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THE MCKAY NUMBERS OF A SUBGROUP OF *GL(N; Q)* CONTAINING *SL(N; Q)*

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1. Introduction

For a finite group G, a prime number p, a non-negative integer k, and a p-block B of G, we put

$$m_p(k, G, B) = |\{\zeta \in \mathrm{Irr}(G) \mid \nu(\zeta(1)) = k, \zeta \in B\}|,$$

where $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G and ν is the exponential valuation of some splitting field of G with $\nu(p) = 1$. The sum $m_p(k, G) = \sum_B m_p(k, G, B)$ over all p-blocks of G is called the k-th McKay number of G.

Let $GL = GL(n, q)$ be the general linear group of degree n over the finite field $GF(q)$ with q elements, where $q = p^e$ is a power of the prime p. Let

$$L_h = L_h(n, q) = \{x \in GL(n, q) \mid \det(x) \in U_h\},$$

where U_h is the subgroup of the multiplicative group F_1 of $GF(q)$ of order h. (Thus h is a divisor of $q - 1$.) In particular, $L_{q-1}(n, q)$ is $GL(n, q)$ and $L_1(n, q)$ is the special linear group $SL(n, q)$. In general, $L_h(n, q)$ satisfies

$$GL(n, q) \trianglerighteq L_h(n, q) \trianglerighteq SL(n, q).$$

Moreover, we denote by $PL_h = PL_h(n, q)$ the factor group of $L_h(n, q)$ modulo its center $Z(L_h(n, q))$.

In Section 4 of this paper, we write $m_p(k, G, B)$ concretely in terms of several invariants of partitions, where $G = L_h(n, q)$ or $G = PL_h(n, q)$.

In Section 5, we show the Alperin-McKay conjecture [1] holds for L_h and PL_h. Note that for L_h or PL_h, every p-block is of defect 0 or maximal defect. Thus it suffices to prove the following. If a p-block B of G is not of defect zero, then for a Sylow p-subgroup P of G and the p-block b of the normalizer $N_G(P)$ of P corresponding to B by Brauer's first main theorem, we have

$$m_p(0, G, B) = m_p(0, N_G(P), b).$$

Section 2 is devoted to stating several preliminary results, and Section 3 is devoted a parametrization of irreducible characters of L_h. Notations are standard. See, for example, [7].

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2. Preliminaries

In this section, we mention several definitions and results which are important when studying irreducible characters of GL and related groups.

2.1. Polynomials and Simplices. Let n be a fixed integer. For each positive integer k, denote by F_k the multiplicative group of $GF(q^k)$. We take K to be a fixed copy of $F_{n!}$ and regard F_k as a subgroup of K for each k with $k \leq n$. For each positive integer k and h, let $\hat{F}_k$ and $\hat{U}_h$ denote the complex character group of F_k and U_h, respectively.

Suppose k and l are positive integers and k divides l. Then $F_k \leq F_l$ and we have a surjective homomorphism $N_{lk} : F_l \to F_k$ given by

$$N_{lk}(\rho) = \rho^{n_{lk}} \quad \text{for all } \rho \in F_l,$$

where $n_{lk} = |F_l|/|F_k| = (q^l - 1)/(q^k - 1)$. Defining $I_{kl} : \hat{F}_k \to \hat{F}_l$ by

$$I_{kl}(\psi)(\rho) = \psi(\rho^{n_{lk}}) \qquad (\psi \in \hat{F}_k, \rho \in F_l),$$

we can embed $\hat{F}_k$ in $\hat{F}_l$. In this way, we embed $\hat{F}_k$ in $\hat{K}$ for each integer k with $1 \leq k \leq n$. This embedding is well-defined (See Lemma 3.1 in [5]).

Lemma 2.1 (Lemma 3.2 in [5]). *For integers k, l with $k \mid l$, under the above identifications, the surjection: $\psi \mapsto \psi^{n_{lk}}$ from $\hat{F}_l$ to $\hat{F}_k$ is the same map as the restriction of characters.*

In the same way, $\hat{U}_h$ is embedded in $\hat{F}_k$ and in $\hat{K}$.

DEFINITION AND NOTATIONS.

(1) Let σ denote the *Frobenius map* $\rho \mapsto \rho^q$ on K, and $\hat{\sigma}$ the corresponding action on $\hat{K}$.

(2) An *irreducible polynomial* f over $GF(q)$ with the degree less than n will be identified with its set of roots in $GF(q^{n!})$, which forms a σ-orbit. If $f(x) \neq x$, then f is a σ-orbit in K. If ρ is an element of this orbit, we write $f = \langle \rho \rangle$.

(3) A *simplex* g over $GF(q)$ is a $\hat{\sigma}$-orbit in $\hat{K}$. If $\psi \in g$, we write $g = \langle \psi \rangle$.

We denote by $\mathcal{F}$ the set of irreducible polynomials regarded as σ-orbits in K, and by $\mathcal{G}$ the set of simplices over $GF(q)$. By the degree $deg(f)$ of an irreducible polynomial f, or $deg(g)$ of simplex g, we mean the cardinality of the orbit concerned.

If we fix an isomorphism between K and $\hat{K}$, then $\hat{F}_k$ and $\hat{U}_h$ correspond to F_k and U_h, respectively. Moreover, $\mathcal{F}$ and $\mathcal{G}$ correspond bijectively, and then a polynomial and the corresponding simplex have the same degree.

2.2. Partitions. Let $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_\delta^{l_\delta})$ be a partition of n. Here we put $a_1 > a_2 > \ldots > a_\delta > 0$ and $l_i \neq 0$ is the multiplicity of a_i as a part of μ. (Thus $n = l_1 a_1 + l_2 a_2 + \ldots + l_\delta a_\delta$.) For convenience sake, we also write $\mu = (j^{m_j})$, where $m_{a_i} = l_i$, and $m_j = 0$ if $j \neq a_i$ for any i.

We write $|\mu| = n$ to indicate that μ is a partition of n. Moreover $l(\mu) = \sum l_i$ is the length of μ, $\Lambda(\mu) = \gcd(l_1, l_2, \cdots, l_\delta)$, $A(\mu) = \gcd(a_1, a_2, \cdots, a_\delta)$, $\delta(\mu) = \delta$ is the number of distinct parts in μ, and $n'(\mu) = \sum \binom{a_i}{2} l_i$. The partition conjugate to μ is denoted by μ'. Let $\mathcal{P}$ be the set of partitions of all nonnegative integers n. Here we regard (0) as the only partition of 0.

