Takegoshi, K.
Osaka J. Math.
36 (1999), 17–26

TORSION FREENESS THEOREMS FOR HIGHER DIRECT IMAGES OF CANONICAL SHEAVES BY A CERTAIN CONVEX KÄHLER MORPHISM

KENSHO TAKEGOSHI

(Received June 12, 1997)

Introduction

Let $f : X \to Y$ be a morphism of analytic spaces. In this paper any analytic space is always assumed to be reduced unless otherwise stated. In [20] we discussed the torsion freeness of higher direct images of canonical sheaves tensorized with Nakano semi-positive vector bundle under the situation that X is non-singular and f is a proper surjective Kähler morphism. In this case the coherency of the higher direct image sheaves is guaranteed by Grauert's direct image theorem (cf. [6]). However not much is known about not only coherency but also torsion freeness of higher direct image sheaves by non-proper morphisms except a few special cases (cf. [3], [5], [13], [15], [16], [17]). In this article we study torsion freeness and vanishing theorems of higher direct image sheaves by a certain non-proper morphism.

Let $f : X \to Y$ be as above. A smooth function $\Phi : X \to [a, b)$, $-\infty < a < b \leq +\infty$, on X is called a relative exhaustion function if $f : \{\Phi \leq c\} \to Y$ is proper for every $c \in (a, b)$. For a positive integer q, $f : X \to Y$ is said to be *strongly q convex* if there exist a relative exhaustion function $\Phi : X \to [a, b)$ and $d \in (a, b)$ such that Φ is strongly q convex in the sense of Andreotti-Grauert,[1] on $\{\Phi > d\}$. The following coherency theorem for strongly q convex morphisms is known (cf. [15], § IV, (IV.8) Théorèm).

Theorem. *Let $f : X \to Y$ be a strongly q convex morphism of analytic spaces provided with a relative exhaustion function Φ. Let $\mathcal{F}$ be a coherent analytic sheaf on X and let r be an integer with $r \geq q$. Then $R^r f_* \mathcal{F}$ is a coherent analytic sheaf on Y and the canonical homomorphism $R^r f_* : H^r(X(S), \mathcal{F}) \to \Gamma(S, R^r f_* \mathcal{F})$ is a topological isomorphism for any relatively compact Stein open subset S of Y and $X(S) := f^{-1}(S)$. In particular, $H^r(X(S), \mathcal{F})$ has a structure of separated topological vector space.*

In order to discuss the torsion freeness of higher direct image sheaves by f we impose the hyper convexity induced by [7] on Φ and show the following theorem.

Theorem 1. *Let $f : X \to Y$ be a strongly q convex surjective morphism of analytic spaces of pure dimension provided with a relative exhaustion function Φ and let E be a holomorphic vector bundle on X. Suppose*

(i) *X is non-singular of pure dimension n and is provided with a Kähler metric ω_X such that Φ is weakly hyper p convex relative to ω_X on $\{\Phi > e\}$ with $e \in (a,b)$; i.e., the sum of any p eigen values of the Levi form of Φ relative to ω_X is non-negative at any point of $\{\Phi > e\}$, and*

(ii) *E is Nakano semi-positive on X (cf.* Definition 1.4).

Then for any $r \geq \max\{p,q\}$ the sheaf homomorphism $\mathcal{L}^r : R^0 f_* \Omega_X^{n-r}(E) \to R^r f_* \Omega_X^n(E)$ induced by the r-times exterior product by ω_X is surjective and the Hodge star operator relative to ω_X yields a splitting sheaf homomorphism $\delta^r : R^r f_* \Omega_X^n(E) \to R^0 f_* \Omega_X^{n-r}(E)$ with $\mathcal{L}^r \circ \delta^r = \mathrm{id}$. In particular, $R^r f_* \Omega_X^n(E)$ is torsion free and vanishes if $r > q_* := \max\{n-m, \max\{p,q\}\}$ with $m := \dim_{\mathbb{C}} Y$. Furthermore $R^s f_! \mathcal{O}_X(E^*) = 0$ if $s < n - q_* - \dim_{\mathbb{C}} Y$, where $R^\bullet f_!$ denotes the direct image with proper supports and E^* is the dual of E.

Theorem 1 can be shown by determining the structure of $H^r(X(S), \Omega_X^n(E))$ as an $\mathcal{O}(S)$−torsion free module, for any relatively compact Stein open subset S of Y, which follows from the weak hyper p convexity of Φ and the separability of cohomology group guaranteed by Theorem (cf. §2, Theorem 2.1). This can be done by an L^2−theory for the $\bar\partial$ operator with $\bar\partial$−Neumann condition on bounded domains with smooth boundary, which does not depend on the existence of complete Kähler metrics on $X(S)$. This is a difference of method from the one used in [20]. As a corollary we obtain the following vanishing theorem which is the relative version of Grauert-Riemenschneider's vanishing theorem for strongly hyper q convex Kähler manifolds (cf. [5], [7], [12] and [18]).

