \Gamma$ and $x = (z, w) \in \mathbb{C}^n$.

5. Extendable line bundles on quasi-abelian varieties

Let $G = \mathbb{C}^n/\Gamma$ be a non-compact quasi-abelian variety of rank $\Gamma = n + m$ ($1 \leq m < n$). Consider a holomorphic line bundle $L_1 \rightarrow G$. Then we have $L_1 \cong L_0 \otimes L_\rho$, where L_0 is a topologically trivial holomorphic line bundle and L_ρ is a holomorphic line bundle given by a reduced generalized theta factor ρ of type (\mathcal{H}, ψ) ([17], see also [1]). Suppose that \mathcal{H} is an ample generalized Riemann form

of kind k ($0 \leq 2k \leq n - m$). Then we may assume that a period matrix P of Γ has the form in (4.4) and ρ is expressed as (4.6) in Proposition 4.5.

The projection $\tau : \mathbb{C}^n \rightarrow \mathbb{C}^{m+k}$ to the first $(m+k)$ variables gives a $\mathbb{C}^k \times (\mathbb{C}^*)^{n-m-2k}$ -bundle $\tau : G \rightarrow A$ on an $(m+k)$ -dimensional abelian variety A . We denote by $\bar{\tau} : \bar{G} \rightarrow A$ the associated \mathbb{P}_1^{n-m-k} -bundle on A . Now we consider the following problem.

Problem 5.1. Which line bundle L_1 is extendable to \bar{G} ?

This problem was considered by M.Stein in his Dissertation[15] in the case that G is a toroidal group, $\tau : G \rightarrow T$ is a $(\mathbb{C}^*)^{n-m}$ -bundle on an m -dimensional complex torus T and L_1 is any holomorphic line bundle. We have to modify his method. Because a part of his arguments does not fit when $k \neq 0$.

First we define a mapping $p : \mathbb{C}^n \rightarrow \mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$ by

$$p(x_1, \dots, x_n) = (x_1, \dots, x_{m+k}, e(x_{m+k+1}), \dots, e(x_{n-k}), x_{n-k+1}, \dots, x_n).$$

Then $p(\Gamma)$ is a subgroup of $\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$. Any $\eta = (\eta_1, \dots, \eta_n) \in p(\Gamma)$ acts on $\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$ by

$$\eta \cdot (y_1, \dots, y_n) = (y_1 + \eta_1, \dots, y_{m+k} + \eta_{m+k}, y_{m+k+1} \cdot \eta_{m+k+1}, \dots, y_{n-k} \cdot \eta_{n-k}, y_{n-k+1} + \eta_{n-k+1}, \dots, y_n + \eta_n).$$

This action is extendable to $\mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k}$. Then we consider $p(\Gamma)$ as a subgroup of the automorphism group of $\mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k}$. We have the isomorphisms

$$\bar{G} \cong (\mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k})/p(\Gamma)$$

$$G \cong (\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k) / p(\Gamma).$$

For the sake of simplicity we write

$$\bar{G} = (\mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k})/p(\Gamma).$$

There exists the inclusion mapping $\iota : G = \mathbb{C}^n/\Gamma \rightarrow \bar{G}$. Let $\pi_1 : \mathbb{C}^n \rightarrow G$ and $\pi_2 : \mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k} \rightarrow \bar{G}$ be projections. We have the following diagram

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-k} \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ G & \xrightarrow{\iota} & \bar{G} \end{array}$$

The meaning of the extendability of L_1 to \overline{G} is that there exists a holomorphic line bundle $L \rightarrow \overline{G}$ such that

$$(\iota^{-1})^*L_1 \cong L|_{\iota(G)},$$

where

$$\begin{aligned} \iota(G) &= \pi_2 (\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k) \\ &= (\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k) / p(\Gamma). \end{aligned}$$

Take a subset $I \subset \{1, \dots, n - m - 2k\}$. Let $I^c := \{1, \dots, n - m - 2k\} \setminus I$. We define

$$X_{m+k+i}(I) := \begin{cases} \mathbb{C} & \text{if } i \in I, \\ \mathbb{P}_1 \setminus \{0\} & \text{if } i \in I^c. \end{cases}$$

We set

$$\begin{aligned} \mathbb{P}(I) &:= X_{m+k+1}(I) \times \dots \times X_{n-k}(I) \times (\mathbb{P}_1 \setminus \{0\})^k, \\ G(I) &:= \pi_2 (\mathbb{C}^{m+k} \times \mathbb{P}(I)). \end{aligned}$$

Then we have $\iota(G) \subset G(I) \subset \overline{G}$.