2.3. Applications of the Clifford theory. Let G be a finite group and H be a normal subgroup of G. For $\zeta \in \mathrm{Irr}(H)$ we denote by $T_G(\zeta)$ the stabilizer of ζ in G and set

$$\mathrm{Irr}(G \mid \zeta) = \left\{ \chi \in \mathrm{Irr}(G) \;\middle|\; (\chi|_H, \zeta)_H = \frac{1}{|H|} \sum_{x \in H} \chi(x)\zeta(x^{-1}) \neq 0 \right\}.$$

For $\chi \in \mathrm{Irr}(G)$, let

$$\mathrm{Irr}(H|\chi) = \{\zeta \in \mathrm{Irr}(H) \mid (\chi|_H, \zeta)_H \neq 0\}.$$

Theorem 2.2 (Chapter 3, Theorem 3.8 in [7]). *Let $\zeta \in \mathrm{Irr}(H)$ and $T = T_G(\zeta)$. For $\chi \in \mathrm{Irr}(G \mid \zeta)$, we have*

$$\chi|_H = c\Big(\sum_{x \in T\backslash G} \zeta^x\Big),$$

where c is some positive integer.

Theorem 2.3 (Chapter 3, Theorem 5.12 in [7]). *Let $\zeta \in \mathrm{Irr}(H)$ and $T = T_G(\zeta)$. If ζ extends to an irreducible character η of T, then we have*

$$\mathrm{Irr}(T \mid \zeta) = \{\theta\eta \mid \theta \in \mathrm{Irr}(T/H)\} \quad \text{and}$$
$$\mathrm{Irr}(G \mid \zeta) = \{(\theta\eta)^G \mid \theta \in \mathrm{Irr}(T/H)\}.$$

Theorem 2.4. *With the above notation, each one of the following conditions implies that ζ is extendible to an irreducible character of T:*
(1) (Chapter 3, Theorem 5.11 in [7]) *T/H is cyclic.*
(2) *ζ is of degree 1, and $T = S \ltimes H$, where S is a certain group.*

Lemma 2.5. *If G/H is cyclic then the following hold.*
(1) *In the restriction of irreducible characters of G to H, the multiplicity of each irreducible constituent is* 1.
(2) *Two irreducible characters of G either have the same restrictions to H, or have restrictions without common irreducible constituents.*
(3) *For $\zeta \in \mathrm{Irr}(H)$, $\chi \in \mathrm{Irr}(G \mid \zeta)$, we have $|\mathrm{Irr}(H \mid \chi)| = |G|/|H||\mathrm{Irr}(G \mid \zeta)|$.*

Proof. Let $T = T_G(\zeta)$. Note that T/H is also cyclic.
(1) $\mathrm{Irr}(T/H)$ has only characters of degree 1. By Theorems 2.4(1) and 2.3, c in Theorem 2.2 is 1.
(2) It is clear from (1) and Theorem 2.2.
(3) By Theorem 2.3, we have $|\mathrm{Irr}(G \mid \zeta)| = |\mathrm{Irr}(T/H)| = |T/H|$, and by Theorem 2.2, we obtain $|\mathrm{Irr}(H \mid \chi)| = |G/T|$. Thus the equality holds. □

2.4. p-blocks. Let G be a finite group, $G_{p'}$ the set of elements of G whose orders are prime to p, and $Cl(G_{p'})$ the set of conjugate classes of G contained in $G_{p'}$. For $C \in Cl(G_{p'})$, let $\bar{C}$ be the sum of all elements of C in the group algebra of G over $\mathbb{C}$, and $d(C)$ the defect of C, i.e., $d(C) = \nu(|C_G(x)|)$ for $x \in C$. For a p-block B of G, let $d(B)$ be the defect of B. For $\chi \in \mathrm{Irr}(G)$ and $C \in Cl(G_{p'})$, we define $\omega_\chi(\bar{C})$ by

$$\omega_\chi(\bar{C}) = |C|\chi(x)/\chi(1)$$

with $x \in C$. Let $\mathfrak{p}$ be the valuation ideal of ν, i.e., $\mathfrak{p}$ is the set of elements in the field such that the values of ν on them are positive.

Theorem 2.6 (Chapter 3, Theorem 6.28 in [7]). *Assume that χ, $\chi' \in \mathrm{Irr}(G)$ belong to p-blocks of the same defect d. Then χ and χ' belong to the same p-block if and only if*

$$\omega_\chi(\bar{C}) \equiv \omega_{\chi'}(\bar{C}) \quad (mod \quad \mathfrak{p})$$

for any $C \in Cl(G_{p'})$ with $d(C) = d$.

Theorem 2.7 (Chapter 3, Theorem 6.29 in [7]). *Let B be a p-block of G and let $\chi \in \mathrm{Irr}(G)$ belong to B. Then the following three conditions are equivalent to each other.*

(1) $d(B) = 0$.
(2) $\nu(\chi(1)) = \nu(|G|)$.
(3) *The number of irreducible characters belonging to B is* 1.

Therefore the number of p-blocks of defect 0 is the number of characters satisfying the condition (2) above.

3. A parametrization of irreducible characters of $L_h(n, q)$

In this section, we treat $\mathrm{Irr}(L_h)$. We remark that the center $Z(L_h)$ of L_h is isomorphic to $U_{\gcd(q-1,nh)}$. In particular, $Z(GL) \simeq F_1$.

3.1. A parametrization of Irr(GL) and Irr(L_h). Green [3] showed in 1955 how an irreducible complex character of GL is given by a partition-valued function $\lambda : \mathcal{G} \to \mathcal{P}$ which satisfies

$$\sum_{g \in \mathcal{G}} |\lambda(g)| \deg(g) = n. \tag{3.1}$$

In this subsection, we identify the set of all such functions with $\mathrm{Irr}(GL)$. An account of how such a function determines an irreducible character may be found in Section 3 in [4] and Chapter IV in [6], too.

We explain properties of characters of GL which we need in this paper. We denote by $deg(\lambda)$ the degree of a character λ. Let $\lambda' : \mathcal{G} \to \mathcal{P}$ be the function such that $\lambda'(g)$ is the partition conjugate to $\lambda(g)$ for all $g \in \mathcal{G}$. We denote by $\hat{\xi}(\langle\psi\rangle)$ the product of all elements of $\langle\psi\rangle \in \mathcal{G}$, i.e. $\hat{\xi}(\langle\psi\rangle) = \psi^{n_{d1}}$ where $d = deg(\langle\psi\rangle)$. It is clear that $\hat{\xi}(\langle\psi\rangle) \in \hat{F}_1$.