Theorem 2. *Let $f : X \to Y$ be a surjective morphism of analytic spaces of pure dimension provided with a relative exhaustion function $\Phi : X \to [a,b)$ and let E be a holomorphic vector bundle on X. Suppose*

(i) *X is non-singular of pure dimension n and is provided with a Kähler metric ω_X such that Φ is strongly hyper q convex relative to ω_X on $\{\Phi > e\}$ with $e \in (a,b)$; i.e., the sum of any p eigen values of the Levi form of Φ relative to ω_X is positive at any point of $\{\Phi > e\}$, and*

(ii) *E is Nakano semi-positive on X.*

Then $R^r f_* \Omega_X^n(E) = 0$ if $r \geq q$, and $R^s f_! \mathcal{O}_X(E^*) = 0$ if $s \leq n - q - \dim_{\mathbb{C}} Y$. Especially $R^r f_* \Omega_X^n = 0$ if $r \geq q$, and $R^s f_! \mathcal{O}_X = 0$ if $s \leq n - q - \dim_{\mathbb{C}} Y$.

1. An L^2 estimate for the $\bar\partial$ operator with $\bar\partial$–Neumann condition on Kähler manifolds

Let M be a complex manifold of dimension n provided with a Kähler metric ω_M and let E be a holomorphic vector bundle on M provided with a smooth hermitian metric h along the fibres of E. The curvature form Θ_h relative to h is defined by $\Theta_h := \bar\partial(h^{-1}\partial h) \in C^{1,1}(M, \mathrm{Hom}(E,E))$.

Let X be a bounded domain with smooth boundary ∂X; i.e., the closure $\overline{X}$ of X is compact and there exists a smooth function Ψ defined on a neighborhood of $\overline{X}$ such that $X = \{\Psi < 0\}$ and $d\Psi \neq 0$ on ∂X. We set $X_t := \{\Psi < t\}$ and $\partial X_t := \{\Psi = t\}$ for sufficiently small $t \in (-1,1)$. X_t is also a bounded domain with smooth boundary ∂X_t, and clearly $X_0 = X$ and $\partial X_0 = \partial X$.

From now on we fix this situation and use the formulations established in [20], § 1. Let $\langle\ ,\ \rangle_h$ denote the pointwise inner product of E-valued differential forms relative to ω_M and h. Let $(\ ,\ )_{h,t}$ (resp. $[\ ,\ ]_{h,t}$) denote the inner product for E-valued differential forms defined by the integral of $\langle\ ,\ \rangle_h$ on X_t (resp. ∂X_t, which is a smooth and compact real hyper surface of M).

The following formula is a variant of [19], §4, Proposition 1 (also cf. [20], §1, Proposition 1.11).

Proposition 1.1. *Let ψ be a real-valued smooth function on a neighborhood of $\bar X$ and set $\eta := e^{\psi}$. If $|t|$ is sufficiently small, then the following holds*:

$$\begin{aligned}\frac{d}{dt}[\sqrt{\eta}\mathbf{e}(\bar\partial\Psi)^*u]^2_{h,t} &= [\eta\sqrt{-1}\mathbf{e}(\partial\bar\partial\Psi)\Lambda u, u]_{h,t} + (\eta\sqrt{-1}\mathbf{e}(\Theta_h + \partial\bar\partial\psi)\Lambda u, u)_{h,t}\\ &\quad + \|\sqrt{\eta}(\vartheta - \mathbf{e}(\partial\psi)^*)u\|^2_{h,t} - \|\sqrt{\eta}(\bar\partial + \mathbf{e}(\bar\partial\psi))u\|^2_{h,t}\\ &\quad - \|\sqrt{\eta}\vartheta_h u\|^2_{h,t} - 2\mathrm{Re}[\eta\vartheta_h u, \mathbf{e}(\bar\partial\Psi)^*u]_{h,t}\end{aligned}$$

for any $u \in C^{n,r}(M,E)$ with $r \geq 1$.

