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DADE'S CONJECTURE FOR 2-BLOCKS OF SYMMETRIC GROUPS

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0. Introduction

Let G be a finite group, p a prime and B a p-block of G. In [4] Dade conjectured that the number of ordinary irreducible characters of B with a fixed defect can be expressed as an alternating sum of the numbers of ordinary irreducible characters of related defects in related blocks B' of certain local p-subgroups of G. This (ordinary) conjecture has been proved by Olsson and Uno for the symmetric groups when p is odd. In this paper, we prove the (ordinary) conjecture for the symmetric groups G when $p = 2$.

In Section 1 we state the ordinary conjecture and fix some notation. In Section 2 we reduce the family of radical 2-chains $\mathcal{R}(G)$ to a G-invariant subfamily $\mathcal{QR}(G)$. In Section 3 we first give several more reductions, and then prove the conjecture for $p = 2$ using results of Olsson and Uno [6].

1. Dade's ordinary conjecture

Throughout this paper we shall follow the notation of Dade [4]. Let C be a p-subgroup chain of a finite group G,

$$C : P_0 < P_1 < \cdots < P_w. \tag{1.1}$$

Then $w = |C|$ is called the *length* of C,

$$N(C) = N_G(C) = N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_w) \tag{1.2}$$

is called the *normalizer* of C in G, and

$$C_k : P_0 < P_1 < \cdots < P_k, \qquad 0 \leq k \leq w \tag{1.3}$$

is called the *k-th initial* p-subchain of C. In addition, C is called a *radical* p-chain if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for all $1 \leq k \leq w$.

Thus P_{k+1} and P_{k+1}/P_k are radical subgroups of $N(C_k)$ and $N(C_k)/P_k$, respectively for $0 \leq k \leq w-1$, where a p-subgroup R of G is *radical* if $R = O_p(N_G(R))$. Let $\mathcal{R} = \mathcal{R}(G)$ be the set of all radical p-chains of G.

Given $C \in \mathcal{R}$, B a p-block of G and u a non-negative integer, let $\mathrm{k}(N(C), B, u)$ be the number of characters of the set

$$\text{(1.4)} \quad \mathrm{Irr}(N(C), B, u) = \left\{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B, \text{ and } \mathrm{d}(\psi) = u\right\},$$

where $B(\psi)$ is the block of $N(C)$ containing ψ and $\mathrm{d}(\psi)$ is the p-defect of ψ (see [4, (5.5)] for the definition). Then the following is Dade's ordinary conjecture, [4, Conjecture 6.3].

Dade's ordinary conjecture. *If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, and if u is a non-negative integer, then*

$$\text{(1.5)} \qquad \sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, u) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

2. The first reduction

In this section we shall first define a G-invariant subfamily $\mathcal{QR}$ of radical 2-chains of a symmetric group and then reduce Dade's conjecture to the family $\mathcal{QR}$. In the rest of the paper we always suppose $p = 2$.

We shall also follow the notation of Alperin and Fong [1]. Given a positive integer n, we denote by $\mathbf{S}(n) = \mathbf{S}(V)$ the symmetric group of degree n acting on the set V of cardinality n. For each non-negative integer c, let A_c denote the elementary abelian group of order 2^c represented by its regular permutation representation. Thus A_c is embedded uniquely up to conjugacy as a transitive subgroup of $\mathbf{S}(2^c)$, $C_{\mathbf{S}(2^c)}(A_c) = A_c$, and

$$N_{\mathbf{S}(2^c)}(A_c) \simeq A_c \rtimes \mathrm{GL}(c, 2).$$

For a sequence $\mathbf{c} = (c_1, c_2, \ldots, c_\ell)$ of non-negative integers, let $|\mathbf{c}| = c_1 + \ldots + c_\ell$ and let $A_{\mathbf{c}}$ be the wreath product $A_{c_1} \wr A_{c_2} \wr \ldots \wr A_{c_\ell}$. Then $A_{\mathbf{c}}$ is embedded uniquely up to conjugacy as a transitive subgroup of $\mathbf{S}(2^{|\mathbf{c}|})$. Moreover,

$$\text{(2.1)} \quad \begin{aligned} N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{\mathbf{c}}) &= N_{\mathbf{S}(2^{c_1})}(A_{c_1}) \otimes N_{\mathbf{S}(2^{|\mathbf{c}'|})}(A_{\mathbf{c}'}) \\ N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{\mathbf{c}})/A_{\mathbf{c}} &\simeq \mathrm{GL}(c_1, 2) \times \mathrm{GL}(c_2, 2) \times \ldots \times \mathrm{GL}(c_\ell, 2), \end{aligned}$$

where $\mathbf{c}' = (c_2, \ldots, c_\ell)$ and $N_{\mathbf{S}(2^{c_1})}(A_{c_1}) \otimes N_{\mathbf{S}(2^{|\mathbf{c}'|})}(A_{\mathbf{c}'})$ is the tensor product of the normalizers $N_{\mathbf{S}(2^{c_1})}(A_{c_1})$ and $N_{\mathbf{S}(2^{|\mathbf{c}'|})}(A_{\mathbf{c}'})$. Suppose R is a radical 2-subgroup of G. By Alperin and Fong [1, (2A)], there exists a corresponding decomposition

$$\text{(2.2)} \quad \begin{aligned} V &= V_0 \cup V_1 \cup \cdots \cup V_t, \\ R &= R_0 \times R_1 \times \cdots \times R_t \end{aligned}$$

such that $R_0 = \langle 1_{V_0} \rangle$ and each R_i for $i \geq 1$ is conjugate to some $A_{\mathbf{c}}$ in $\mathbf{S}(V_i)$. Let $A(R)$ be the subgroup generated by all normal abelian subgroups of R, and let $B(R) = C_{A(R)}([A(R), A(R)])$, where $[A(R), A(R)]$ is the commutator subgroup of $A(R)$. Then $B(R)$ is a characteristic subgroup of R and $N_G(R) \leq N_G(B(R))$. By [2, (2A)],

$$B(A_{\mathbf{c}}) = \begin{cases} (A_{c_1})^{2^{|\mathbf{c}'|}} & \text{if } c_1 \neq 1 \text{ or } c_2 \neq 1, \\ (D_8)^{2^{|\mathbf{w}|}} & \text{if } c_1 = c_2 = 1, \end{cases}$$

where $D_8 = A_1 \wr A_1$ is a dihedral group of order 8 and $\mathbf{w} = (c_3, \dots, c_\ell)$.

Let $\Psi = \{A_{\mathbf{c}} : \mathbf{c} = (1, c_2, c_3 \dots, c_\ell),\ c_2 \geq 2\}$, $\Psi' = \{A_{\mathbf{c}} : \mathbf{c} = (1, 1, c_3 \dots, c_\ell)\}$ and $\Psi^* = \{A_{\mathbf{c}} : \mathbf{c} = (0) \text{ or } \mathbf{c} = (1, 1, \dots, 1)\}$. A 2-subgroup R with a decomposition (2.2) is radical in G if and only if $m_R(P) \neq 2, 4$ for all $P \in \Psi^*$, where $m_R(P)$ is the multiplicity of the components P in R.

Let S_d be a Sylow 2-subgroup of $N_{\mathbf{S}(2^d)}(A_d)$. Then $S_1 = A_1$ and $S_2 = D_8$. Let $\Delta(1) = \{A_1 \wr A_2\}$, $\Delta(d) = \{S_d \wr A_1, S_d \wr A_2, S_d \wr A_1 \wr A_1\}$ for $d \geq 2$,

$$\Delta^+ = \bigcup_{d \geq 1} \Delta(d) \qquad \text{and} \qquad \Delta = \Delta(1) \cup \Delta(2).$$

Suppose R is a radical subgroup of G with a decomposition (2.2). Then $m_R(D_8) \notin \{2, 4\}$, $m_R(A_1) \neq 2$ and $m_R(D_8 \wr A_1) \neq 2$. So $B(R) = \prod_{i=1}^{t} B(R_i)$ is non-radical in G if and only if $m_{B(R)}(A_1) = 4$ or $m_{B(R)}(D_8) \in \{2, 4\}$, which is equivalent to
(a) $m_R(A_1 \wr A_2) = 1$ but $m_R(P) = 0$ for $P \in \Psi \backslash \{A_1 \wr A_2\}$, or
(b) For $X \in \Delta(2)$, $m_R(X) = 1$ but $m_R(P) = 0$ for all $P \in \Psi' \backslash \{X\}$.
If $B(R)$ is radical, then define $K(R) = B(R)$. Suppose $B(R)$ is non-radical. Define

$$K(R) = \begin{cases} A_1 \wr A_2 \times \prod_{R_j \neq A_1 \wr A_2} B(R_j) & \text{if only case (a) occurs,} \\ X \times \prod_{R_j \neq X} B(R_j) & \text{if only case (b) occurs,} \\ A_1 \wr A_2 \times X \times \prod_{R_j \neq A_1 \wr A_2, X} B(R_j) & \text{if both cases (a) and (b) occur.} \end{cases}$$

Thus $K(R) \trianglelefteq R$, $K(R)$ is a radical subgroup of G and

$$N_G(R) \leq N_G(K(R)) \leq N_G(B(R)).$$

In addition, if two radical subgroups R and W are G-conjugate, then $K(R)$ and $K(W)$ are G-conjugate, since $B(R)$ and $B(W)$ are G-conjugate. We also need the following lemma to define the chains in $\mathcal{QR}$.