We denote

$$\Gamma^* := \tau(p(\Gamma)) = \tau(\Gamma),$$

where $\tau : \mathbb{C}^{m+k} \times \mathbb{P}_1^{n-m-2k} \rightarrow \mathbb{C}^{m+k}$ is the projection to the space of the first $(m+k)$ -variables. Then Γ^* is a discrete subgroup of rank $2(m+k)$ and $A = \mathbb{C}^{m+k} / \Gamma^*$. A period matrix P^* of Γ^* is

$$P^* = \begin{bmatrix} & d_1 & & \\ W' & & \ddots & \\ & & & \\ & & & d_{m+k} \end{bmatrix} = [W' \quad D'].$$

We define a theta factor $\rho_0 : \Gamma^* \times \mathbb{C}^{m+k} \rightarrow \mathbb{C}^*$ by

$$\rho_0(\gamma^*, z) = e \left[- {}^t a^* z - \frac{1}{2} {}^t a^* W' a^* \right]$$

for $\gamma^* = W' a^* + D' b^*$ and $z \in \mathbb{C}^{m+k}$. Let $L_{\rho_0} \rightarrow A$ be the holomorphic line bundle on A defined by ρ_0 . Let $\tilde{L} := \bar{\tau}^* L_{\rho_0}$ be the pull-back of L_{ρ_0} by $\bar{\tau} : \overline{G} \rightarrow A$. For any $I \subset \{1, \dots, n - m - 2k\}$ we have $\mathbb{C}^{m+k} \times \mathbb{P}(I) \cong \mathbb{C}^n$. Then $\tilde{L}|_{G(I)}$ is given by a factor of automorphy

$$\begin{aligned} \rho_I : p(\Gamma) \times (\mathbb{C}^{m+k} \times \mathbb{P}(I)) &\longrightarrow \mathbb{C}^*, \\ \rho_I(\eta, y) &= \rho_0(\tau(\eta), \tau(y)). \end{aligned}$$

On the other hand, $\rho(\gamma, x) = \rho_0(\tau \circ p(\gamma), \tau \circ p(x))$. Hence we have $(\iota^{-1})^* L_\rho \cong \tilde{L}|_{\iota(G)}$. Thus we obtain the following proposition.

Proposition 5.2. *For L_ρ there exists a holomorphic line bundle $L_{\rho_0} \rightarrow A$ given by a theta factor $\rho_0 : \Gamma^* \times \mathbb{C}^{m+k} \rightarrow \mathbb{C}^*$ such that $L_\rho \cong \tau^* L_{\rho_0}$.*

Let $L \rightarrow G(I)$ be a holomorphic line bundle on $G(I)$, where $I \subset \{1, \dots, n-m-2k\}$. Assume that L is given by a factor of automorphy $\alpha : p(\Gamma) \times (\mathbb{C}^{m+k} \times \mathbb{P}(I)) \rightarrow \mathbb{C}^*$. Then $\iota^* L := \iota^*(L|_{\iota(G)})$ is given by a factor of automorphy $\tilde{\alpha} : \gamma \times \mathbb{C}^n \rightarrow \mathbb{C}^*$, $\tilde{\alpha}(\gamma, x) := \alpha(p(\gamma), p(x))$.