Proof. Similarly to the proof of [20], §1, Proposition 1.11, if $u \in C^{n,r}(M,E)$ and $|t|$ is sufficiently small, then we obtain the following by integration by parts:

$$\begin{aligned}(*)\qquad &\|\sqrt{\eta}\bar\partial u\|^2_{h,t} + \|\sqrt{\eta}\vartheta_h u\|^2_{h,t} - \|\sqrt{\eta}\vartheta u\|^2_{h,t}\\ &= (\eta\sqrt{-1}\mathbf{e}(\Theta_h + \partial\bar\partial\psi)\Lambda u, u)_{h,t} - \|\sqrt{\eta}\mathbf{e}(\bar\partial\psi)u\|^2_{h,t} + \|\sqrt{\eta}\mathbf{e}(\partial\psi)^*u\|^2_{h,t}\\ &\quad - 2\mathrm{Re}\{(\eta\mathbf{e}(\bar\partial\psi)u, \bar\partial u)_{h,t} + (\eta\mathbf{e}(\partial\psi)^*u, \vartheta u)_{h,t}\}\\ &\quad - [\eta\vartheta_h u, \mathbf{e}(\bar\partial\Psi)^*u]_{h,t} + [\eta\mathbf{e}(\bar\partial\Psi)^*\bar\partial u, u]_{h,t} + [\eta\mathbf{e}(\partial\Psi)\vartheta u, u]_{h,t}\\ &\quad + [\eta\mathbf{e}(\bar\partial\psi)u, \mathbf{e}(\bar\partial\Psi)u]_{h,t} - [\eta\mathbf{e}(\partial\psi)^*u, \mathbf{e}(\partial\Psi)^*u]_{h,t}.\end{aligned}$$

On the other hand, by integration by parts we obtain the following:

$$(\bar\partial\mathbf{e}(\bar\partial\Psi)^*u, \eta u)_{h,t} = (\eta\mathbf{e}(\bar\partial\Psi)^*u, \vartheta_h u)_{h,t}$$

$$- (\eta\mathbf{e}(\bar{\partial}\Psi)^*u, \mathbf{e}(\bar{\partial}\psi)^*u)_{h,t} + [\sqrt{\eta}\mathbf{e}(\bar{\partial}\Psi)^*u]^2_{h,t}.$$

Substituting the formula [20], §1, (1.9) to the left hand side of the above equality and differentiating in t, we obtain the following:

$$\begin{aligned}\frac{d}{dt}[\sqrt{\eta}\mathbf{e}(\bar{\partial}\Psi)^*u]^2_{h,t} &= [\eta\sqrt{-1}\mathbf{e}(\partial\bar{\partial}\Psi)\Lambda u, u]_{h,t} - [\eta\mathbf{e}(\bar{\partial}\Psi)^*u, \vartheta_h u]_{h,t} - [\eta\mathbf{e}(\partial\Psi)\bar{\vartheta}u, u]_{h,t} \\ &- [\eta\mathbf{e}(\partial\Psi)^*\bar{\partial}u, u]_{h,t} + [\eta\mathbf{e}(\bar{\partial}\Psi)^*u, \mathbf{e}(\bar{\partial}\psi)^*u]_{h,t}.\end{aligned}$$

By the formula [20], §1, (1.4), if $u \in C^{n,r}(M, E)$, then we have the following:

$$(**) \qquad \langle\mathbf{e}(\partial\varphi)^*u, \mathbf{e}(\partial\Psi)^*u\rangle_h = \langle\mathbf{e}(\bar{\partial}\varphi)u, \mathbf{e}(\bar{\partial}\Psi)u\rangle_h + \langle\mathbf{e}(\bar{\partial}\Psi)^*u, \mathbf{e}(\bar{\partial}\varphi)^*u\rangle_h.$$

By substituting the above two equalities to $(*)$ we can obtain the desired equality. □

Lemma 1.2 (cf. [11], §1.4 and [18], Fact 2.7). *Let $\{\lambda_j\}$ be the eigen-values of a smooth $(1,1)$ differential form Θ on M relative to ω_M with $\lambda_1 \leq \lambda_2 \leq, \dots, \leq \lambda_n$ (which are continuous functions on M); i.e., $\Theta(x) = \sum_{j=1}^n \lambda_j(x) dz^j \wedge d\bar{z}^j$ with $\omega_X(x) = \sqrt{-1}\sum_{j=1}^n dz^j \wedge d\bar{z}^j$, at $x \in M$. Then if $v(x) = \sum v_{A_n,B_r} dz^{A_n} \wedge d\bar{z}^{B_r} \in C^{n,r}(M, E)$ with $r \geq 1$, the following holds*:

$$\langle\sqrt{-1}\mathbf{e}(\Theta)\Lambda v, v\rangle_h(x) = \sum_{|A_n|=n,|B_r|=r} \Big(\sum_{j\in B_r} \lambda_j(x)\Big) |v_{A_n,B_r}|^2_h.$$

In particular setting $\delta_r := \sum_{j=1}^r \lambda_j$ with $r \geq 1$ the following holds

$$\langle\sqrt{-1}\mathbf{e}(\Theta)\Lambda v, v\rangle_h \geq \delta_r\langle v, v\rangle_h \text{ if } v \in C^{n,r}(M, E).$$

As a consequence we can obtain the following L^2-estimate.