(2A). *Given integer $d \geq 1$, let $G = \mathbf{S}(2^d) = \mathbf{S}(V)$ and $N = N(A_d) = N_G(A_d)$.*
(a) *There exists a bijection between the classes of radical subgroups R of N and the compositions $\mathbf{c} = (c_1, c_2, \cdots, c_\ell)$ of d such that*

$$N_N(R)/R \simeq \mathrm{GL}(c_1, 2) \times \mathrm{GL}(c_2, 2) \times \cdots \times \mathrm{GL}(c_\ell, 2).$$

In particular, the subset $[V, R]$ *of* V *consisting of all points moved by* R *is* V *itself.*

(b) *Let* R *be a radical subgroup of* N *and* Q *a radical subgroup of* $N_N(R)$. *Then* Q *is radical in* N *and* $N_N(Q) \leq N_N(R)$.

Proof. (a) Since R is radical in N, $A_d \trianglelefteq R$ and R/A_d is a radical subgroup of $N/A_d \simeq \mathrm{GL}(d, 2)$. Since $N_N(R)/A_d \simeq N_{N/A_d}(R/A_d)$, it follows by Borel-Tits theorem [3] that R/A_d is the unipotent radical of a parabolic subgroup of N/A_d. The classes of parabolic subgroups of $\mathrm{GL}(d, 2)$ are labelled by compositions of d, and so (a) follows easily.

(b) Suppose Q is a radical subgroup of $N_N(R)$. Then $R \leq Q$, and the proof of (b) is also straightforward by applying the Borel-Tits theorem to $N_N(R)/A_d$. □

REMARK. Follow the notation of (2A). Then R is radical in G if and only if $R = A_d$ except when $d = 2$ and $c_1 = c_2 = 1$, in which case either $R = A_d$ or $R = D_8$. Indeed, we may suppose $d \geq 2$. Since $A_d \leq R$, it follows that R acts transitively on V, and $R = A_{\mathbf{w}}$ for some sequence $\mathbf{w} = (w_1, \ldots, w_\ell)$ of positive integers with $|\mathbf{w}| = d$. Note that $A_d \leq A(R)$. If $w_1 \geq 2$, then each $A(R)$-orbit in V has 2^{w_1} elements, so that $d = w_1$. If $w_1 = 1$, then each $A(R)$-orbit in V is contained in some $A_1 \wr A_{w_2}$-orbit, so that $\mathbf{w} = (1, w_2)$ and $d = 1 + w_2$. But $|R/A_d| = 2^{w_2}$ and $|A_1 \wr A_{w_2}| = 2^{2^{w_2} + w_2}$, so $2^{w_2} = d = w_2 + 1$ and $w_2 = 0$ or 1. Thus $w_2 = 1$ and $R = D_8$.

The radical subgroup R of $N_{\mathbf{S}(2^d)}(A_d)$ determined by the composition $\mathbf{c}$ in (2A) (a) will be denoted by $Q_{\mathbf{c}}$ if R is not a radical subgroup of $\mathbf{S}(2^d)$. This holds in particular if $d \geq 3$. We set $B(Q_{\mathbf{c}}) = A_d$. Now we can define the family $\mathcal{QR}$.

Let $\mathcal{QR} = \mathcal{QR}(G)$ be the G-invariant subfamily of $\mathcal{R}$ consisting of radical 2-chains

$$(2.3) \qquad C: \ 1 < P_1 < \cdots < P_w$$

such that $P_1 = K(P_1)$ and each P_i has a decomposition $\prod_{j=1}^{t_i} Q_{i,j}$ with $Q_{i,j} \in \Delta^+ \cup \{A_d, Q_{\mathbf{c}}, D_8\}$ for all i, j. Let $\mathcal{M}$ be the complement $\mathcal{R} \backslash \mathcal{QR}$ of $\mathcal{QR}$ in $\mathcal{R}$, so that

$$\mathcal{R} = \mathcal{QR} \bigcup \mathcal{M} \qquad \text{(disjoint)}.$$

In the following we shall show that Dade's conjecture can be reduced to the family $\mathcal{QR}$. First of all, we consider the structure of the subgroup P_2. By definition, P_2 is a radical subgroup of $N_G(P_1)$.

Let D be a radical subgroup of G such that $D = K(D)$. Then

$$(2.4) \qquad V = V^+ \cup V^* \qquad \text{and} \qquad D = D^+ \times D^*,$$

where $D^+ = \prod_{X\in\Delta}(X)^{\alpha_X}$ with $\alpha_X \in \{0,1\}$, $D^* = (D_8)^{m'_2} \times \prod_{d\geq 0}(A_d)^{m_d}$, $V^+ = [V, D^+]$ and $V^* = V \backslash V^+$. Let U_X be the underlying set of $X \in \Delta^+$ such that $U_X = [U_X, X]$, and $N_X = N_{\mathbf{S}(U_X)}(X)$. Then

$$N(D) = N_G(D) = N(D)^+ \times N(D)^*,$$

where $N(D)^+ = \prod_{X\in\Delta}(N_X)^{\alpha_X}$ and $N(D)^* = D_8 \wr \mathbf{S}(m'_2) \times \prod_{d\geq 0} N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(m_d)$. If $X = A_1 \wr A_2$, then $N_X = A_1 \wr \mathbf{S}(4)$ and it has exactly two radical subgroups, $A_{(1,1,1)}$ and $A_1 \wr A_2$ up to conjugacy. Similarly, if $X = D_8 \wr A_2$, then $N_X = D_8 \wr \mathbf{S}(4)$ and it has exactly two radical subgroups, $A_{(1,1,1,1)}$ and $D_8 \wr A_2$ up to conjugacy.

(2B). *Let D be a subgroup of $G = \mathbf{S}(V)$ with a decomposition* (2.2) *such that $D = B(D)$ and $[V, D] = V$. In addition, let R be a radical subgroup of $N = N(R)$. Suppose $D = D(1) = (A_1)^{m_1}$, $D(2) = (A_2)^{m_2}$ or $D(2)' = (D_8)^{m'_2}$. Then R is radical in G and $K(R)$ is radical in N. If $L = N_N(K(R))$, then*

$$L = \begin{cases} (A_1) \wr \mathbf{S}(t_1) \times (D_8) \wr \mathbf{S}(t'_2) \times \prod_{X\in\Delta}(N_X)^{\beta_X} & \mathit{if}\ D = D(1), \\ N_{\mathbf{S}(4)}(A_2) \wr \mathbf{S}(t_2) \times (D_8) \wr \mathbf{S}(t'_2) \times \prod_{X\in\Delta(2)}(N_X)^{\beta_X} & \mathit{if}\ D = D(2), \\ (D_8) \wr \mathbf{S}(t'_2) \times \prod_{X\in\Delta(2)}(N_X)^{\beta_X} & \mathit{if}\ D = D(2)', \end{cases}$$

where t_1, t_2 and t'_2 are some non-negative integers and $\beta_X = 0, 1$. Moreover, $N_N(R) \leq L$.

Proof. Suppose $D = D(1)$, so that $N = A_1 \wr \mathbf{S}(m_1)$. It follows by [5, Proposition 4.7] or [6, Proposition 2.3 and the Remark 2.5] that $R = \prod_{i=1}^m R_i$, where $R_i = A_1 \wr R'_i$ with $R'_i = A_{\mathbf{z}}$. Thus $R_i \in \Psi \cup \Psi'$ and $B(R) = (A_1)^\alpha \times (D_8)^\beta$ for some integers $\alpha, \beta \geq 0$. Since $R/D = \prod_{i=1}^m R'_i$ is radical in $\mathbf{S}(m_1)$, it follows that $m_{R/D}(A_{\mathbf{c}}) \notin \{2,4\}$, and hence $m_R(A_{\mathbf{c}}) \notin \{2,4\}$ for all $A_{\mathbf{c}} \in \Psi^*$. Thus R and then $K(R)$ are radical in G, $B(R) = (A_1)^{t_1} \times (D_8)^{t'_2}$ with $t_1 + 2t'_2 = m_1$ and $N(B(R)) = (A_1) \wr \mathbf{S}(t_1) \times D_8 \wr \mathbf{S}(t'_2)$. Since $N(K(R)) \leq N(B(R)) \leq N$, $K(R)$ is radical in N. If $B(R)$ is radical in G, then $K(R) = B(R)$ and $N(K(R)) = N(B(R))$. Suppose $B(R)$ is non-radical in G. Then