Lemma 5.3. *Suppose that holomorphic line bundles $L \rightarrow G(I)$ and $L_1 \rightarrow G$ are given by factors of automorphy $\alpha : \Gamma \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ and $\beta : p(\Gamma) \times (\mathbb{C}^{m+k} \times \mathbb{P}(I)) \rightarrow \mathbb{C}^*$, respectively. Then*

$$(\iota^{-1})^* L_1 \cong L|_{\iota(G)}$$

if and only if there exists a holomorphic function $c : \mathbb{C}^n \rightarrow \mathbb{C}^$ such that*

$$c(x + \gamma)\beta(\gamma, x)c(x)^{-1} = \alpha(p(\gamma), p(x)) \quad \text{for } (\gamma, x) \in \Gamma \times \mathbb{C}^n.$$

Proof. It holds that $(\iota^{-1})^* L_1 \cong L|_{\iota(G)}$ if and only if $L_1 \cong \iota^* L$. Since $\iota^* L$ is given by a factor of automorphy $\alpha(p(\gamma), p(x))$, we obtain the conclusion. \square

Proposition 5.4. *Let $L_1 \rightarrow G$ be a topologically trivial holomorphic line bundle on G . Suppose that there exists a holomorphic line bundle $L \rightarrow \bar{G}$ on \bar{G} such that*

$$(\iota^{-1})^* L_1 \cong L|_{\iota(G)}.$$

Then there exists a homomorphism $\psi : \Gamma^ \rightarrow \mathbb{C}^*$ such that $L_1 \rightarrow G$ is given by the homomorphism $\varphi := \psi \circ \tau : \Gamma \rightarrow \mathbb{C}^*$.*

Proof. Let

$$D_{n-k} := \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & & d_{n-k} \end{bmatrix}.$$

We assume that L_1 is defined by a factor of automorphy $\alpha(\gamma, x) = \exp(a(\gamma, x))$. The summand of automorphy $a(\gamma, x)$ has the following properties.

- (i) $a(\gamma, x) = 0$ for $\gamma \in D_{n-k}\mathbb{Z}^{n-k}$.
- (ii) $a(\gamma, x)$ is $D_{n-k}\mathbb{Z}^{n-k}$ -periodic with respect to x_1, \dots, x_{n-k} .

Take a subset $I \subset \{1, \dots, n-m-2k\}$. By Lemma 5.3 there exist a factor of automorphy $\beta^I : p(\Gamma) \times (\mathbb{C}^{m+k} \times \mathbb{P}(I)) \rightarrow \mathbb{C}^*$ and a holomorphic function $c^I : \mathbb{C}^n \rightarrow \mathbb{C}^*$ such that

$$(5.5) \quad \beta^I(p(\gamma), p(x)) = c^I(x + \gamma)\alpha(\gamma, x)c^I(x) \quad \text{for all } (\gamma, x) \in \Gamma \times \mathbb{C}^n.$$

Take a holomorphic function $f^I(x)$ on \mathbb{C}^n such that $c^I(x) = \exp(f^I(x))$. Let e_j be the j -th unit vector of \mathbb{C}^n . Then $\alpha(d_j e_j, x) = 1$ ($j = 1, \dots, n-k$). We have a

mapping $b^I : p(\Gamma) \times (\mathbb{C}^{m+k} \times \mathbb{P}(I)) \rightarrow \mathbb{C}$ such that $\beta^I(p(\gamma), y) = \exp(b^I(p(\gamma), y))$. Let $E = c_1(L_{\beta^I}) \in H^2(G(I), \mathbb{Z})$ be the first Chern class of the line bundle L_{β^I} on $G(I)$ given by β^I . Then it is well-known that

$$E(p(\gamma_1), p(\gamma_2)) = \frac{1}{2\pi\sqrt{-1}} [b^I(p(\gamma_2), p(\gamma_1)y) + b^I(p(\gamma_1), y) - b^I(p(\gamma_1), p(\gamma_2)y) - b^I(p(\gamma_2), y)].$$

Since L_1 is topologically trivial, $E(p(\gamma_1), p(\gamma_2)) = 0$ for $\gamma_1, \gamma_2 \in \Gamma$ by (5.5). Then L_{β^I} is topologically trivial. Therefore we may assume that $\beta^I(p(d_j e_j), p(x)) = 1$ for $j = 1, \dots, n - k$. Hence we have

$$f^I(x + d_j e_j) - f^I(x) = k_j \in 2\pi\sqrt{-1}\mathbb{Z}, \quad j = 1, \dots, n - k.$$