Proposition 1.3. *Suppose the defining function Ψ of X is weakly hyper p-convex relative to ω_M on a neighborhood of ∂X and ψ is a smooth function on $\overline{X}$. Then the following holds*:

$$\begin{aligned}&(\eta\sqrt{-1}\mathbf{e}(\Theta + \partial\bar{\partial}\psi)\Lambda u, u)_{h,X} + \|\sqrt{\eta}(\bar{\vartheta} + \mathbf{e}(\partial\psi)^*)u\|^2_{h,X} \\ &\leq \|\sqrt{\eta}(\bar{\partial} + \mathbf{e}(\bar{\partial}\psi))u\|^2_{h,X} + \|\sqrt{\eta}\vartheta_h u\|^2_{h,X}\end{aligned}$$

for any $u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\vartheta_h) \subset L^{n,r}(X, E)$ with $r \geq p$ and $\eta := e^{\psi}$.

Proof. Since ψ and its derivatives are bounded on X, and $C^{n,r}(\bar{X}, E) \cap \mathrm{Dom}(\vartheta_h) := \{u \in C^{n,r}(\bar{X}, E); \mathbf{e}(\bar{\partial}\Psi)^*u = 0 \text{ on } \partial X\}$ is dense in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\vartheta_h)$

relative to the graph norm $\|v\|_{h,X} + \|\bar{\partial} v\|_{h,X} + \|\vartheta_h v\|_{h,X}$ (cf. [8], Chap 1), we have only to show the above estimate for the forms contained in $C^{n,r}(\bar{X}, E) \cap \mathrm{Dom}(\vartheta_h)$. By Lemma 1.2 and the weak hyper r-convexity of Ψ, if $u \in C^{n,r}(\bar{X}, E)$, then $\langle \sqrt{-1}\mathbf{e}(\partial\bar{\partial}\Psi)\Lambda u, u\rangle_h$ is non-negative on ∂X. Hence the desired estimate follows from Proposition 1.1 immediately in view of the boundary condition $\mathbf{e}(\bar{\partial}\Psi)^* u = 0$ on ∂X. □

DEFINITION 1.4. (E, h) is said to be Nakano semi-positive if the curvature form Θ_h relative to h is a positive semi-definite quadratic form on each fibre of $E \bigotimes TM$, where TM is the holomorphic tangent bundle of M.

In line bundle case the Nakano semi-positivity coincides with the semi-positivity in the sense of Kodaira. The following lemma is used in the next section.

Lemma 1.5 (cf. [11], § 1.4). *Suppose (E, h) is Nakano semi-positive on M. Then there exists a non-negative continuous function ε_r on M such that*

$$\langle \sqrt{-1}\mathbf{e}(\Theta_h)\Lambda u, u\rangle_h \geq \varepsilon_r \langle u, u\rangle_h$$

for any $u \in C^{n,r}(X, E)$ with $r \geq 1$.

2. A criterion for the separability for cohomology groups of canonical sheaves on a certain non-compact Kähler manifold

In this section we show the following theorem.

Theorem 2.1. *Let X be a complex manifold of dimension n provided with a Kähler metric ω_X and let (E, h) be a holomorphic vector bundle on X. Suppose*

(i) *There exist non-negative smooth functions Φ and φ on X such that*
   (1) *Φ is weakly hyper p convex relative to ω_X on $\{\Phi > 0\}$ and φ is plurisubharmonic on X,*
   (2) *$\Psi := \Phi + \varphi$ is an exhaustion function of X; i.e., $X_c := \{\Psi < c\}$ is relatively compact for any c with $0 < c < \sup_X \Psi \leq +\infty$, and*

(ii) *(E, h) is Nakano semi-positive on X.*

Then for any $r \geq p$, the space of E-valued harmonic (n, r) forms $\mathcal{H}^{n,r}(X, E, \Psi)$ defined by

$$\mathcal{H}^{n,r}(X, E, \Psi) := \{u \in C^{n,r}(X, E); \bar{\partial} u = \vartheta_h u = 0 \text{ and } \mathbf{e}(\bar{\partial}\Psi)^* u = 0 \text{ on } X\}$$

represents $H^r(X, \Omega_X^n(E))$ if and only if $H^r(X, \Omega_X^n(E))$ has a structure of separated topological vector space.

We need the following propositions to show Theorem 2.1.