$$K(R) = \begin{cases} A_1 \wr A_2 \times Y = R & \text{if } t_1 = 4 \text{ and } t'_2 \in \{2,4\}, \\ (A_1)^{t_1} \times Y & \text{if } t_1 \notin \{2,4\} \text{ and } t'_2 \in \{2,4\}, \\ A_1 \wr A_2 \times (D_8)^{t'_2} & \text{if } t_1 = 4 \text{ and } t'_2 \notin \{2,4\} \end{cases}$$

for some $Y \in \Delta(2)$. Thus $N_N(K(R))$ is given by (2B). Since $N(R) \leq N(K(R))$, it follows that

$$N_N(R) = N(R) \cap N \leq N(K(R)) \cap N = N_N(K(R)) = L.$$

Suppose $D = D(2)$, so that $N = N_{\mathbf{S}(4)}(A_2) \wr \mathbf{S}(m_2)$. Since A_2 and D_8 are the only radical subgroups (up to conjugacy) in $N_{\mathbf{S}(4)}(A_2)$, it follows that $R = \prod_{i=1}^{m} R_i$, where $R_i = A_2 \wr R_i'$ or $D_8 \wr R_i'$ with $R_i' = A_{\mathbf{z}}$. Let $B(R) = (A_2)^{t_2} \times (D_8)^{t_2'}$, $R(2) = \prod_i R_i$ and $R(2)' = \prod_j R_j$, where i and j run over the indices such that $R_i = A_2 \wr R_i'$ and $R_j = D_8 \wr R_j'$, respectively. Then $R(2)'/(A_2)^{t_2'}$ is radical in $\mathrm{GL}(2,2) \wr \mathbf{S}(t_2')$, since R/D is radical in $\mathrm{GL}(2,2) \wr \mathbf{S}(m_2)$. Thus $m_{R(2)'/(D_8)^{t_2'}}(A_{\mathbf{c}}) \notin \{2,4\}$, and hence $m_{R(2)'}(A_{\mathbf{c}}) \notin \{2,4\}$ for each $A_{\mathbf{c}} \in \Psi^*$. It follows that R is radical in G. If $B(R)$ is non-radical in G, then $t_2' \in \{2,4\}$ and so $R = R(2) \times Y$ for some $Y \in \Delta(2)$, and $K(R) = (A_2)^{t_2} \times Y$. Since $N(B(R)) = N_{\mathbf{S}(4)}(A_2) \wr \mathbf{S}(t_2) \times (D_8) \wr \mathbf{S}(t_2') \leq N$, it follows that $N(K(R))$ is given as (2B) and $K(R)$ is radical in N. A proof similar to above shows that $N_N(R) \leq L$.

Suppose $D = D(2)'$, so that $N = D_8 \wr \mathbf{S}(m_2')$. A proof similar to above shows that each component of R is an element of Ψ' and $m_R(A_{\mathbf{c}}) \notin \{2,4\}$ for all $A_{\mathbf{c}} \in \Psi^*$. It follows that R is radical in G and $B(R) = D$. If $m_2' \notin \{2,4\}$, then $K(R) = B(R)$. If $m_2' = 2$ or 4, then $K(R) = R \in \Delta(2)$. This proves (2B). $\square$

Given sequences $\mathbf{c} = (c_1, \dots, c_\ell)$ and $\mathbf{z} = (z_1, \dots, z_v)$ of non-negative integers, let $Q_{\mathbf{c},\mathbf{z}}$ be the wreath product $X \wr A_{\mathbf{z}}$ in $\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})$, where $X = A_{\mathbf{c}}$ or $Q_{\mathbf{c}}$. If $X = A_{\mathbf{c}}$, then $Q_{\mathbf{c},\mathbf{z}} = A_{\mathbf{w}}$ and $N_{\mathbf{S}(2^{|\mathbf{w}|})}(Q_{\mathbf{c},\mathbf{z}})/Q_{\mathbf{c},\mathbf{z}}$ is given by (2.1) with some obvious modifications, where $\mathbf{w} = (c_1, \cdots, c_\ell, z_1, \cdots, z_v)$. Suppose $Q_{\mathbf{c},\mathbf{z}} = X \wr A_{\mathbf{z}}$ with $X = Q_{\mathbf{c}}$. Let $d = |\mathbf{c}|$ and M the underlying set of X. Then we may suppose $A_d \trianglelefteq X$ and $[M, X] = M$. Let $X_1, X_2, \cdots, X_{2^{|\mathbf{z}|}}$ be copies of X, and let $U_1, U_2, \cdots, U_{2^{|\mathbf{z}|}}$ be disjoint underlying sets of $X_1, X_2, \cdots, X_{2^{|\mathbf{z}|}}$. Then $U = U_1 \cup U_2 \cup \cdots \cup U_{2^{|\mathbf{z}|}}$ can be taken as the underlying set of $X \wr A_{\mathbf{z}}$, and $(\prod_{i=1}^{2^{|\mathbf{z}|}} X_i) \rtimes A_{\mathbf{z}} = X \wr A_{\mathbf{z}}$. Let W_i be a normal subgroup of X_i isomorphic to A_d. Then $[U_i, W_i] = U_i$ and $W = \prod_{i=1}^{2^{|\mathbf{z}|}} W_i$ is a normal abelian subgroup of $X \wr A_{\mathbf{z}}$, so that $W \leq A(Q_{\mathbf{c},\mathbf{z}})$. If A is a normal abelian subgroup of X, then $(A)^{2^{|\mathbf{z}|}}$ is a normal abelian subgroup of $Q_{\mathbf{c},\mathbf{z}}$. It follows that $(A)^{2^{|\mathbf{z}|}} \leq A(Q_{\mathbf{c},\mathbf{z}})$ and $\prod_{i=1}^{2^{|\mathbf{z}|}} A(X_i) \leq A(Q_{\mathbf{c},\mathbf{z}})$. Since $Q_{\mathbf{c}}$ is nonabelian, it follows by [2, (2A)] that each normal abelian subgroup of $Q_{\mathbf{c},\mathbf{z}}$ is a subgroup of $\prod_{i=1}^{2^{|\mathbf{z}|}} X_i$. Thus $A(Q_{\mathbf{c},\mathbf{z}}) \leq \prod_{i=1}^{2^{|\mathbf{z}|}} A(X_i)$, so that $A(Q_{\mathbf{c},\mathbf{z}}) = \prod_{i=1}^{2^{|\mathbf{z}|}} A(X_i)$ and $U_1, U_2, \cdots, U_{2^{|\mathbf{z}|}}$ are the orbits of $A(Q_{\mathbf{c},\mathbf{z}})$ in U. Since $N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(Q_{\mathbf{c},\mathbf{z}})$ normalizes $A(Q_{\mathbf{c},\mathbf{z}})$, $N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(Q_{\mathbf{c},\mathbf{z}})$ permutes $U_1, U_2, \cdots, U_{2^{|\mathbf{z}|}}$ among themselves, so that

$$N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(Q_{\mathbf{c}} \wr A_{\mathbf{z}}) = N_{\mathbf{S}(2^{|\mathbf{c}|})}(Q_{\mathbf{c}}) \otimes N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}}). \tag{2.5}$$

In particular, $N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(Q_{\mathbf{c},\mathbf{z}})$ normalizes the subgroup $\prod_{i=1}^{2^{|\mathbf{z}|}} X_i = (Q_{\mathbf{c}})^{2^{|\mathbf{z}|}}$ of $Q_{\mathbf{c},\mathbf{z}}$. We claim that

$$N_{N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(W)}(Q_{\mathbf{c},\mathbf{z}}) \simeq N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(Q_{\mathbf{c}}) \otimes N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}}), \tag{2.6}$$

where $W = \prod_{i=1}^{2^{|\mathbf{z}|}} W_i$ is a normal abelian subgroup of $Q_{\mathbf{c},\mathbf{z}}$ such that each W_i is a normal subgroup of X_i isomorphic to $A_{|\mathbf{c}|}$. Indeed, let

$$N = N_{N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(W)}(Q_{\mathbf{c},\mathbf{z}}), \qquad H = N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(Q_{\mathbf{c}}) \otimes N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}}).$$

If $g \in N$, then g normalizes $Q_{\mathbf{c},\mathbf{z}}$, so that by (2.5) $g = \operatorname{diag}\{g_1, g_2, \cdots, g_{2^{|\mathbf{z}|}}\}\sigma$, where $g_i \in N_{\mathbf{S}(M)}(Q_{\mathbf{c}})$ and $\sigma \in N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}})$. Since $W_i \leq X_i$ and g normalizes W, it follows that g_i normalizes W_i and $g \in H$. Conversely, if $g \in H$, then $g = \operatorname{diag}\{g_1, g_2, \cdots, g_{2^{|\mathbf{z}|}}\}\sigma$, where $\sigma \in N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}})$ and $g_i \in N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(Q_{\mathbf{c}})$. Thus g normalizes $Q_{\mathbf{c},\mathbf{z}}$ and $g \in N$, so that $H = N$.