We define a $D_{n-k}\mathbb{Z}^{n-k}$ -periodic function

$$k^I(x) := f^I(x) - \sum_{j=1}^{n-k} \frac{k_j}{d_j} x_j.$$

Let $x = (x', x'', x''') \in \mathbb{C}^{m+k} \times \mathbb{C}^{n-m-2k} \times \mathbb{C}^k$. Considering $k^I(x)$ as a periodic function of x'' , we obtain the Fourier expansion

$$f^I(x) = \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} f^I_{\sigma}(x', x''') e({}^t \sigma(x''/d'')) + \sum_{j=m+k+1}^{n-k} \frac{k_j}{d_j} x_j,$$

where we set

$$x''/d'' = ({}^t(x_{m+k+1}/d_{m+k+1}, \dots, x_{n-k}/d_{n-k})).$$

Similarly we have

$$a(\gamma, x) = \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} a_{\gamma, \sigma}(x', x''') e({}^t \sigma(x''/d'')),$$

$$b^I(p(\gamma), p(x)) = \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} b^I_{\gamma, \sigma}(x', x''') e({}^t \sigma(x''/d'')).$$

By the right hand side of (5.5), $b^I(p(\gamma), p(x))$ is holomorphic on \mathbb{C} with respect to x_{n-k+j} ($j = 1, \dots, k$). On the other hand, $b^I(p(\gamma), y)$ is holomorphic on $\mathbb{P}_1 \setminus \{0\}$ with respect to y_{n-k+j} . Since the $(n - k + j)$ -th element $(p(x))_{n-k+j}$ of $p(x)$ is equal to x_{n-k+j} , $b^I(p(\gamma), p(x))$ is holomorphic on \mathbb{P}_1 with respect to x_{n-k+j} ($j = 1, \dots, k$). Hence it does not depend on x''' . Thus we obtain

$$b^I(p(\gamma), p(x)) = \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} b^I_{\gamma, \sigma}(x') e({}^t \sigma(x''/d'')).$$

For $i \in I$, $X_{m+k+i}(I) = \mathbb{C}$. Then $b^I(p(\gamma), y)$ is holomorphic on \mathbb{C} with respect to y_{m+k+i} . If $j \in I^c$, then $X_{m+k+j}(I) = \mathbb{P}_1 \setminus \{0\}$. Therefore $b^I(p(\gamma), y)$ is holomorphic on $\mathbb{P}_1 \setminus \{0\}$ with respect to y_{m+k+j} . Hence we have

$$(5.6) \quad b_{\gamma, \sigma}^I(x') = 0$$

for all $\sigma = (\sigma_1, \dots, \sigma_{n-m-2k})$ with $\sigma_i < 0$ for some $i \in I$ or with $\sigma_j > 0$ for some $j \in I^c$. It follows from (5.5) that

$$(5.7) \quad \begin{aligned} & \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} b_{\gamma, \sigma}^I(x') e({}^t\sigma(x''/d'')) = \\ & \sum_{\sigma \in \mathbb{Z}^{n-m-2k}} [a_{\gamma, \sigma}(x', x''') + f_{\sigma}^I(x' + \gamma', x''' + \gamma''') e({}^t\sigma(\gamma''/d'')) - f_{\sigma}^I(x', x''')] \\ & \times e({}^t\sigma(x''/d'')) + \sum_{j=m+k+1}^{n-k} \frac{k_j}{d_j} \gamma_j + 2\pi\sqrt{-1}n_{\gamma}, \quad n_{\gamma} \in \mathbb{Z}. \end{aligned}$$

By (5.6) and (5.7) we obtain

$$(5.8) \quad a_{\gamma, \sigma}(x', x''') + f_{\sigma}^I(x' + \gamma', x''' + \gamma''') e({}^t\sigma(x''/d'')) - f_{\sigma}^I(x', x''') = 0$$

for all $\sigma = (\sigma_1, \dots, \sigma_{n-m-2k})$ with $\sigma_i < 0$ for some $i \in I$ or with $\sigma_j > 0$ for some $j \in I^c$, and

$$(5.9) \quad b_{\gamma, 0}^I(x') = a_{\gamma, 0}(x', x''') + f_0^I(x' + \gamma', x''' + \gamma''') - f_0^I(x', x''') + \text{constant}.$$