Theorem 3.1. *Let $\lambda : \mathcal{G} \to \mathcal{P}$ be an irreducible character of GL.*
(1) (p.444 in [3], (6.7) in IV of [6]) $\nu(\deg(\lambda)) = e \sum_{g \in \mathcal{G}} \deg(g) n'(\lambda'(g))$.
(2) (Example 2 in IV of [6], Theorem 5.4 in [5]) *The restriction of λ to $Z(GL)$ is a multiple of $\prod_{g \in \mathcal{G}} \hat{\xi}(g)^{|\lambda(g)|}$.*

Let $\alpha \in \hat{F}_1$ and $\langle\psi\rangle \in \mathcal{G}$. We define the parallel translation $\tau_\alpha : \mathcal{G} \to \mathcal{G}$ as $\tau_\alpha\langle\psi\rangle = \langle\alpha\psi\rangle$. Moreover, we define an action of $\alpha \in \hat{F}_1$ on $\mathrm{Irr}(GL)$ as follows. For any irreducible character $\lambda : \mathcal{G} \to \mathcal{P}$ of GL, the character λ^α is defined by $\lambda^\alpha(\langle\psi\rangle) = \lambda(\tau_\alpha\langle\psi\rangle) = \lambda(\langle\alpha\psi\rangle)$ for any $\langle\psi\rangle \in \mathcal{G}$.

Theorem 3.2 (Proposition 5.2 in [5]). *Let λ, $\chi \in \mathrm{Irr}(GL)$. Then λ and χ have the same restrictions to L_h if and only if $\lambda^\alpha = \chi$ for some $\alpha \in \hat{U}_{(q-1)/h}$.*

For any irreducible character $\lambda : \mathcal{G} \to \mathcal{P}$ of GL, we denote by λ_∞ the $\hat{U}_{(q-1)/h}$-

orbit of $\mathrm{Irr}(GL)$ containing λ. Let $\mathrm{Irr}(L_h \mid \lambda_\infty)$ denote $\mathrm{Irr}(L_h \mid \lambda)$. (This notation is well defined by virtue of Theorem 3.2.)

Theorem 3.3.

(1) $\mathrm{Irr}(L_h) = \bigcup_{\lambda_\infty} \mathrm{Irr}(L_h \mid \lambda_\infty)$ (disjoint), *where this union is over all $\hat{U}_{(q-1)/h}$-orbits in* $\mathrm{Irr}(GL)$. *Moreover, for each $\lambda \in \mathrm{Irr}(GL)$, we have*

$$|\mathrm{Irr}(L_h \mid \lambda_\infty)| = \frac{q-1}{h|\lambda_\infty|}.$$

(2) *For $\lambda \in \mathrm{Irr}(GL)$, the followings hold.*

(i) *For any $\varphi \in \mathrm{Irr}(L_h \mid \lambda_\infty)$,*

$$\nu(\deg(\varphi)) = e \sum\nolimits_{g \in \mathcal{G}} \deg(g) n'(\lambda'(g)).$$

(ii) *The restriction of each character in $\mathrm{Irr}(L_h \mid \lambda_\infty)$ to $Z(L_h)$ is a multiple of*

$$\prod\nolimits_{g \in \mathcal{G}} \hat{\xi}(g)^{|\lambda(g)|(q-1)/\gcd(q-1,hn)}.$$

Proof. (1) Since GL/L_h is a cyclic group, the first half is clear from Lemma 2.5(2).

Therefore, for $\lambda \in \mathrm{Irr}(GL)$ and $\zeta \in \mathrm{Irr}(L_h \mid \lambda)$, we have $\mathrm{Irr}(GL \mid \zeta) = \lambda_\infty$. So we have the latter half by Lemma 2.5(3).

(2)(i) By Theorem 2.2 and Lemma 2.5(1), for $\zeta \in \mathrm{Irr}(L_h \mid \lambda_\infty)$,

$$\deg(\lambda) = |T_{GL}(\zeta)\backslash GL| \deg(\zeta).$$

Here, $|T_{GL}(\zeta)\backslash GL|$ divides $|L_h\backslash GL| = (q-1)/h$ which is prime to p. So the p-part of $\deg(\zeta)$ equals that of $\deg(\lambda)$. From Theorem 3.1(1), we have (i).

(ii) The irreducible constituent of a restriction of each character in $\mathrm{Irr}(L_h \mid \lambda_\infty)$ to $Z(L_h)$ equals the irreducible constituent of $\lambda|_{Z(L_h)}$. By $Z(L_h) \simeq U_{\gcd(q-1,nh)}$, Lemma 2.1 and Theorem 3.1(2), (ii) holds. $\square$

By using the above, we can count the number of characters of L_h.

3.2. A parametrization of $\mathrm{Irr}(L_h)$ by polynomials. In order to count irreducible characters effectively, we parametrize $\mathrm{Irr}(GL)$ and $\mathrm{Irr}(L_h)$ by polynomials over $GF(q)$.

Fix an isomorphism from K to $\hat{K}$. Then, as is seen in 2.1, we have the bijection from $\mathcal{F}$ to $\mathcal{G}$. Therefore elements in $\mathrm{Irr}(GL)$ are parametrized by partition-valued functions $\lambda : \mathcal{F} \to \mathcal{P}$ which satisfy

$$\sum_{f\in\mathcal{F}} |\lambda(f)| \deg(f) = n. \tag{3.2}$$

We denote polynomials in $\mathcal{F}$ by $f_1, f_2, \cdots$. Let $\lambda'(f_i) = (j^{m(i,j)})$ where $m(i,j)$ is a non negative integer. For $\lambda : \mathcal{F} \to \mathcal{P}$ with (3.2), we define the sequence of polynomials

$$\left(\prod_i f_i^{m(i,1)}, \prod_i f_i^{m(i,2)}, \cdots\right). \tag{3.3}$$

Then it is easy to see that this gives a bijection from the set of partition-valued functions with (3.2) to the set of sequences $(h_1, h_2, \cdots)$ of monic polynomials over $GF(q)$ which satisfy the following.

(1) The constant term of each h_i does not equal to 0, and

(2) $\sum_j j \deg(h_j) = n$, i.e., $\mu = (j^{\deg(h_j)})$ is a partition of n.

From now on, we identify the set of such sequences with $\mathrm{Irr}(GL)$.