Let $R = X \wr A_{\mathbf{z}}$ be a subgroup of $\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})$, where $X = A_{\mathbf{c}}$ or $Q_{\mathbf{c}}$. If $R = A_{\mathbf{c}} \wr A_{\mathbf{z}}$, then set $QB(R) = B(R)$; If $R = Q_{\mathbf{c}} \wr A_{\mathbf{w}}$, then set $QB(R) = (Q_{\mathbf{c}})^{2^{|\mathbf{w}|}}$ and $B(R) = (A_{|\mathbf{c}|})^{2^{|\mathbf{w}|}}$. By (2.5)

$$N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(Q_{\mathbf{c},\mathbf{z}}) \leq N_{\mathbf{S}(2^{|\mathbf{c}|+|\mathbf{z}|})}(QB(Q_{\mathbf{c},\mathbf{z}})).$$

(2C). *Let $G = \mathbf{S}(n) = \mathbf{S}(V)$, and let Q decompose as (2.2) with $Q = B(Q)$ or $Q = K(Q)$. Suppose R a radical subgroup of $N(Q)$. Then there exists a corresponding decomposition*

$$\begin{aligned} V &= M_0 \cup M_1 \cup \cdots \cup M_v, \\ R &= R_0 \times R_1 \times \cdots \times R_v \end{aligned} \tag{2.7}$$

such that $R_0 = \langle 1_{M_0} \rangle$ and $R_i = Q_{\mathbf{c},\mathbf{z}} \leq \mathbf{S}(M_i)$ for $i \geq 1$.

Proof. By (2B) and the remark before (2B), we may suppose $Q = \prod_{d \geq 3}(A_d)^{m_d}$ and

$$N = N(Q) = \prod_{d \geq 3} N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(m_d).$$

By [6, Lemma (2.2)], $R = \prod_{d\geq 3} R_d$, where R_d is a radical subgroup of $N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(m_d)$ for all $d \geq 3$. By induction, we may suppose N acts transitively on V, so that $Q = (A_d)^{m_d}$. Thus $R = Z_1 \times Z_2 \times \cdots \times Z_m$ and each $Z_i = X \wr Y$ for some subgroup $Y = A_{\mathbf{z}}$ of $\mathbf{S}(m_d)$ and a radical subgroup X of $N_{\mathbf{S}(2^d)}(A_d)$. By (2A) (a), $X \in \{A_d, Q_{\mathbf{c}}\}$, where $\mathbf{c}$ is a composition of d. So $Z_i = Q_{\mathbf{c},\mathbf{z}}$ and this proves (2C). $\square$

Suppose R has a decomposition (2.7). Define $QB(R) = R_0 \times \prod_{i=1}^{v} QB(R_i)$ and $B(R) = R_0 \times \prod_{i=1}^{v} B(R_i)$.

(2D). *Let R be a subgroup of $G = \mathbf{S}(n) = \mathbf{S}(V)$ such that R decomposes as (2.7). Given sequences $\mathbf{c} = (c_1, c_2, \cdots, c_\ell)$, $\mathbf{z} = (z_1, z_2, \cdots, z_u)$, and*

$\mathbf{w} = (w_1, w_2, \cdots, w_m)$ *of positive integers, let* $M(\mathbf{c}) = \cup_i M_i$, $R(\mathbf{c}) = \prod_i R_i$, $M(\mathbf{w}, \mathbf{z}) = \cup_j M_j$, *and* $R(\mathbf{w}, \mathbf{z}) = \prod_j R_j$, *where* i *and* j *run over the indices such that* $R_i = A_\mathbf{c}$ *and* $R_j = Q_\mathbf{w} \wr A_\mathbf{z}$, *respectively. Then*

$$N(R) = N_G(R) = \mathbf{S}(M_0) \times \prod_{\mathbf{c}} N_{\mathbf{S}(M(\mathbf{c}))}(R(\mathbf{c})) \times \prod_{\mathbf{w},\mathbf{z}} N_{\mathbf{S}(M(\mathbf{w},\mathbf{z}))}(R(\mathbf{w}, \mathbf{z})).$$

Moreover,

$$N_{\mathbf{S}(M(\mathbf{c}))}(R(\mathbf{c})) = N_{\mathbf{S}(M_\mathbf{c})}(A_\mathbf{c}) \wr \mathbf{S}(t_\mathbf{c}),$$
$$N_{\mathbf{S}(M(\mathbf{w},\mathbf{z}))}(M(\mathbf{w}, \mathbf{z})) = N_{\mathbf{S}(M_{\mathbf{w},\mathbf{c}})}(Q_\mathbf{w} \wr A_\mathbf{z}) \wr \mathbf{S}(t_{\mathbf{w},\mathbf{z}}),$$

where $M_\mathbf{c}$ *and* $M_{\mathbf{w},\mathbf{z}}$ *are the underlying sets of* $A_\mathbf{c}$ *and* $Q_\mathbf{w} \wr A_\mathbf{z}$, *respectively, and* $t_\mathbf{c}$, $t_{\mathbf{w},\mathbf{z}}$ *are the numbers of components* $A_\mathbf{c}$ *and* $Q_\mathbf{w} \wr A_\mathbf{z}$ *in* $R(\mathbf{c})$ *and* $R(\mathbf{w}, \mathbf{z})$, *respectively. In particular, if* $D = QB(R)$, *then* $N(R) \leq N(D)$.

Proof. Let $D_i = QB(R_i)$, so that $D = R_0 \times \prod_{i=1}^{v} D_i$, where v is given by the decomposition (2.7). If M is an R-orbit with $|M| \geq 2$, then $M = M_i$ for some $i \geq 1$ and $R_i = \{g \in R : gy = y \text{ for all } y \in V \backslash M_i\}$. Thus $N(R)$ acts as a permutation group on the set of pairs (M_i, R_i). Suppose a component R_i is conjugate to a component R_j, where $1 \leq i, j \leq v$. Then $|M_i| = |M_j|$, so that $\mathbf{S}(M_i)$ is conjugate to $\mathbf{S}(M_j)$ in G. If $R_i = A_\mathbf{c}$, then R_i is radical in $\mathbf{S}(M_i)$, so is R_j in $\mathbf{S}(M_j)$. Thus $R_j = A_{\mathbf{c}'}$ for some sequence $\mathbf{c}'$ of non-negative integers. Since $|M_i| = |M_j|$, it follows that $|\mathbf{c}| = |\mathbf{c}'|$ and so $\mathbf{c} = \mathbf{c}'$ as shown in the proof of [1, (2B)]. In particular, D_i is conjugate to D_j. If $R_i = Q_\mathbf{w} \wr A_\mathbf{z}$, then by the remark of (2A), R_i is non-radical in $\mathbf{S}(M_i)$, so is R_j in $\mathbf{S}(M_j)$. Thus $R_j = Q_{\mathbf{w}'} \wr A_{\mathbf{z}'}$ for some sequences $\mathbf{w}'$ and $\mathbf{z}'$ of non-negative integers. Moreover, $D_i = (Q_\mathbf{w})^{2^{|\mathbf{z}|}}$ and $D_j = (Q_{\mathbf{w}'})^{2^{|\mathbf{z}'|}}$.

As shown in the proof of (2.5), an $A(R_i)$-orbit of M_i has $2^{|\mathbf{w}|}$ elements and it is a underlying set of a factor $Q_\mathbf{w}$ of D_i. Since $A(R_i)$ is conjugate to $A(R_j)$, it follows that $|\mathbf{w}| = |\mathbf{w}'|$, so that $|\mathbf{z}| = |\mathbf{z}'|$. Moreover, R_i induces a permutation group $A_\mathbf{z}$ on the set of $A(R_i)$-orbits and R_j induces a permutation group $A_{\mathbf{z}'}$ on the set of $A(R_j)$-orbits. Thus $\mathbf{z} = \mathbf{z}'$ by [1, (2B)]. Let $W = \prod_{k=1}^{2^{|\mathbf{z}|}} W_k$ be a normal subgroup of D_i such that $W_k \simeq A_{|\mathbf{w}|}$. Then W is a normal abelian subgroup of R_i and the underlying set U_k of W_k is an $A(R_i)$-orbit of M_i. Suppose $\sigma \in N(R)$ such that $\sigma(M_i) = M_j$ and $R_i^\sigma = R_j$. Then $\mathbf{S}(M_i)^\sigma = \mathbf{S}(M_j)$ and $A(R_i)^\sigma = A(R_j)$. Thus W^σ is a normal abelian subgroup of R_j, so that $W^\sigma \leq A(R_j)$. The image of an $A(R_i)$-orbit of M_i is an $A(R_j)$-orbit of M_j. In particular, each $\sigma(U_k)$ is an $A(R_j)$-orbit and it is the underlying set of a factor of D_j. Thus $W^\sigma = \prod_{k=1}^{2^{|\mathbf{z}|}} L_k$ is a normal subgroup of R_j such that $L_k \simeq A_{|\mathbf{w}|}$. So σ induces an isomorphism between $N_{N_{\mathbf{S}(M_i)}(W)}(R_i)/R_i$ and $N_{N_{\mathbf{S}(M_j)}(W^\sigma)}(R_j)/R_j$. By (2.6),

$$N_{N_{\mathbf{S}(M_i)}(W)}(R_i)/R_i \simeq N_{N_{\mathbf{S}(2^{|\mathbf{w}|})}(A_{|\mathbf{w}|})}(Q_\mathbf{w})/Q_\mathbf{w} \times N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_\mathbf{z})/A_\mathbf{z}$$

$$N_{N_{\mathbf{S}(M_j)}(W^\sigma)}(R_j)/R_j \simeq N_{N_{\mathbf{S}(2^{|\mathbf{w}'|})}(A_{|\mathbf{w}'|})}(Q_{\mathbf{w}'})/Q_{\mathbf{w}'} \times N_{\mathbf{S}(2^{|\mathbf{z}|})}(A_{\mathbf{z}})/A_{\mathbf{z}}.$$