We define

$$g_0(x) := f_0^I(x', x'''),$$

$$\begin{aligned} g^I(x) := & \sum_{\substack{\sigma \in \mathbb{Z}^{n-m-2k} \\ \sigma_i < 0 \text{ for some } i \in I}} f_{\sigma}^I(x', x''') e({}^t\sigma(x''/d'')) \\ & + \sum_{\substack{\sigma \in \mathbb{Z}^{n-m-2k} \\ \sigma_j > 0 \text{ for some } j \in I^c}} f_{\sigma}^I(x', x''') e({}^t\sigma(x''/d'')). \end{aligned}$$

Then $g_0(x)$ and $g^I(x)$ are holomorphic functions on \mathbb{C}^n .

Take disjoint subsets I_1, \dots, I_N of $\{1, \dots, n - m - 2k\}$ such that

$$\bigcup_{\alpha=1}^N I_{\alpha} = \{1, \dots, n - m - 2k\}.$$

And we set

$$g(x) := g_0(x) + \sum_{\alpha=1}^N g^{I_\alpha}(x).$$

Then $c(x) := \exp(g(x))$ is a $D_{n-k}\mathbb{Z}^{n-k}$ -periodic \mathbb{C}^* -valued holomorphic function on \mathbb{C}^n . We define

$$\tilde{\alpha}(\gamma, x) := c(x + \gamma)\alpha(\gamma, x)c(x)^{-1}.$$

Then $\tilde{\alpha}$ is a factor of automorphy equivalent to α . It follows from (5.8) and (5.9) that $\tilde{\alpha}(\gamma, x)$ does not depend on (x'', x''') . Therefore there exists a factor of automorphy $\alpha_0 : \Gamma^* \times \mathbb{C}^{m+k} \rightarrow \mathbb{C}^*$ such that

$$\tilde{\alpha}(\gamma, x) = \alpha_0(\tau(\gamma), \tau(x)).$$

The holomorphic line bundle L_{α_0} on $A = \mathbb{C}^{m+k}/\Gamma^*$ given by α_0 is topologically trivial. And A is an abelian variety. Then there exists a homomorphism $\psi : \Gamma^* \rightarrow \mathbb{C}^*$ such that α_0 and ψ are equivalent. \square

Theorem 5.10. *Let $L_1 \rightarrow G$ be the holomorphic line bundle given in the beginning of this section. Suppose that there exists a holomorphic line bundle $L \rightarrow \overline{G}$ such that*

$$(\iota^{-1})^*L_1 \cong L|_{\iota(G)}.$$

Then there exists a holomorphic line bundle $L' \rightarrow A$ given by a theta factor such that

$$L_1 \cong \tau^*L'.$$

Proof. Let $L_1 = L_0 \otimes L_\rho$. By Proposition 5.2 there exists a holomorphic line bundle $L_{\rho_0} \rightarrow A$ given by a theta factor $\rho_0 : \Gamma^* \times \mathbb{C}^{m+k} \rightarrow \mathbb{C}^*$ such that $L_\rho \cong \tau^*L_{\rho_0}$. Then L_ρ is extendable to \overline{G} . Since $L_0 = L_1 \otimes L_\rho^{-1}$, there exists a holomorphic line bundle $\tilde{L}_0 \rightarrow \overline{G}$ such that

$$(\iota^{-1})^*L_0 \cong \tilde{L}_0|_{\iota(G)}.$$

Thus we obtain a homomorphism $\psi : \Gamma^* \rightarrow \mathbb{C}^*$ by Proposition 5.4 such that L_0 is given by the homomorphism $\varphi := \psi \circ \tau : \Gamma \rightarrow \mathbb{C}^*$. \square