Let $g \in \mathcal{G}$ correspond to an irreducible monic polynomial $f(x) = x^d + b_1x^{d-1} + \cdots + b_d$ over $GF(q)$ such that $b_d \neq 0$, and let $\alpha \in \hat{F}_1$ correspond to $\rho \in F_1$. As in 2.1, we regard an irreducible polynomial as the σ-orbit consisting of its roots in K. Because $\tau_\alpha(g)$ is the σ-orbit obtained by multiplying all elements of the σ-orbit g by α, it corresponds to the σ-orbit obtained by multiplying all roots of f by ρ, i.e., we have

$$\tau_\rho(f(x)) = x^d + \rho b_1 x^{d-1} + \rho^2 b_2 x^{d-2} + \cdots + \rho^d b_d. \tag{3.4}$$

We apply this notation when $f(x)$ is reducible, too. If ρ is a primitive m-th root of unity, then

$$f(x) = \tau_\rho(f(x)) \Leftrightarrow b_k = 0 \quad \text{if } m \nmid k. \tag{3.5}$$

In particular, if this condition holds, then we have $m \mid d$ because $b_d \neq 0$.

Let $\lambda = (h_1, h_2, \cdots) \in \mathrm{Irr}(GL)$ and put $h_i(x) = x^{d_i} + b_{i,1}x^{d_i-1} + \cdots + b_{i,d_i}$. Note that $b_{i,d_i} \neq 0$ for any i. The action of α on λ corresponds to the action of ρ in such a way that

$$\lambda^\rho = (\tau_{\rho^{-1}}(h_1), \tau_{\rho^{-1}}(h_2), \cdots). \tag{3.6}$$

If ρ is a primitive m-th root of 1, then

$$\lambda = \lambda^\rho \Leftrightarrow b_{i,j} = 0 \quad \text{if } m \nmid j. \tag{3.7}$$

In particular, if ρ stabilizes λ, then $m \mid \gcd(\deg(h_1), \deg(h_2), \cdots)$.

By our identification, λ_∞ equals the $U_{(q-1)/h}$-orbit of $\mathrm{Irr}(GL)$ containing λ.
We restate Theorems 3.2 and 3.3 by using the above notation.

Corollary 3.4.

(1) λ, $\chi \in \mathrm{Irr}(GL)$ *have the same restrictions to* L_h *if and only if* $\lambda^\rho = \chi$ *for some* $\rho \in U_{(q-1)/h}$.

(2) $\mathrm{Irr}(L_h) = \bigcup_{\lambda_\infty} \mathrm{Irr}(L_h \mid \lambda_\infty)$ (*disjoint*), *where this union is over all* $U_{(q-1)/h}$-*orbits in* $\mathrm{Irr}(GL)$. *Moreover, we have*

$$|\mathrm{Irr}(L_h \mid \lambda_\infty)| = \frac{q-1}{h|\lambda_\infty|}$$

(3) *For* $\lambda = (h_1, h_2, \cdots) \in \mathrm{Irr}(GL)$ *with* $h_i(x) = x^{d_i} + b_{i,1}x^{d_i-1} + \cdots + b_{i,d_i}$ *and* $b_{i,d_i} \neq 0$, *the following hold.*

(i) *For any* $\varphi \in \mathrm{Irr}(L_h \mid \lambda_\infty)$,

$$\nu(\deg(\varphi)) = e\sum_j \binom{j}{2} \deg(h_j) = en'(\mu),$$

where μ *is the partition* $(j^{\deg(h_j)})$ *of* n.

(ii) *The restriction of each character in* $\mathrm{Irr}(L_h \mid \lambda_\infty)$ *to* $Z(L_h)$ *is a multiple of the irreducible character of* $U_{\gcd(q-1,nh)}$ *corresponding to*

$$\left\{(-1)^n \prod_j (b_{j,d_j})^j\right\}^{(q-1)/\gcd(q-1,nh)}.$$

Proof. (1) and (2) are clear from Theorems 3.2 and 3.3.

(3) Let us regard λ as a function from $\mathcal{F}$ to $\mathcal{P}$. We denote polynomials in $\mathcal{F}$ by $f_1, f_2, \cdots$. We write $\lambda'(f_i) = (j^{m(i,j)})$ where $m(i,j)$ is a non negative integer. Then $h_j = \prod_i f_i^{m(i,j)}$ and $\deg(h_j) = \sum_i \deg(f_i)m(i,j)$.

(i) By Theorem 3.3(2)(i), the p-part of the degree of each character in $\mathrm{Irr}(L_h|\lambda_\infty)$ equals

$$\begin{aligned} e\sum_{f\in\mathcal{F}} \deg(f)n'(\lambda'(f)) &= e\sum_i \deg(f_i)\sum_j \binom{j}{2} m(i,j) \\ &= e\sum_j \binom{j}{2}\sum_i \deg(f_i)m(i,j) = e\sum_j \binom{j}{2} \deg(h_j). \end{aligned}$$

(ii) If $g \in \mathcal{G}$ corresponds to $f \in \mathcal{F}$, then $\hat{\xi}(g)$ corresponds to the product of all roots of f, i.e., $(-1)^{\deg(f)}f(0)$. We remark that $(j^{\deg(h_j)})$ is a partition of n. Then the irreducible constituent of the restriction of λ to $Z(L_h)$ corresponds to

$$\prod_{f\in\mathcal{F}} \{(-1)^{\deg(f)}f(0)\}^{|\lambda(f)|(q-1)/\gcd(q-1,nh)}$$

$$= \prod_i \{(-1)^{\deg(f_i)} f_i(0)\}^{\left(\sum_j jm(i,j)\right)(q-1)/\gcd(q-1,nh)}$$
$$= \prod_j \prod_i \{(-1)^{\deg(f_i)m(i,j)j} f_i(0)^{m(i,j)j}\}^{(q-1)/\gcd(q-1,nh)}$$
$$= \left\{\prod_j (-1)^{\deg(h_j)j} h_j(0)^j\right\}^{(q-1)/\gcd(q-1,nh)}$$
$$= \left\{(-1)^n \prod_j (b_{j,d_j})^j\right\}^{(q-1)/\gcd(q-1,nh)}. \qquad \square$$

3.3. p-blocks of $L_h(n,q)$. By Theorem 4 of [2], a defect group of any p-block of $L_1 = SL(n,q)$ is a Sylow p-subgroup or trivial subgroup. The same argument as for GL in the last paragraph of Section 4 in [2] yields that the same is true for L_h. Therefore the defect $d(B)$ of the block B of L_h equals to 0 or $\nu(|L_h|) = en(n-1)/2$. In the later case, we say that B is of maximal defect. By Theorem 2.7, any p-block of defect 0 has a character the p-part of whose degree equals that of $|L_h|$, i.e. $p^{en(n-1)/2}$. On the other hand, characters in any p-block of the maximal defect have p-parts of degree less than $p^{en(n-1)/2}$.