It follows that $\mathbf{w} = \mathbf{w}'$ as $|\mathbf{w}'| = |\mathbf{w}|$. In particular, D_i is conjugate to D_j. The remaining assertions of (2D) now follows easily. $\square$

Suppose $R = \prod_{i=1}^{v} R_i$ is a subgroup of G with a decomposition (2.7). We define

$$QK(R) = \prod_i QB(R_i) \times \prod_j R_j,$$

where i runs over the indices such that either $R_i \notin \Delta^+$ or $R_i = S_d \wr A_{\mathbf{c}} \in \Delta^+$ but $m_{QB(R)}(S_d) \notin \{2,4\}$, and j runs over the indices such that $R_j = S_d \wr A_{\mathbf{c}} \in \Delta^+$ and $m_{QB(R)}(S_d) \in \{2,4\}$. If R and W are subgroups given by (2C) and they are G-conjugate, then $QK(R)$ and $QK(W)$ are also G-conjugate. Since P_2 is radical in $N(P_1)$, it follows that $P_2 = QK(P_2)$. Next, we study the structure of P_i for $i \geq 3$.

Let $G = \mathbf{S}(n) = \mathbf{S}(V)$ and let

$$H = \prod_{X \in \Delta^+} (N_X)^{\alpha_X} \times \prod_{\mathbf{c} \in \Omega} N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(X_{\mathbf{c}}) \wr \mathbf{S}(t_{\mathbf{c}}) \tag{2.8}$$

be a subgroup of G, where $N_X = N_{\mathbf{S}(U_X)}(X)$, α_X and $t_{\mathbf{c}}$ are non-negative integers, $X_{\mathbf{c}} \in \{A_{|\mathbf{c}|}, Q_{\mathbf{c}}, D_8\}$ and $\Omega = \Omega(H) = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$ is a subset of sequences $\mathbf{w}_i$ of non-negative integers. (It may happen that $\mathbf{w}_i = \mathbf{w}_j$ for $i \neq j$). In addition, let $H^+ = \prod_{X \in \Delta^+} (N_X)^{\alpha_X}$, $H_{\mathbf{c}} = N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(X_{\mathbf{c}}) \wr \mathbf{S}(t_{\mathbf{c}})$ and $H^* = \prod_{\mathbf{c} \in \Omega} H_{\mathbf{c}}$.

(2E). *Suppose W is a radical subgroup of H. Then $W = W^+ \times W^*$ such that $W^+ = \prod_{Y \in \Delta^+} Y^{\beta_Y}$ and $W^* = \prod_{\mathbf{c} \in \Omega} W_{\mathbf{c}}$, where Y and $W_{\mathbf{c}}$ are radical subgroups of N_X and $H_{\mathbf{c}}$, respectively and β_Y is a non-negative integer.*

(a) *Each $W_{\mathbf{c}}$ has a decomposition* (2.7)*, and if $|\mathbf{c}| \in \{0,1,2\}$, then $W_{\mathbf{c}}$ is a radical subgroup of $\mathbf{S}(2^{|\mathbf{c}|+t_{\mathbf{c}}})$. Thus W has a deomposition* (2.7).

(b) *Let $QK_H(W) = W^+ \times \prod_{|\mathbf{c}|=0,1,2} K(W_{\mathbf{c}}) \times \prod_{|\mathbf{c}| \geq 3} QK(W_{\mathbf{c}})$, where $\mathbf{c}$ runs over Ω. In addition, let $Q = QK_H(W)$ and $L = N_H(Q)$. Then Q is radical in H and $N_H(W) \leq N_H(Q)$. In particular, $O_2(H) \leq Q$ and $QK_H(Q) = Q$. Moreover, $L = L^+ \times L^*$ such that $L^+ = \prod_{Y \in \Delta^+} (N_Y)^{\delta_Y}$ and*

$$L^* = \prod_{\mathbf{w} \in \Omega(L)} N_{N_{\mathbf{S}(2^{|\mathbf{w}|})}(A_{|\mathbf{w}|})}(Y_{\mathbf{w}}) \wr \mathbf{S}(t_{\mathbf{w}}),$$

where $Y_{\mathbf{w}} \in \{A_{|\mathbf{w}|}, Q_{\mathbf{w}}, D_8\}$, and δ_Y and $t_{\mathbf{w}}$ are non-negative integers. In particular, L has a decomposition (2.8).

(c) *Let R be a radical subgroup of $L = N_H(Q)$. If $QK_L(R) = Q$, then R is radical in H and $N_H(R) = N_L(R)$.*

Proof. The decomposition of W follows by [6, Lemma 2.2]. We now prove (a) and (b). If Y is a radical subgroup of N_X and $X \in \Delta(d)$ for some $d \geq 1$, then $N_X = X$ or $S_d \wr \mathbf{S}(4)$, so $Y \in \{X, S_d \wr A_1 \wr A_1\} \subseteq \Delta^+$. Thus $W^+ = \prod_{Y \in \Delta^+} (Y)^{\beta_Y}$ for some integers β_Y. If $|\mathbf{c}| \in \{0,1,2\}$, then by (2B), $W_\mathbf{c}$ is radical in $\mathbf{S}(2^{|\mathbf{c}|+t_\mathbf{c}})$, so that $K(W_\mathbf{c})$ is a radical subgroup of both $\mathbf{S}(2^{|\mathbf{c}|+t_\mathbf{c}})$ and $H_\mathbf{c}$. In particular, $W_\mathbf{c}$ has a decomposition (2.2). The normalizer $N_{H_\mathbf{c}}(K(W_\mathbf{c}))$ is given by [1, (2B)] or (2B).

Suppose $d = |\mathbf{c}| \geq 3$. Then $W_\mathbf{c} = W_1 \times \ldots \times W_m$ such that $W_i = Z_\mathbf{w} \wr A_\mathbf{z}$, where $Z_\mathbf{w}$ is a radical subgroup of $N_{N_{\mathbf{S}(2^{|\mathbf{c}|})}(A_{|\mathbf{c}|})}(X_\mathbf{c})$. By (2A) (b), $Z_\mathbf{w} \in \{X_\mathbf{c}, Q_\mathbf{w}\}$ with $|\mathbf{w}| = |\mathbf{c}|$, and so $W_\mathbf{c}$ has a decomposition (2.7) and $Q = QK(W_\mathbf{c})$ is well-defined. By induction, we may suppose $\Omega = \{\mathbf{c}\}$ and $d = |\mathbf{c}|$. Thus $W = W_\mathbf{c}$ and $H = H_\mathbf{c}$. Suppose $m_{QB(W)}(S_d) \in \{2,4\}$. Since W is radical in H, it follows that $m_W(S_d) \notin \{2,4\}$. If $m_{QB(W)}(S_d) = 2$, then $m_W(S_d \wr A_1) = 1$. If $m_{QB(W)}(S_d) = 4$, then $m_W(E) = 1$ for one $E \in \{S_d \wr A_2, S_d \wr A_1 \wr A_1\}$. It follows that

$$N_H(Q) = \prod_{Z \in \Delta(d)} (N_Z)^{\gamma_Z} \times \prod_{\mathbf{w}} N_{N_{\mathbf{S}(2^{|\mathbf{w}|})}(A_{|\mathbf{w}|})}(Z_\mathbf{w}) \wr \mathbf{S}(t_\mathbf{w}),$$

where $t_\mathbf{w}$ is an integer and $\gamma_Z = 0,1$. If $Z_\mathbf{w}$ is not a Sylow 2-subgroup of $N_{\mathbf{S}(2^{|\mathbf{w}|})}(A_{|\mathbf{w}|})$, then $Z_\mathbf{w}$ is not a self-normalizer. If $Z_\mathbf{w}$ is a Sylow 2-subgroup of $N_{\mathbf{S}(2^{|\mathbf{w}|})}(A_{|\mathbf{w}|})$, then $t_\mathbf{w} \notin \{2,4\}$. It follows that $Q = O_2(N_H(Q))$ and so Q is radical in H. The rest of the proof of (b) is straightforward.