Proposition 2.2. *For any non-critical value $c > 0$ of Ψ and $r \geq p$ if $u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\vartheta_h) \subset L_2^{n,r}(X_c, E)$ satisfies $\bar{\partial}u = \vartheta_h u = 0$, then u satisfies the following*:

$$\langle \sqrt{-1}\mathbf{e}(\Theta_h)\Lambda u, u\rangle_h \equiv 0,\ \langle \sqrt{-1}\mathbf{e}(\partial\bar{\partial}\Phi)\Lambda u, u\rangle_h \equiv 0,\ \langle \sqrt{-1}\mathbf{e}(\partial\bar{\partial}\varphi)\Lambda u, u\rangle_h \equiv 0,$$
$$\mathbf{e}(\bar{\partial}\Phi)^* u \equiv 0,\ \mathbf{e}(\bar{\partial}\varphi)^* u \equiv 0 \ \text{ and } \ \bar{\vartheta} u \equiv 0 \text{ on } X_c.$$

Proof. Since Ψ is weakly hyper p convex relative to ω_X on the whole space X in view of the plurisubharmonicity of φ, setting $\psi \equiv 0$ in Proposition 1.3 we obtain the first and sixth equations by Lemma 1.5. By setting $\psi = \Phi$ in Proposition 1.3 the second and fourth ones can be derived from Lemma 1.2 and the equality $(**)$ used in the proof of Proposition 1.1. The third and fifth ones can be obtained similarly. □

Proposition 2.3. *For any $r \geq p$ let $\mathcal{H}^{n,r}(X, E, \Psi)$ be the space of E-valued harmonic forms defined in Theorem 2.1. Then the following assertions hold*:

(i) *Assume $u \in C^{n,r}(X, E)$ satisfies $\mathbf{e}(\bar{\partial}\Psi)^* u = 0$ on X. Then $\bar{\partial}u = \vartheta_h u = 0$ if and only if $\bar{\vartheta} u = 0$ and $\sqrt{-1}\langle \mathbf{e}(\Theta_h + \partial\bar{\partial}\Psi)\Lambda u, u\rangle_h = 0$ on X*

(ii) *If $u \in \mathcal{H}^{n,r}(X, E, \Psi)$, then $\langle \sqrt{-1}\mathbf{e}(\partial\bar{\partial}e^{\psi})\Lambda u, u\rangle_h \equiv 0$ on X for any smooth plurisubharmonic function ψ on X. In particular $\mathcal{H}^{n,r}(X, E, \Psi)$ does not depend on the choice of φ.*

(iii) *$\mathcal{H}^{n,r}(X, E, \Psi)$ is a torsion free $\mathcal{O}(X)$-module and the Hodge star operator $*$ relative to ω_X yields an injective $\mathcal{O}(X)$-homomorphism from $\mathcal{H}^{n,r}(X, E, \Phi)$ to $\Gamma(X, \Omega_X^{n-r}(E))$.*

(iv) *The canonical homomorphism $\iota^r : \mathcal{H}^{n,r}(X, E, \Psi) \longrightarrow H^r(X, \Omega_X^n(E))$ induced by Dolbeault's isomorphism theorem is injective (this property depends on neither the curvature condition of E nor the Kähler property of ω_X and depends only on the condition $\mathbf{e}(\bar{\partial}\Psi)^* u = 0$).*

Since Proposition 2.3 can be shown similarly to [20], §4, Theorem 4.3 in view of Proposition 1.1, the details is left to the reader.

Proof of Theorem 2.1. We first show the necessity of Theorem. If the canonical homomorphism $\iota^r : \mathcal{H}^{n,r}(X, E, \Psi) \longrightarrow H^r(X, \Omega_X^n(E))$ induced by Dolbeault's isomorphism theorem yields an isomorphism, then any $\bar{\partial}$-closed form $v \in C^{n,r}(X, E)$ has the following decomposition:

$$(\sharp) \qquad v = u + \bar{\partial}w \quad \text{for} \quad u \in \mathcal{H}^{n,r}(X, E, \Psi) \quad \text{and} \quad w \in C^{n,r-1}(X, E)$$

Suppose the above v is contained in the closure of $\bar{\partial}C^{n,r-1}(X, E)$ relative to the Fréchet-Schwartz topology. Then there exists a sequence of smooth forms $\{w_k\}_{k \geq 1} \in C^{n,r-1}(X, E)$ such that $\bar{\partial}w_k$ converges strongly to v in L^2-sense on every compact subset of X. Hence for any non-critical value c of Ψ, by integration by parts on X_c we obtain

$$(u,u)_h = (v - \bar\partial w, u)_h = (v,u)_h = \lim_{k\to\infty} (\bar\partial w_k, u)_h = \lim_{k\to\infty} (w_k, \vartheta_h u)_h = 0$$

Here we note that every boundary integral on $\partial X_c = \{\Psi = c\}$ arising from integration by parts vanishes in view of the equation $\mathbf{e}(\bar\partial\Psi)^* u = 0$. Therefore $u \equiv 0$ on X and so $v = \bar\partial w$. This implies that $\bar\partial C^{n,r-1}(X,E)$ is closed and so the cohomology group is Hausdorff.