Lemma 3.5. *The number of blocks of L_h of defect 0 is h. Moreover, for a non-negative integer k and any block B of L_h of defect 0, we have*

$$m_p(k, L_h, B) = \begin{cases} 1, & \textit{if} \quad k = en(n-1)/2; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. The latter half is clear by Theorem 2.7.

Let $\lambda_a = (\underbrace{1, 1, \cdots, 1}_{(n-1)\text{ times}}, x-a) \in \mathrm{Irr}(GL)$ for $a \in F_1$. By Theorem 2.7 and Corollary 3.4(3)(i), the set of p-blocks of defect 0 corresponds bijectively to

$$\begin{aligned} &\{\zeta \in \mathrm{Irr}(L_h) \mid \nu(\zeta(1)) = \nu(|L_h|) = en(n-1)/2\} \\ &= \bigcup_{a\in F_1} \mathrm{Irr}(L_h \mid \lambda_a) \\ &= \bigcup_{(\lambda_a)_\infty} \mathrm{Irr}(L_h \mid (\lambda_a)_\infty) \quad \text{(disjoint)} \end{aligned}$$

where the last union is over all $U_{(q-1)/h}$-orbit consisting of characters λ_a. Because λ_a is stabilized only by 1, $|(\lambda_a)_\infty| = (q-1)/h$. Therefore, the number of $U_{(q-1)/h}$-orbit consisting of characters λ_a is h, and $|\mathrm{Irr}(L_h \mid (\lambda_a)_\infty)| = 1$ by Corollary 3.4(2). Therefore the number of blocks of defect 0 is h. $\square$

For characters belonging to p-blocks of maximal defect, we can determine their distribution to p-blocks by looking at the values at $\bar{C}$'s for all $C \in Cl((L_h)_{p'})$

with $C_{L_h}(x)(x \in C)$ containing a Sylow p-subgroup of L_h. This is possible because of Theorem 2.6. Since an element of L_h satisfying this condition is in the center $Z(L_h)$ of L_h, it is enough to see the character values on $Z(L_h)$. Moreover $Z(L_h) \simeq U_{\gcd(q-1,nh)}$ is a cyclic group whose order is prime to p. So, it is enough to look at their actual values, not those modulo $\mathfrak{p}$. Therefore we have the following.

Lemma 3.6. *Let ζ, $\zeta' \in \mathrm{Irr}(L_h)$ belong to p-blocks of non-zero defect. Then ζ and ζ' belong to the same block if and only if $\omega_\zeta(x) = \omega_{\zeta'}(x)$ for all $x \in Z(L_h)$.*

By this lemma, we can determine distribution of characters to p-blocks of L_h of maximal defect by looking at the irreducible constituent of their restriction to $Z(L_h)$. Therefore p-blocks of maximal defect are parametrized by the element of $\widehat{Z(L_h)}$. Because $\widehat{Z(L_h)} \simeq \hat{U}_{\gcd(q-1,nh)} \simeq U_{\gcd(q-1,nh)}$, p-blocks of L_h of maximal defect are parametrized by the element of $U_{\gcd(q-1,nh)}$. The number of blocks of L_h of maximal defect is $\gcd((q-1), nh)$.

We fix an isomorphism $\widehat{Z(L_h)} \simeq U_{\gcd(q-1,nh)}$, and identify them via the isomorphism. We denote by B_a the p-block of L_h of maximal defect corresponding to $a \in U_{\gcd(q-1,nh)}$.

In particular, the principal block is B_1. Moreover, B_1 is the set of characters in blocks of non-zero defect of L_h such that restrictions of those to $Z(L_h)$ equal to multiples of the trivial character. So these characters are regarded as characters of $L_h/Z(L_h) = PL_h$. Therefore, we can identify B_1 with the only p-block $\tilde{B}_0$ of maximal defect of PL_h. On the other hand, by Corollary 3.4 and the proof of Lemma 3.5, the number of p-blocks of defect zero of PL_h is $\gcd(q-1,n)(q-1)/\gcd(q-1,nh)$.

Let $\lambda = (h_1, h_2, \cdots) \in \mathrm{Irr}(GL)$ and let a_i be the constant term of h_i. All characters in $\mathrm{Irr}(L_h|\lambda_\infty)$ have the same restrictions to $Z(L_h)$. So, all constituents belong to the same p-block. By Corollary 3.4(3), characters in $\mathrm{Irr}(L_h|\lambda_\infty)$ belong to B_a if and only if

$$a = \left((-1)^n \prod\nolimits_j (a_j)^j\right)^{(q-1)/\gcd(q-1,nh)}. \tag{3.8}$$

Lemma 3.7 (Lemma 2.5 in [8]). *Let a_i $(1 \le i \le \delta)$ be positive integers, $A = \gcd(a_1, a_2, \cdots, a_\delta)$, and $a \in F_1$. Then*

$$|\{(x_1, x_2, \cdots, x_\delta) \in F_1^\delta \mid x_1^{a_1} x_2^{a_2} \cdots x_\delta^{a_\delta} = a\}| = (q-1)^{\delta-1}\beta(A, a)$$

where $\beta(A,a)$ is the number of solutions in F_1 to the equation $x^A = a$, i.e.,

$$\beta(A,a) = \begin{cases} \gcd(q-1, A), & \text{if } a \in U_{(q-1)/\gcd(q-1,A)}; \\ 0, & \text{otherwise.} \end{cases}$$

Let a be in $U_{\gcd(q-1,nh)}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \cdots, a_\delta^{l_\delta})$ be a partition. By the above lemma, we have

$$\begin{aligned}
(3.9)\, &|\{(x_1, x_2, \cdots, x_\delta) \in F_1^\delta \mid ((-1)^n x_1^{a_1} x_2^{a_2} \cdots x_\delta^{a_\delta})^{(q-1)/\gcd(q-1,nh)} = a\}| \\
&= |\{(x_1, x_2, \cdots, x_\delta) \in F_1^\delta \mid \{((-1)^{l_1} x_1)^{a_1} \cdots ((-1)^{l_\delta} x_\delta)^{a_\delta}\}^{(q-1)/\gcd(q-1,nh)} = a\}| \\
&= (q-1)^{\delta-1} \beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)}, a\right)
\end{aligned}$$

4. The McKay numbers of L_h

For a partition $\mu = (a_1^{l_1}, a_2^{l_2}, \cdots, a_\delta^{l_\delta})$ of n, a be in $U_{\gcd(q-1,nh)}$, and a positive integer s, we denote by $\mathrm{Irr}(GL, \mu, a, s)$ the set of irreducible characters $\lambda = (h_1, h_2, \cdots)$ of GL satisfying the following.