(c) In the notation above, $Q = \prod_{Y \in \Delta^+} Y^{\beta_Y} \times \prod_{\mathbf{c} \in \Omega} Q(\mathbf{c})$, where Y and $Q(\mathbf{c})$ are radical subgroups of N_X and $H_\mathbf{c}$, respectively. In addition, $Q(\mathbf{c}) = \prod_{Z \in \Delta^+} (Z)^{\gamma_Z} \times \prod_\mathbf{w} (Z_\mathbf{w})^{t_\mathbf{w}}$ for some $\gamma_Z = 0,1$. Since R is radical in L, it follows that $R = \prod_E (E)^{\epsilon_E} \times \prod_\mathbf{c} R_\mathbf{c}$, where E and $R_\mathbf{c}$ are radical subgroups of N_Y and $L_\mathbf{c} = N_{H_\mathbf{c}}(Q(\mathbf{c}))$, respectively. But $QK_L(R) = Q$, so $E = Y$ and $\epsilon_E = \beta_Y$. By induction, we may suppose $\Omega = \{\mathbf{c}\}$ and $Q = Q(\mathbf{c})$

If $|\mathbf{c}| = 0$, then $H = \mathbf{S}(t_\mathbf{c})$, $Q = Q^+ \times Q^*$ with $Q = K(Q)$ and R is given by (2C), where $Q^+ = \prod_{Z \in \Delta} Z^{\gamma_Z}$ with $\gamma_Z = 0,1$ and $Q^* = (D_8)^{m_2'} \times \prod_{d \geq 0} (A_d)^{m_d}$. Thus $L = L^+ \times L^*$ and $R = R^+ \times R^*$, where $L^+ = \prod_{Z \in \Delta} (N_Z)^{\gamma_Z}$, $L^* = D_8 \wr \mathbf{S}(m_2') \times \prod_{d \geq 0} N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(m_d)$, $R^+ = \prod_{E \in \Delta} (E)^{\epsilon_E}$ and $R^* = R_2' \times \prod_{d \geq 0} R_d$. So E is a radical subgroup of N_Z, R_2' is a radical subgroup of $D_8 \wr \mathbf{S}(m_2')$ and R_d is a radical subgroup of $N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(m_d)$. Since $QK_L(R) = Q$, it follows that $E = Z$ and $\epsilon_E = \gamma_Z$, so that $R^+ = Q^+$. By (2B), R_d and R_2' are radical in $\mathbf{S}(2^{d+m_d})$ and $\mathbf{S}(2^{2+m_2'})$, respectively, where $d = 1,2$. By definition,

$$Q = QK_L(R) = Q^+ \times K(R_2') \times \prod_{d=0,1,2} K(R_d) \times \prod_{d \geq 3} QK(R_d).$$

Thus $R_0 = (A_0)^{m_0}$, $K(R_1) = (A_1)^{m_1}$, $K(R_2') = (D_8)^{m_2'}$, $K(R_2) = (A_2)^{m_2}$, and $m_i, m_2' \notin \{2,4\}$, since Q is radical in H. By definition, $K(R_2') = B(R_2')$ and $K(R_d) = B(R_d)$ for $d = 1,2$. Similarly, since $QK(R_d) = (A_d)^{m_d}$, it follows that

R_d has a decomposition (2.2) and $QK(R_d) = QB(R_d) = B(R_d)$ for $d \geq 3$. If $\gamma_Z = 1$ for some $Z \in \Delta$, then $Z = A_1 \wr A_2$ or $Z \in \Delta(2)$. In the former case $m_{B(Q)}(A_1) = 4$, since $Q = QK_H(Q) = K(Q)$. So $m_1 = 0$ and $R_1 = 1$. In the latter cases $m_{B(Q)}(D_8) \in \{2, 4\}$, so that $m_2' = 0$ and $R_2' = 1$. In particular, $m_{R^*}(Z) = 0$. Since R is radical in L, it follows that R is radical in G and $QK_L(R) = K(R) = Q$. Thus $N_H(R) \leq N_H(Q) = L$ and $N_L(R) = N_H(R)$.

If $|\mathbf{c}| = 1$, then $H = A_1 \wr \mathbf{S}(t_\mathbf{c})$ and $Q = Q^+ \times Q^*$, where $Q^+ = \prod_{Z\in\Delta}(Z)^{\gamma_Z}$ with $\gamma_Z = 0, 1$ and $Q^* = (A_1)^{t_1} \times (D_8)^{t_2}$. If $|\mathbf{c}| = 2$, then $H = N_{\mathbf{S}(4)}(X_\mathbf{c}) \wr \mathbf{S}(t_\mathbf{c})$ and $Q = Z^{\gamma_Z} \times Q^*$, where $Z \in \Delta(2)$, $\gamma_Z = 0, 1$ and $Q^* = (A_2)^{t_2} \times (D_8)^{t_2'}$ or $(D_8)^{t_2'}$ according as $X_\mathbf{c} = A_2$ or D_8. The same proof as above shows that $K(R) = Q$, $N_H(R) = N_L(R)$ and R is radical in H.

If $|\mathbf{c}| = d \geq 3$, then $H = N_{N_{\mathbf{S}(2^d)}(A_d)}(X_\mathbf{c}) \wr \mathbf{S}(t_\mathbf{c})$ and $Q = Q^+ \times Q^*$, where $Q^+ = \prod_{Z\in\Delta(d)}(Z)^{\gamma_Z}$ with $\gamma_Z = 0, 1$ and $Q^* = \prod_\mathbf{w}(Z_\mathbf{w})^{t_\mathbf{w}}$ with $Z_\mathbf{w} = A_d$ or $Q_\mathbf{w}$. It is clear that $t_\mathbf{w} = m_Q(Z_\mathbf{w})$ and $|\mathbf{w}| = d$. Thus $L = L^+ \times L^*$ and $R = R^+ \times R^*$, where $L^+ = \prod_{Z\in\Delta(d)}(N_Z)^{\gamma_Z}$, $L^* = \prod_\mathbf{w} N_{N_{\mathbf{S}(2^d)}(A_d)}(Z_\mathbf{w}) \wr \mathbf{S}(t_\mathbf{w})$, and R^+ and R^* are radical in L^+ and L^*, respectively. Since $QK_L(R) = Q$, it follows that $R^+ = Q^+$. Let $R^* = \prod_\mathbf{w} R_\mathbf{w}$, where $R_\mathbf{w}$ is a radical subgroup of $L_\mathbf{w} = N_{N_{\mathbf{S}(2^d)}(A_d)}(Z_\mathbf{w}) \wr \mathbf{S}(t_\mathbf{w})$. Then $QK_L(R) = R^+ \times \prod_\mathbf{w} QK(R_\mathbf{w})$, so that $QK(R_\mathbf{w}) = QB(R_\mathbf{w}) = (Z_\mathbf{w})^{t_\mathbf{w}}$. In particular, each component of $R_\mathbf{w}$ has the form $Z_\mathbf{w} \wr A_\mathbf{z}$. Thus $QK_L(R) = R^+ \times QB(R^*)$. If $m_{Q^+}(Z) \neq 0$, then $Z = S_d \wr A_\mathbf{z}$ for some $A_\mathbf{z}$ and $m_{QB(Q)}(S_d) = m_{QB(Z)}(S_d) \in \{2, 4\}$, since $QK_H(Q) = QK(Q) = Q$. So $m_{Q^*}(S_d) = 0$ and $Z_\mathbf{w} \neq S_d$. It follows that $QK_L(R) = QK(R) = Q$, so that $N(R) \leq N(Q)$. Thus $N_H(R) = N_L(R)$ and R is radical in H. □

REMARK. In the notation of (2E), suppose $O_2(H) = \prod_{X\in\Delta^+}(X)^{\alpha_X} \times \prod_{\mathbf{c}\in\Omega}(X_\mathbf{c})^{t_\mathbf{c}}$. Then N_X, α_X, $H_\mathbf{c}$ and $t_\mathbf{c}$ are determined uniquely by H. In particular, $QK_H(R)$ is independent of the choice of decompositions of H. Indeed, the underlying set U of $H_\mathbf{c}$ and U_X are H-orbits of V, and $N_X = \{g \in H : gy = y \text{ for all } y \in V \backslash U_X\}$ and $H_\mathbf{c} = \{g \in H : gy = y \text{ for all } y \in V \backslash U\}$. Thus $H_\mathbf{c}$ and N_X are determined by H. In addition, $X = O_2(N_X) \in \Delta^+$, $(X_\mathbf{c})^{t_\mathbf{c}} = O_2(H_\mathbf{c}) \notin \Delta^+$ and $\alpha_X = m_{O_2(H)}(X)$. So they are determined by H. This proves the remark.