6. Main result

Consider a subfield K of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$ which is non-degenerate and admits (AAT). Take a non-degenerate function $f \in K$. Let $L_1 := [(f)_0]$ be the holomorphic line bundle on $G = \mathbb{C}^n/\Gamma$ given by the zero-divisor $(f)_0$ of f . Here we have

$$G = \mathbb{C}^p \times (\mathbb{C}^*)^q \times X,$$

where X is a quasi-abelian variety. Consider a compactification \overline{G} of G and complex coordinates (x_1, \dots, x_n) of \mathbb{C}^n as in Section 3. We assume that K is algebraically non-degenerate with respect to coordinates (x_1, \dots, x_n) . Then f is extendable meromorphically to \overline{G} , hence L_1 is extendable holomorphically to \overline{G} . We know that the meromorphic functions on $\mathbb{C}^p \times (\mathbb{C}^*)^q$ which are extendable meromorphically to \mathbb{P}_1^{p+q} are the rational functions. The restriction $g := f|_X$ of f to X is a non-degenerate meromorphic function on the quasi-abelian variety X .

Proposition 6.1. *Let $X = \mathbb{C}^r / \Gamma$ be a quasi-abelian variety of rank $\Gamma = r + s$ ($1 \leq s \leq r$). Suppose that there exists a non-degenerate meromorphic function g on X . Let \overline{X} be the compactification of X determined by $L_1 := [(g)_0]$ as in the previous section. If g is extendable meromorphically to \overline{X} , then $\text{rank } \Gamma = 2r$ and X is an abelian variety.*

Proof. Suppose that $1 \leq s < r$. Then we have a bundle $\tau : X \rightarrow A$ on an abelian variety A with $\dim A < \dim X$. By Theorem 5.10 there exists a holomorphic line bundle $L' \rightarrow A$ such that $L_1 = \tau^* L'$. Let $\overline{\tau} : \overline{X} \rightarrow A$ be the associated \mathbb{P}_1^{r-s-k} -bundle. We denote by \overline{g} and \overline{L}_1 the meromorphic extension of g to \overline{X} and the holomorphic extension of L_1 to \overline{X} , respectively. There exist $\varphi, \psi \in H^0(\overline{X}, \mathcal{O}(\overline{L}_1))$ such that $\overline{g} = \psi / \varphi$. Since $\overline{L}_1 = \overline{\tau}^* L'$, we have

$$H^0(\overline{X}, \mathcal{O}(\overline{L}_1)) = \overline{\tau}^* H^0(A, \mathcal{O}(L')).$$

Then \overline{g} is degenerate. This is a contradiction. \square

Theorem 6.2 (Main Theorem). *Let K be a non-degenerate subfield of $\mathfrak{M}(\mathbb{C}^n)$ with $\text{Trans } K = n$ which admits (AAT). Let Γ be the discrete subgroup of \mathbb{C}^n in Theorem 2.6. We take complex coordinates (x_1, \dots, x_n) of \mathbb{C}^n as in Section 3. Suppose that K is algebraically non-degenerate with respect to (x_1, \dots, x_n) . Then we have*

$$\mathbb{C}^n / \Gamma = \mathbb{C}^p \times (\mathbb{C}^*)^q \times A,$$

where A is an r -dimensional abelian variety ($n = p + q + r$), and K is a subfield of $\mathbb{C}(z_1, \dots, z_p, w_1, \dots, w_q, g_0, g_1, \dots, g_r)$, where z_1, \dots, z_p and w_1, \dots, w_q are coordinate functions of \mathbb{C}^p and $(\mathbb{C}^*)^q$ respectively, and g_0, g_1, \dots, g_r are generators of the abelian function field $\mathfrak{M}(A)$ on A .

Proof. By Theorem 2.6 we have that K is a subfield of $\mathfrak{M}(\mathbb{C}^n / \Gamma)$ and

$$\mathbb{C}^n / \Gamma = \mathbb{C}^p \times (\mathbb{C}^*)^q \times X,$$

where $X = \mathbb{C}^r / \Gamma^*$ is a quasi-abelian variety. Since K is non-degenerate, X is an abelian variety by Proposition 6.1. Furthermore any $f \in K$ is extendable meromorphically to $\mathbb{P}_1^{p+q} \times X$ by Theorem 3.3. Then we obtain the conclusion. \square

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