(1) The partition $(j^{\deg(h_j)})$ equals μ,

(2) $\mathrm{Irr}(L_h \mid \lambda) \subseteq B_a$, and

(3) λ is stabilized by s-th roots of 1 in $U_{(q-1)/h}$, but is not stabilized by s'-th roots of 1 for any $s' > s$ with $s \mid s'$, i.e., the restriction of λ to L_h has s irreducible constituents.

Note that by (3.6) and (3.7) $\mathrm{Irr}(GL, \mu, a, s)$ is closed under the action of $U_{(q-1)/h}$.

We denote by $\mathrm{Irr}(GL, \mu, a)$ the set of irreducible characters λ of GL satisfying (1) and (2) of the above, i.e.,

$$\mathrm{Irr}(GL, \mu, a) = \bigcup_{s \mid (q-1)/h} \mathrm{Irr}(GL, \mu, a, s) \quad \text{(disjoint)}.$$

And we denote by $\widetilde{\mathrm{Irr}}(GL, \mu, a, s)$ the set of irreducible characters λ of GL satisfying (1),(2) above and the following.

(4) λ is stabilized by s-th roots of 1 in $U_{(q-1)/h}$. (Thus λ is stabilized by s'-th roots of 1 for any $s' > s$ with $s \mid s'$.)

This means that

$$\widetilde{\mathrm{Irr}}(GL, \mu, a, s) = \bigcup_{s \mid s'} \mathrm{Irr}(L_h, \mu, a, s').$$

Moreover, we put

$$\begin{aligned}
\mathrm{Irr}(L_h, \mu, a, s) &= \{\zeta \in \mathrm{Irr}(L_h) \mid \zeta \in \mathrm{Irr}(L_h|\chi),\ \chi \in \mathrm{Irr}(GL, \mu, a, s)\}, \\
\mathrm{Irr}(L_h, \mu, a) &= \{\zeta \in \mathrm{Irr}(L_h) \mid \zeta \in \mathrm{Irr}(L_h|\chi),\ \chi \in \mathrm{Irr}(GL, \mu, a)\}, \\
m(\mu, a, s) &= |\mathrm{Irr}(L_h, \mu, a, s)|, \quad \text{and} \\
m(\mu, a) &= |\mathrm{Irr}(L_h, \mu, a)|.
\end{aligned}$$

For an integer $t > 1$, we define $\Pi(t)$ by

$$\Pi(t) = \prod \left(1 - \frac{1}{r^2}\right),$$

where r runs over all prime numbers that divide t. For example, for any positive integers i, j, $\Pi(2^i) = 3/4$, $\Pi(3^i) = 8/9$, $\Pi(2^i 3^j) = 24/36$, etc. For convenience, we put $\Pi(1) = 1$.

At first, we show the following lemma. For a divisor s of $\Lambda(\mu)$, we put

$$\gamma(\mu, s) = q^{l(\mu)/s - \delta(\mu)}.$$

Note that if $\lambda = (h_1, h_2, \cdots) \in \mathrm{Irr}(GL)$ is stabilized by s-th roots of 1 in $U_{(q-1)/h}$, then s divides $\gcd((q-1)/h, \deg(h_1), \deg(h_2), \cdots)$.

Lemma 4.1.
(1) $|\widetilde{\mathrm{Irr}}(GL, \mu, a, s)| = \gamma(\mu, s)(q-1)^{\delta(\mu)-1}\beta((q-1)A(\mu)/\gcd(q-1, nh), a)$.
(2) *Let* $\gcd(\Lambda(\mu), (q-1)/h) = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ *be the prime decomposition of* $\gcd(\Lambda(\mu), (q-1)/h)$, s *be a divisor of* $\gcd(\Lambda(\mu), (q-1)/h)$ *with the prime decomposition* $s = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, *and set* $c_i (1 \leq i \leq k)$ *as follow. We put* $c_i = 0$ *if* $s_i = r_i$ *and* $c_i = 1$ *if* $s_i < r_i$. *Then*

$$m(\mu, a, s) = hs^2 \sum_{\substack{0 \leq d_i \leq c_i \\ 1 \leq i \leq k}} (-1)^{d_1 + \cdots + d_k} \gamma(\mu, p_1^{s_1 + d_1} \cdots p_k^{s_k + d_k})$$
$$\times (q-1)^{\delta(\mu)-2} \beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1, nh)}, a\right)$$

Proof. (1) If $(h_1, h_2, \cdots) \in \mathrm{Irr}(GL, \mu, a)$ is stabilized by s-th roots of 1 in $U_{(q-1)/h}$, then we may write

$$h_{a_j}(x) = x^{l_j} + \sum_{i=0}^{l_j/s-1} b_{j,i} x^{is} \tag{4.1}$$

for all j by (3.7). Moreover, because this character belongs to B_a, by Corollary 3.4(3)(ii) we have

$$((-1)^n b_{1,0}^{a_1} b_{2,0}^{a_2} \cdots b_{\delta,0}^{a_\delta})^{(q-1)/\gcd(q-1,nh)} = a. \tag{4.2}$$

If $i \neq 0$, then the possible of $b_{j,i}$ is any element in $GF(q)$, and the number of all possible of the set of $b_{j,0}$ is determined by (3.9). Thus

$$|\widetilde{\mathrm{Irr}}(GL,\mu,a,s)| = \left(\prod_{j=1}^{\delta} q^{l_j/s-1}\right)(q-1)^{\delta-1}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right)$$
$$= \gamma(\mu,s)(q-1)^{\delta(\mu)-1}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right).$$