If $C \in \mathcal{QR}$ is the chain given by (2.3), then either $w = 0$ or $P_i = QK_{N(C_{i-1})}(P_i)$ for all $1 \leq i \leq w$. We also need the following lemma.

(2F). *Let $G = \mathbf{S}(n) = \mathbf{S}(V)$ and let $C \in \mathcal{QR}$ be the chain given by* (2.3) *with $w \geq 1$. In addition, let R be a radical subgroup of $N(C)$. Then R has a decomposition* (2.7). *If $D = QK_{N(C)}(R)$, then D is radical in $N(C)$, $P_w \trianglelefteq D$, and $N_{N(C)}(R) \leq N_{N(C)}(D)$. In addition, if $P_w \neq D$ and*

$$C' : P_0 < P_1 < \cdots < P_w < D,$$

then $C' \in \mathcal{QR}$, R is radical in $N(C')$, and $N_{N(C')}(R) = N_{N(C)}(R)$. If $P_w = D$, then R is radical in $N(C_{w-1})$ and $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$.

Proof. Since P_1 is radical in G and $QK_G(P_1) = K(P_1) = P_1$, it follows that $N(C_1) = N(P_1)$ has a decomposition (2.8) and P_2 is radical in $N(C_1)$ with $QK_{N(C_1)}(P_2) = P_2$. By (2E) (b), $N(C_2) = N_{N(C_1)}(P_2)$ has a decomposition (2.8) and by induction, $N(C)$ has a decomposition (2.8). Thus R decomposes as (2.7) and $D = QK_{N(C)}(R)$ is well-defined. By (2E) (b) again, D is radical in $N(C)$ and moreover, $N_{N(C)}(R) \leq N_{N(C)}(D)$. Thus $N_{N(C)}(R) \leq N(C')$ and $N_{N(C)}(R) = N_{N(C')}(R)$. If $P_w = D$, then apply (2E) (c) to $H = N(C_{w-1})$ and $Q = P_w$. Thus R is radical in $N(C_{w-1})$ and $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$. This proves (2F). □

We can now prove the main result of this section.

(2G). *Let $G = \mathbf{S}(n) = \mathbf{S}(V)$ with $O_2(G) = \{1_V\}$, and let B be a positive defect 2-block of G and u an integer. Then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, u) = \sum_{C \in \mathcal{QR}/G} (-1)^{|C|} \mathrm{k}(N(C), B, u)$$

where $\mathcal{QR}/G$ is a set of representatives for the G-orbits in $\mathcal{QR}$.

Proof. It suffices to show that

$$\sum_{C \in \mathcal{M}/G} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0, \tag{2.9}$$

where $\mathcal{M} = \mathcal{R} \backslash \mathcal{QR}$. Suppose $C \in \mathcal{M}$ is given by (1.1). Then $C_0 \in \mathcal{QR}$ and $C = C_w \notin \mathcal{QR}$, so that there must be some minimal $m = m(C) \in \{0, 1, \ldots, w-1\}$ such that $C_m \in \mathcal{QR}$ and $C_{m+1} \notin \mathcal{QR}$. Since P_{m+1} is radical in $N(C_m)$, P_{m+1} has a decomposition (2.7). We can apply (2F) to C_m. If $D = QK_{N(C_m)}(P_{m+1})$, then $D \neq P_{m+1}$, D is radical in $N(C_m)$ and $N_{N(C_m)}(P_{m+1}) \leq N_{N(C_m)}(D)$, so that $P_m \trianglelefteq D$. Moreover, if $P_m = D$, then P_{m+1} is radical in $N(C_{m-1})$ and $N_{N(C_{m-1})}(P_{m+1}) = N_{N(C_m)}(P_{m+1})$. Define

$$\varphi(C) : \begin{cases} 1 < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_w & \text{if } P_m = D, \\ 1 < P_1 < \ldots < P_m < D < P_{m+1} < \ldots < P_w & \text{if } P_m < D. \end{cases}$$

Then $\varphi(C) \in \mathcal{M}$ and $N(C) = N(\varphi(C))$. Moreover, $\varphi(\varphi(C)) = C$ and $|\varphi(C)| = |C| \pm 1$. Thus φ is a bijection from $\mathcal{M}$ to itself. This implies (2.9). □

3. More reductions and the proof of the conjecture

In this section we shall follow the notation of Sections 1 and 2. Let $\mathcal{QR}^0$ be the G-invariant subfamily of $\mathcal{QR}$ consisting of chains C given by (2.3) such that $m_{P_i}(S_d \wr D_8) = 0$ for all $d \geq 1$ except when $d = 1$, in which case if $m_{P_i}(A_1 \wr D_8) \neq 0$, then $(A_2)^2$ is a component of some P_k for $k < i$, and $[V, (A_2)^2] = [V, D_8 \wr A_1]$ and $(A_2)^2 \trianglelefteq D_8 \wr A_1 = A_1 \wr D_8$. If $\mathcal{QR}^1 = \mathcal{QR} \backslash \mathcal{QR}^0$, then

$$\mathcal{QR} = \mathcal{QR}^0 \cup \mathcal{QR}^1 \qquad \text{(disjoint)}.$$

We shall first reduce Dade's conjecture to the family $\mathcal{QR}^0$.

Fix integer $d \geq 1$. Let $X \in \{S_d \wr A_2, S_d \wr D_8\}$, and let $X \times Q$ be a subgroup of $G = \mathbf{S}(n) = \mathbf{S}(V)$ with a decomposition (2.7). If $U_X = [V, X]$ and $U_Q = V \backslash U_X$, then $V = U_X \cup U_Q$. Suppose $C(0) \in \mathcal{QR}^0$ is a fixed radical chain with $|C(0)| = s$. Let $\mathcal{QR}(C(0), X \times Q)$ be the subfamily of $\mathcal{QR}$ consisting of all chains C given by (2.3) such that its s-th subchain C_s is $C(0)$ and its $(s+1)$-st subgroup P_{s+1} is $X \times Q$ up to conjugacy in G. Since $X \in \Delta^+$ and $N(C_{s+1})$ has a decomposition (2.8), it follows that $N(C_{s+1}) = N_X \times N(s+1)$, where $N_X = N_{\mathbf{S}(U_X)}(X)$ and $N(s+1) \leq \mathbf{S}(U_Q)$. Let P_t be the t-th subgroup of C with $t \geq s+1$. Then $P_t = Y(t) \times Z(t)$, where $Y(t) \in \{X, S_d \wr A_1 \wr A_1\}$ and $Z(t) \leq N(s+1)$. Note that $\mathcal{QR}(C(0), S_d \wr D_8 \times Q) \subseteq \mathcal{QR}^1$ whenever $d \geq 2$.

Let $\mathcal{M} = \mathcal{M}(C(0), S_d \wr A_2 \times Q)$ be the subset of $\mathcal{QR}(C(0), S_d \wr A_2 \times Q)$ consisting of all chains C such that $Y(t) = S_d \wr D_8$, that is, $P_t = S_d \wr D_8 \times Z(t)$ (up to conjugacy) for some $t \geq s+2$. In particular, $\mathcal{M}(C(0), S_d \wr A_2 \times Q) \subseteq \mathcal{QR}^1$ and

$$\mathcal{QR}^1 = \bigcup_{C(0), S_d \wr A_2 \times Q} \mathcal{S}(C(0), S_d \wr A_2 \times Q) \qquad \text{(disjoint)},$$

where $\mathcal{S}(C(0), S_d \wr A_2 \times Q) = \mathcal{M}(C(0), S_d \wr A_2 \times Q) \cup (\mathcal{QR}(C(0), S_d \wr D_8 \times Q) \cap \mathcal{QR}^1)$, $C(0)$ runs over $\mathcal{QR}^0$ and $S_d \wr A_2 \times Q$ runs over subgroup of G with a decomposition (2.7).

For $C \in \mathcal{M}$, denote by $m = m(C)$ the smallest integer such that $P_m = S_d \wr D_8 \times Z(m)$, so that $Q \leq Z(m)$. Let $\mathcal{M}_0$ and $\mathcal{M}_+$ be the subsets of $\mathcal{M}$ consisting of all chains C such that $Z(m) = Q$ and $Z(m) \neq Q$, respectively.

(3A). *In the notation above, suppose $\mathcal{S} = \mathcal{S}(C(0), S_d \wr A_2 \times Q)$. Then*

$$\sum_{C \in \mathcal{S}/G} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0 \tag{3.1}$$

for all 2-blocks B and integers $u \geq 0$.