The sufficiency of Theorem is shown as follows. In view of Proposition 2.3, (iv) we have only to show that any $\bar\partial$-closed form $v \in C^{n,r}(X,E)$ admits the decomposition ($\sharp$) under the Hausdorff property of $H^r(X,\Omega_X^n(E))$. From now on we fix an increasing sequence $\{c_k\}_{k\geq 1}$ of non-critical values of Ψ such that $\lim_{k\to\infty} c_k = \sup_X \Psi$. Setting $X_k := X_{c_k}$, let $N_k^{n,r}(\bar\partial)$ (resp. $N_k^{n,r}(\vartheta_h)$) be the null space of $\bar\partial$ (resp. ϑ_h) in $\mathrm{Dom}(\bar\partial)$ (resp. $\mathrm{Dom}(\vartheta_h)$) $\subset L_2^{n,r}(X_k,E)$. $N_k^{n,r}(\bar\partial)$ is decomposed as follows:

$$N_k^{n,r}(\bar\partial) = H_k^{n,r}(E) \bigoplus [\mathrm{Range}(\bar\partial)] \quad \text{for} \quad H_k^{n,r}(E) := N_k^{n,r}(\bar\partial) \cap N_k^{n,r}(\vartheta_h)$$

Hence setting $v_k := v|_{X_k}$, v_k is decomposed as follows:

$$v_k = u_k + v_k^* \quad \text{with} \quad u_k \in H_k^{n,r}(E) \quad \text{and} \quad v_k^* \in [\mathrm{Range}(\bar\partial)]$$

Applying Proposition 2.2 to X_k, it follows that $H_k^{n,r}(E) \subset \mathcal{H}^{n,q}(X_k,E,\Psi)$ and $u|_{X_k} \in H_k^{n,r}(E)$ if $u \in H_l^{n,r}(E)$ and $l > k \geq 1$ (cf. [4], Chap. 1). In particular $u_{k+1} = u_k$ and $v_{k+1}^* = v_k^*$ on X_k for any $k \geq 1$. Setting $u := u_k$ and $v^* := v_k^*$ on X_k for any $k \geq 1$ we obtain $v = u + v^*$ and $u \in \mathcal{H}^{n,r}(X,E,\Psi)$. Since Ψ is an exhaustion function of X, we can take a smooth strictly increasing function $\lambda : [0, \sup\Psi) \to [0,+\infty)$ such that v and $u \in L_2^{n,r}(X,E,he^{-\lambda(\Psi)})$. Setting $g := he^{-\lambda(\Psi)}$, u satisfies $\bar\partial u = \vartheta_g u = 0$ in $L_2^{n,r}(X,E,g)$ by $\vartheta_g = \vartheta_h + \lambda'(\Psi)\mathbf{e}(\bar\partial\Psi)^*$, which implies $v^* \in [\mathrm{Range}(\bar\partial)] \subset L_2^{n,r}(X,E,g)$. Therefore there exists $w \in C^{n,r-1}(X,E)$ with $v^* = \bar\partial w$ by the Hausdorff property of $H^r(X,\Omega_X^n(E))$ by [20], Proposition 4.6. Finally we have obtained the decomposition ($\sharp$). □

Setting $\Phi \equiv 0$ in Theorem 2.1 we obtain the following theorem.

Theorem 2.4. *Let X be a weakly 1-complete manifold of dimension n; i.e., X admits a smooth plurisubharmonic exhaustion function Ψ. Suppose X admits a Kähler metric ω_X and E is a Nakano semi-positive vector bundle on X. Then for any $r \geq 1$, $\mathcal{H}^{n,r}(X,E,\Psi)$ represents $H^r(X,\Omega_X^n(E))$ if and only if $H^r(X,\Omega_X^n(E))$ has a structure of separated topological vector space.*

REMARK 2.5. If X is holomorphically convex, then the sufficiency of Theorem

2.1 has already shown in [20], Theorem 5.2. On the other hand it is interesting that there exists a class of weakly 1-complete Kähler manifolds X being not holomorphically convex whose canonical line bundle is flat and $H^r(X, \mathcal{O}_X)$ is either Hausdorff or not (cf. [9], [10], [21])