(2) The above number includes characters stabilized by s'-th roots of 1 for some $s < s'$ with $s \mid s'$. Thus

$$|\mathrm{Irr}(GL,\mu,a,s)|$$
$$= \sum_{\substack{0\le d_i\le c_i\\ 1\le i\le k}} (-1)^{d_1+\cdots+d_k}\gamma(\mu,p_1^{s_1+d_1}\cdots p_k^{s_k+d_k})(q-1)^{\delta(\mu)-1}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right).$$

Each $U_{(q-1)/h}$-orbit in $\mathrm{Irr}(GL,\mu,a,s)$ has $(q-1)/hs$ elements. So each orbit gives s characters of L_h by Corollary 3.4(2). Consequently, all characters in $\mathrm{Irr}(GL,\mu,a,s)$ give

$$hs^2 \sum_{\substack{0\le d_i\le c_i\\ 1\le i\le k}} (-1)^{d_1+\cdots+d_k}\gamma(\mu,p_1^{s_1+d_1}\cdots p_k^{s_k+d_k})(q-1)^{\delta(\mu)-2}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right)$$

irreducible characters of L_h. □

Theorem 4.2. *For a partition $\mu = (a_1^{l_1}, a_2^{l_2}, \cdots, a_\delta^{l_\delta})$ of n and $a \in U_{\gcd(q-1,nh)}$,*

$$m(\mu,a) = h\left\{\sum_{t\mid\gcd(\Lambda(\mu),(q-1)/h)} t^2\Pi(t)q^{l(\mu)/t-\delta(\mu)}\right\}$$
$$\times (q-1)^{\delta(\mu)-2}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right)$$

Proof. We obtain $m(\mu,a)$ by summing $m(\mu,a,s)$ for all s dividing $\gcd(\Lambda(\mu), (q-1)/h)$, i.e., for all $(s_1,\cdots,s_k)$ $(0\le s_i\le r_i)$. Hence we may write by the previous lemma,

$$m(\mu,a) = h\left\{\sum_{t\mid\gcd(\Lambda(\mu),(q-1)/h)} e_t\gamma(\mu,t)\right\}(q-1)^{\delta(\mu)-2}\beta\left(\frac{(q-1)A(\mu)}{\gcd(q-1,nh)},a\right),$$

for some e_t. If $t = p_1^{t_1}\cdots p_k^{t_k}$, then e_t is in fact, obtained as follows.

$$e_t = \sum_{\substack{0\le d_i\le c_i'\\ 1\le i\le k}} (-1)^{d_1+\cdots+d_k}p_1^{2(t_1-d_1)}\cdots p_k^{2(t_k-d_k)},$$

where $c_i' = 0$ if $t_i = 0$, and $c_i' = 1$ if $t_i > 0$. Therefore,

$$e_t = \prod_{t_i \neq 0} (p_i^{2t_i} - p_i^{2t_i - 2}) = t^2 \Pi(t)$$

Consequently, we have the statement of the theorem. $\square$

Note that each character ζ in $\mathrm{Irr}(L_h, \mu, a)$ satisfies $\nu(\zeta(1)) = en'(\mu)$. Therefore, we have the following theorem.

Theorem 4.3. *For* $0 \leq k < n(n-1)/2$,

$$m_p(ek, L_h(n,q), B_a) = \sum{}' m(\mu, a),$$

where the sum is taken over all partitions μ *of* n *such that* $n'(\mu) = k$. *And if* $i \neq ek$ *for any* k *with* $0 \leq k < n(n-1)/2$, *then* $m_p(ek, L_h(n,q), B_a) = 0$.

Recall that $\tilde{B}_0$ is the unique p-block of maximal defect of PL_h. Because we can identify B_1 with $\tilde{B}_0$, by Lemmas 3.7, 4.1, and Theorem 4.2, we have the following.

Corollary 4.4. *For* $0 \leq k < n(n-1)/2$,

$$\begin{aligned} m_p(ek, PL_h(n,q), \tilde{B}_0) &= \sum{}' m(\mu, 1) \\ &= \sum{}' h \Big\{ \sum_{t \mid \gcd(\Lambda(\mu), (q-1)/h)} t^2 \Pi(t) q^{l(\mu)/t - \delta(\mu)} \Big\} (q-1)^{\delta(\mu)-1} \frac{\gcd(q-1, A(\mu))}{\gcd(q-1, nh)}, \end{aligned}$$

where the first sum is the same as in Theorem 4.3. *And if* $i \neq ek$ *for any* k *with* $0 \leq k < n(n-1)/2$, *then* $m_p(ek, PL_h(n,q), \tilde{B}_0) = 0$.

5. The Alperin-McKay conjecture for L_h

In this section, we show the following theorem, i.e., we prove the Alperin-McKay conjecture for L_h. The notations are the same as in the previous sections.

Theorem 5.1. *For a Sylow* p*-subgroup* P *of* L_h, *let* b_a *be the* p*-block of* $N = N_{L_h}(P)$ *corresponding to the* p*-block* B_a *of maximal defect of* L_h. *Then we have*

$$m_p(0, L_h, B_a) = m_p(0, N, b_a).$$

Proof. We classify irreducible characters of L_h and N respectively by sequences $\iota = (s_0, s_1, s_2, \cdots, s_k)$ of integers s_i such that $0 = s_0 < s_1 < s_2 < \cdots < s_k = n$ for some $k \leq n$.

By Corollary 3.4, the degree of $\zeta \in \mathrm{Irr}(L_h)$ is not divisible by p if and only if ζ is in $\mathrm{Irr}(L_h|\lambda_\infty)$ for some $\lambda = (h(x), 1, 1, \cdots) \in \mathrm{Irr}(GL)$ where $h(x)$ is a polynomial of degree n. For given $\iota = (s_0, s_1, \cdots, s_k)$, we consider characters $(h(x), 1, 1, \cdots) \in \mathrm{Irr}(GL)$ with

$$h(x) = x^n + \sum_{i=0}^{n-1} a_i x^i, \quad \text{where} \begin{cases} a_i \neq 0, & \text{if } i = s_j \text{ for some } 0 \leq j \leq k-1; \\ a_i = 0, & \text{otherwise.} \end{cases}$$

Thus the number of characters of this type is $(q-1)^k$. By (3.7), an element of $U_{(q-1)/h}$ stabilizes characters of this type if and only if it is a $\gcd(s_1 - s_0, \cdots, s_k - s_{k-1}, (q-1)/h)$-th roots of 1. By Corollary 3.4(2) the number of characters in $\mathrm{Irr}(L_h)$ given by ι is

$$\gcd\left(s_1 - s_0, \cdots, s_k - s_{k-1}, \frac{q-1}{h}\right)^2 h(q-1)^{k-1}.$$

By (3.8) the above characters belong to a p-block $B_a(a \in U_{\gcd(q-1,nh)})$ if and only if $a_0^{(q-1)/\gcd(q-1,nh)} = a$. Note that for any $a \in U_{\gcd(q-1,nh)}$ the number of solutions a_0 in U_{q-1} to this equation is $(q-1)/\gcd(q-1,nh)$. Since this number does not depend on a, all B_a's have the same number of characters of this type given by ι.