3. Proof of Theorems 1 and 2

Let the situation be the same as in Theorem 1 stated in the introduction. We fix the Kähler metric ω_X and the metric h of E satisfying the hypothesis respectively. By composing an arbitrarily smooth convex increasing function with Φ we may assume that (1) $\Phi \geq 0$ on X, and (2) Φ is strongly q-convex and weakly hyper p-convex on $\{\Phi > 0\}$ relative to ω_X. We take a Stein open covering $\{V_\alpha, \tau_\alpha, S_\alpha, \mathbb{C}^{d(\alpha)}\}_{\alpha \in A}$ of Y such that τ_α is an isomorphism from V_α to a subvariety $S_\alpha \subset (\mathbb{C}^{d(\alpha)}, (z^1, ..., z^{d(\alpha)}))$ for any $\alpha \in A$. Setting $\varphi_\alpha := (\tau^\alpha \circ f)^*\left(\sum_{j=1}^{d(\alpha)} |z^j|^2\right)$, $\Psi_\alpha := \Phi + \varphi_\alpha$ and $X(V_\alpha) := f^{-1}(V_\alpha)$, each pair $\{X(V_\alpha), \Psi_\alpha\}$ satisfies the condition of Theorem 2.1, (i).

For any $r \geq \max\{p, q\}$, by the theorem stated in the introduction and Theorem 2.1, the homomorphism $\iota^r : \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha) \to H^r(X(V_\alpha), \Omega_X^n(E))$ induces an isomorphims as an $\mathcal{O}(V_\alpha)$-module. Furthermore for any Stein open subset $W \subset V_\alpha$ provided with an strictly plurisubharmonic exhaustion function ψ_W, we claim that the restriction homomorphism $r_{V_\alpha, W} : \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha) \to \mathcal{H}^{n,r}(f^{-1}(W), E, \Phi + f^*\psi_W)$ can be well-defined and commutes with the restriction homomorphism of cohomology group. By the surjectivity of f, for any α there exists an open dense subset $U_\alpha \subset V_\alpha$ such that U_α is non-singular and $f : f^{-1}(U_\alpha) \to U_\alpha$ is smooth. By Proposition 2.3, (ii) and § 1, (1.4) in [20], $u \in \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha)$ satisfies the equation: $\sqrt{-1}\langle \mathbf{e}(\partial\bar{\partial}\varphi_\alpha)\Lambda u, u\rangle_h = \sum_{j=1}^{d(\alpha)} |\mathbf{e}(\overline{\partial(\tau^\alpha \circ f)^* z^j})^* u|_h^2 \equiv 0$ on $X(V_\alpha)$ for any α. Hence $d(\tau^\alpha \circ f)^* z^j \wedge *u \equiv 0$ on $X(V_\alpha)$ for any j and α, where $*$ is the star operator relative to ω_X. This implies that (1) $\mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha) = 0$ if $r > \max\{n - m, \max\{p, q\}\}$ with $m = \dim_{\mathbb{C}} Y$, (2) any point $x \in U_\alpha$ admits a neighborhood $V_x \subset U_\alpha$ and a non-vanishing holomorphic m form θ_x on V_x so that $*u$ can be divided by $f^*\theta_x$ on $f^{-1}(V_x)$ for any $u \in \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha)$ if $\max\{p, q\} \leq r \leq n - m$. Hence $u \in \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha)$ satisfies $\mathbf{e}(\bar{\partial}(f^*\psi_W))^* u \equiv 0$ on $X(W)$; i.e., $u|_{X(W)} \in \mathcal{H}^{n,r}(X(W), E, \Phi + f^*\psi_W)$, which implies our claim.

Denoting the sheafification of the data $\{\mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha), r_{V_\alpha, W}\}$ with the restriction homomorphism $r_{V_\alpha, W} : \mathcal{H}^{n,r}(X(V_\alpha), E, \Psi_\alpha) \to \mathcal{H}^{n,r}(f^{-1}(W), E, \Phi + f^*\psi_W), W \subset V_\alpha$ by $R^0 f_* \mathcal{H}^{n,r}(E, \Phi)$, we obtain a sheaf isomorphism $\iota^r : R^0 f_* \mathcal{H}^{n,r}(E, \Phi) \to R^r f_* \Omega_X^n(E)$ of $\mathcal{O}_Y$-module. Furthermore for any relatively compact Stein open subset S provided with a smooth strictly plurisubharmonic exhaustion function ψ_S clearly the canonical homomorphism from $\mathcal{H}^{n,r}(f^{-1}(S), E, \Phi + f^*\psi_S)$ to $\Gamma(S, R^0 f_* \mathcal{H}^{n,r}(E, \Phi))$ is an isomorphism. By Proposition 2.3, (iii), the operator $*$ induces a sheaf homomorphism $\sigma^r : R^0 f_* \mathcal{H}^{n,r}(E, \Phi) \to R^0 f_* \Omega_X^{n-r}(E)$ with $\mathcal{L}^r \circ \sigma^r = \mathrm{id}$ because $L^r \circ * = c(n, r)\mathrm{id}$, $c(n, q) \neq 0 \in \mathbb{C}$, on (n, r) forms. Finally $\delta^r := \sigma^r \circ (\iota^r)^{-1} : R^r f_* \Omega_X^n(E) \to R^0 f_* \Omega_X^{n-r}(E)$ is the desired splitting sheaf homomorphism. The vanishing theorems follow from the above observation and the duality theorem by Ramis and Ruget (cf. [13] and also [3]). This completes the proof of Theorem 1.

To show Theorem 2 we have only to show $\mathcal{H}^{n,r}(f^{-1}(S), E, \Phi + f^*\psi_S) = 0$ for any Stein open subset (S, ψ_S) of Y because $f : X \to Y$ is a strongly q convex morphism. By the strong hyper q convexity of Φ, this follows from Lemma 1.2 and Proposition 2.2 (cf. [2], [14]). This completes the proof of Theorem 2.

References

[1] A. Andreotti and H. Grauert: *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull.Soc. Math. France, **90** (1960), 193–259.

[2] N. Aronszajn: *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957), 235–249.

[3] J.L. Ermine: *Cohérence de certaines images directes à supports propres dans le cas d'un morphisme fortment p-convexe*, Ann. Scuola Norm. Sup. Pisa, **6** (1979), 1–18.

[4] G.B. Folland and J.J. Kohn: The Neumann problem for the Cauchy-Riemann complex, Princeton Univ. Press, 1972.

[5] A. Fujiki: *A coherency theorem for direct images with proper supports in the case of a 1-convex map*, Publ. Res. Inst. Math. Sci. **18** (1982), 451–476 (31–56).

[6] H. Grauert: *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Publ. Math. I. H. E. S, **5** (1960).

[7] H. Grauert and O. Riemenschneider: *Kählersche Mannigfaltigkeiten mit hyper-q-konvexen Rand*,Problems in Analysis (Lecture Symp. in honor of S. Bochner, 1969),Princeton Univ. N. J. (1970), 61–79.

[8] L. Hörmander: *L^2 estimates and existence theorems for the $\bar\partial$ operator*, Acta math. **113** (1965), 89–152.

[9] H. Kazama: *On pseudoconvexity of complex abelian Lie groups*, J. Math. Soc. Japan, **25** (1973), 329–333.

[10] H. Kazama: *$\bar\partial$ cohomology of (H, C) groups*, Publ. RIMS, Kyoto Univ. **20** (1984), 297–317.

[11] T. Ohsawa: *Cohomology vanishing theorems on weakly 1-complete manifolds*, Publ. RIMS, Kyoto Univ. **19** (1983), 1181–1201.

[12] T. Ohsawa: *A vanishing theorem for proper direct images*, RIMS, Kyoto Univ. **23** (1987), 243–250.

[13] J.P. Ramis and G. Ruget: *Residus et dualite*, Invent. Math. **26** (1974), 89–131.

[14] O. Riemenschneider: *A generalization of Kodaira's embedding theorem*, Math. Ann. **200** (1973), 99–102.

[15] P. Siegfried: *Un Théorème de finitude pour les morphisms q-convexes*, Comment. Math. Helv. **49** (1974), 417–459.

[16] T.Y. Siu: *The 1-convex generalization of Grauert's direct image theorem*, Math. Ann. **190** (1971), 203–214.

[17] T.Y. Siu: *A pseudoconvex-pseudoconcave generalization of Grauert's direct image theorem*, Ann. Scuola Norm. Sup. Pisa, **26** (1972), 649–664.

[18] K. Takegoshi: *Relative vanishing theorems in analytic spaces*, Duke Math.J. **52** (1985), 273–279.

[19] K. Takegoshi: *Application of a certain integral formula to complex analysis*, Prospects in Comple Analysis, Proc. 25-th Taniguchi International Symposium, Katata/Kyoto, Lecture Notes in Math. **1469** (1991), Springer, 94–114.

[20] K. Takegoshi: *Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms*, Math. Ann. **303** (1995), 389–416.

[21] C. Vogt: *Line bundles on toroidal groups*, J. Reine Angew. Math. **335** (1982), 197–215.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560–0043, Japan
e-mail: kensho@math.wani.osaka-u.ac.jp