On the other hand, a Sylow p-subgroup P of L_h is conjugate to the subgroup of upper triangle matrices all of whose diagonal entries are 1. Thus we may assume that N is the subgroup of upper triangle matrices in L_h.

But the degree of a character χ of N is not divisible by p if and only if the kernel of χ contains the commutator subgroup P' of P. Therefore we may consider such characters as those of $M = N/P'$.

Let $Q = P/P'$ and let D be the set of elements in N/P' corresponding to diagonal matrices in N. Then we have $M = D \ltimes Q$. We denote an element a in D by $(a_1, a_2, \cdots, a_n)$ where $a_i \in F_1$ and $a_1 a_2 \cdots a_n \in U_h$, in such a way that the product of elements in D is the component-wise product. We denote an element b in Q by $(b_1, b_2, \cdots, b_{n-1})$ where $b_i \in GF(q)$, and the product of elements in Q is the component-wise sum. Thus the action a on b is given by

$$b^a = a^{-1}ba = (a_1^{-1}b_1a_2, a_2^{-1}b_2a_3, \cdots, a_{n-1}^{-1}b_{n-1}a_n).$$

Since D and Q are Abelian groups, every irreducible character of these groups is of degree 1, and we fix an isomorphism from D (resp. Q) to the group of characters of D (resp. Q).

We construct characters of M by using Theorems 2.3 and 2.4.

For the above sequence $\iota = (s_0, \cdots, s_k)$, we consider $b = (b_1, b_2, \cdots, b_{n-1}) \in$

$\mathrm{Irr}(Q)$ such that

$$\begin{cases} b_i = 0, & \text{if } i = s_j \text{ for some } 1 \leq j \leq k-1; \\ b_i \neq 0, & \text{otherwise.} \end{cases}$$

The number of such characters is $(q-1)^{n-k}$. Then, for $a = (a_1, a_2, \cdots, a_n) \in \mathrm{Irr}(D)$, $b^a = b$ if and only if $a_{s_j+1} = a_{s_j+2} = \cdots = a_{s_{j+1}}$ $(0 \leq j \leq k-1)$. And since $a \in \mathrm{Irr}(D)$, it is necessary that $a_{s_1}^{s_1 - s_0} a_{s_2}^{s_2 - s_1} \cdots a_{s_k}^{s_k - s_{k-1}} \in U_h$. By Lemma 3.7, the order of the stabilizer of b in D is

$$\begin{aligned} &(q-1)^{k-1} \sum_{c \in U_h} \beta(m, c) \\ &= (q-1)^{k-1} \sum_{c \in U_{\gcd(mh, q-1)/\gcd(m, q-1)}} \gcd(m, q-1) \\ &= (q-1)^{k-1} \gcd(mh, q-1) \end{aligned}$$

where $m = \gcd(s_1 - s_0, s_2 - s_1, \cdots, s_k - s_{k-1})$. Since the order of D is $h(q-1)^{n-1}$, the number of elements contained in each orbit is $(q-1)^{n-k}/\gcd(m, (q-1)/h)$. Hence the number of orbits in the set of irreducible characters given by ι is $\gcd(m, (q-1)/h)$. From Theorems 2.3 and 2.4, the number of characters χ of M such that the restriction of χ to Q is a sum of certain irreducible characters all of which have the type given by ι is

$$\begin{aligned} &\gcd\left(m, \frac{q-1}{h}\right)^2 h(q-1)^{k-1} \\ &= \gcd\left(s_1 - s_0, s_2 - s_1, \cdots, s_k - s_{k-1}, \frac{q-1}{h}\right)^2 h(q-1)^{k-1}. \end{aligned}$$

The distribution of irreducible characters of N to p-blocks of maximal defect can be seen by comparing the irreducible constituent of the restriction to the center $Z(M)$ of M. Note that $Z(N) = Z(M)$. Recall that the same is true for L_h. See Lemma 3.6. We fix an irreducible character b of Q given by ι, and consider the distribution of the characters in $\mathrm{Irr}(M \mid b)$ to p-blocks. The center $Z(M)$ of M is contained in the stabilizer T of b in D and on the other hand we have $Q \cap Z(M) = \{1\}$. Thus, from Theorem 2.3, for an irreducible character χ in $\mathrm{Irr}(M \mid b)$, there exists an extension $\tilde{b}$ of b to T and an irreducible character η of T such that $\chi = (\tilde{b}\eta)^M$. So, in order to look at the restriction of χ to $Z(M)$, we may consider that of η to $Z(M)$. Since T is Abelian, by Theorems 2.3, 2.4, the characters in $\mathrm{Irr}(M \mid b)$ are distributed into p-blocks in such a way that all blocks of M of maximal defect have the same numbers of characters in $\mathrm{Irr}(M \mid b)$. Since the above argument can be applied for any character b of Q given by ι, all p-blocks of M of maximal defect have the same numbers of characters given by ι.

Let B_a and b_a be the same as in the statement of Theorem 5.1. For a fixed ι, the above argument shows that the numbers of characters of L_h given by ι belonging to B_a and that of N given by ι belonging to b_a are equal. Since $\iota = (s_0, s_1, \cdots, s_k)$ is arbitrary, we have

$$m_p(0, L_h, B_a) = m_p(0, N, b_a). \qquad \square$$

We identify B_1 with $\tilde{B}_0$, and in the same way as we identify b_1 with the p-block $\tilde{b}_0$ of $N_{PL_h}(P)$. Therefore, we have the following.

Corollary 5.2. *For a Sylow p-subgroup P of $L_h/Z(L_h)$, let $\tilde{b}_0$ be the p-block of $\tilde{N} = N_{L_h/Z(L_h)}(P)$ corresponding to the p-block $\tilde{B}_0$ of maximal defect of $L_h/Z(L_h)$. Then we have*

$$m_p(0, L_h/Z(L_h), \tilde{B}_0) = m_p(0, \tilde{N}, \tilde{b}_0).$$

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