Proof. Set $X = S_d \wr A_2$. Suppose $C \in \mathcal{M}_+$ is given by (2.3). Then $m = m(C) \geq s+2$ and $P_{m-1} = X \times Z(m-1)$. So $Z(m-1) \leq Z(m)$ and $N(C_t) = N_X \times N(t)$

for $s+1 \leq t \leq m-1$. In particular, $Z(m-1)$ is a radical subgroup of $N(m-2)$ and moreover, if $m = s+2$, then $Q = Z(m-1) < Z(m)$. Define a map φ such that

$$\varphi(C): \begin{cases} 1 < P_1 < \ldots < P_{m-2} < P_m < \ldots < P_w & \text{if } Z(m-1) = Z(m), \\ 1 < P_1 < \ldots < P_{m-1} & \\ \qquad\qquad < X \times Z(m) < P_m < \ldots < P_w & \text{if } Z(m-1) < Z(m). \end{cases}$$

Then $\varphi(C) \in \mathcal{M}_+$, $N(C) = N(\varphi(C))$, $\varphi(\varphi(C)) = C$ and $|\varphi(C)| = |C| \pm 1$. Thus

$$\sum_{C \in (\mathcal{M}_+)/G} (-1)^{|C|} \operatorname{k}(N(C), B, u) = 0.$$

Suppose $C \in \mathcal{M}_0$ is given by (2.3). Since X and $S_d \wr D_8$ are the only two radical subgroups of $N_X = S_d \wr \mathbf{S}(4)$ up to conjugacy containing $(S_d)^4$, it follows that $m(C) = s+2$, that is, $P_{s+2} = S_d \wr D_8 \times Q$. Thus

$$g(C): 1 < P_1 < \ldots < P_s < P_{s+2} < \ldots < P_w$$

is a chain of $\mathcal{QR}(C(0), S_d \wr D_8 \times Q) \cap \mathcal{QR}^1$ and $N(C) = N(g(C))$. Conversely, suppose

$$C': 1 < P'_1 < \ldots < P'_s < P'_{s+1} < \ldots < P'_{w'}$$

is a chain of $\mathcal{QR}(C(0), S_d \wr D_8 \times Q) \cap \mathcal{QR}^1$, then $P'_{s+1} = S_d \wr D_8 \times Q$ and

$$h(C'): 1 < P'_1 < \ldots < P'_s < X \times Q < P'_{s+1} < \ldots < P'_{w'}$$

is a chain of $\mathcal{M}_0$. It is clear that $g(h(C')) = C'$, $h(g(C)) = C$ and $|g(C)| = |C| - 1$. Thus

$$\sum_{C \in (\mathcal{M}_0 \cup (\mathcal{QR}(C(0), S_d \wr D_8 \times Q) \cap \mathcal{QR}^1))/G} (-1)^{|C|} \operatorname{k}(N(C), B, u) = 0.$$

This proves (3A). □

It follows by (3A) that Dade's conjecture can be reduced to the family $\mathcal{QR}^0$. Let $Z = (A_1)^{m_1}$ be a radical subgroup of $\mathbf{S}(2^{m_1}) = \mathbf{S}(U_Z)$, and $W \neq Z$ a radical subgroup of $N_Z = N_{\mathbf{S}(U_Z)}(Z)$ such that $K(W) = W$. As shown in the proof of (2B)

$$W \in \Phi = \{D_8 \wr A_2 \times A_1 \wr A_2, D_8 \wr A_2 \times (A_1)^{t_1}, A_1 \wr A_2 \times (D_8)^{t_2}, (A_1)^{t_1} \times (D_8)^{t_2}\},$$

where $t_i \notin \{2, 4\}$. If $W = A_1 \wr A_2 \times (D_8)^{t_2}$, then $t_2 \neq 0$, since Z is radical in $\mathbf{S}(U_Z)$. Similarly, if $W = (A_1)^{t_1} \times (D_8)^{t_2}$ and $t_1 = 0$, then $t_2 \neq 1$. Thus $N_{N_{\mathbf{S}(U_Z)}(Z)}(W) =$

$N_{\mathbf{S}(U_Z)}(W)$. Let $\mathcal{QR}^0(C(0), Z\times Q) = \mathcal{QR}(C(0), Z\times Q)\cap\mathcal{QR}^0$, and let $\mathcal{M}(C(0), Z\times Q, W)$ be the subset of $\mathcal{QR}^0(C(0), Z\times Q)$ consisting of chians C given by (2.3) such that $P_m = W\times Z(m)$ (up to conjugacy) for some $m\geq s+2$ and $P_t = Z\times Z(t)$ for $s+1\leq t\leq m$, where $\mathcal{QR}(C(0), Z\times Q)$ is defined as in (3A) and $Z(t)\leq \mathbf{S}(U_Q)$.

(3B). *In the notation above, let $\mathcal{S} = \mathcal{M}(C(0), Z\times Q, W)\cup\mathcal{QR}^0(C(0), W\times Q)$, where $W\in\Phi$. Then* (3.1) *holds for $\mathcal{S}$.*

Proof. Replacing $X = S_d \wr A_2$ by Z, $S_d\wr D_8$ by W and some obvious modifications in the proof of (3A), we have (3B). □

Let $\mathcal{QR}^+$ be the complement of $\bigcup_{C(0),Z,W,Q}(\mathcal{M}(C(0), Z\times Q, W)\cup\mathcal{QR}^0(C(0), W\times Q))$ in $\mathcal{QR}^0$, where $C(0)$ runs over $\mathcal{QR}^0$, $Z = (A_1)^{m_1}$ with $m_1\notin\{2,4\}$, W runs over Φ, and Q runs over subgroups of $\mathbf{S}(U_Q)$ with a decomposition (2.7). It follows by (3A) and (3B) that

$$\sum_{C\in\mathcal{QR}/G}(-1)^{|C|}\,\mathrm{k}(N(C), B, u) = \sum_{C\in\mathcal{QR}^+/G}(-1)^{|C|}\,\mathrm{k}(N(C), B, u).$$

Let $D = P_1$ be the first non-trivial subgroup of $C\in\mathcal{QR}^+$. Then $D = K(D)$ and $D = D^+\times D^*$ decomposes as (2.4). Now

$$\Delta = \{A_1\wr A_2, D_8\wr A_1, D_8\wr A_2, D_8\wr D_8\}.$$

By (3A), $m_D(D_8\wr D_8) = 0$. Since $D_8\wr A_2\in\Phi$, it follows by (3B) that $m_D(D_8\wr A_2) = 0$. Similarly, $m_D(D_8) = 0, 1$ and $m_Q(A_1\wr A_2) = 0, 1$. If $m_D(D_8\wr A_1)\neq 0$, then D is not the first non-trivial subgroup of any chain in $\mathcal{QR}^0$. Suppose $m_D(A_1\wr A_2)\neq 0$. Since $A_1\wr A_2\times(D_8)^{t_2}\in\Phi$ for $t_2\geq 1$, it follows by (3B) that $m_D(D_8) = 0$. But $K(D) = D$, so $m_{B(D)}(A_1) = 4$ and $m_{B(D^+)}(A_1) = 0$. Similarly, if $m_D(D_8)\neq 0$, then $m_D(A_1\wr A_2) = m_D(A_1) = 0$. Thus

$$D = D(0)\times X\times\prod_{d\geq 2} D(d), \tag{3.2}$$

where $D(d) = (A_d)^{m_d}$ for $d\neq 1$ and $X\leq\mathbf{S}(2^{m_1})$ such that

$$X = \begin{cases} D_8 & \text{if } m_1 = 2,\\ A_1\wr A_2 & \text{if } m_1 = 4,\\ (A_1)^{m_1} & \text{if } m_1\notin\{2,4\}.\end{cases}$$

For simplicity, we denote by $D(1)$ the subgroup X. Thus $N(D) = \prod_{d\geq 0} N(D)_d$ such that $N(D)_d = N_{\mathbf{S}(2^d)}(A_d)\wr\mathbf{S}(m_d)$.

Suppose $Q = P_2$ is the second subgroup of C. Then Q is a radical subgroup of $N(D)$, so that $Q = \prod_{d\geq 0} Q_d$, where Q_d is a radical subgroup of $N(D)_d$. Thus Q_0 is of form (3.2). It follows by (3B) that $Q_1 = D(1)$. In general, if $W = P_i$ is the i-th subgroup of C for $i \geq 1$, then $W = \prod_{d\geq 0} W_d$ with $W_1 = D(1)$ and $W_d \leq N(D)_d$ for all $d \geq 1$. By (3B) again, if $m_W(D_8 \wr A_2) \neq 0$, then there is some $1 \leq k \leq i-1$ such that $(A_2)^4$ is a component of P_k, $[V, (A_2)^4] = [V, (D_8) \wr A_2]$ and $(A_2)^4 \trianglelefteq (D_8) \wr A_2$.

Let $\Delta' = \{D_8, A_1 \wr A_2\}$ and let

$$P = \prod_{X\in\Delta'} (X)^{\alpha_X} \times \prod_{d=0}^{s} (A_d)^{m_d}, \tag{3.3}$$

be a subgroup of G, where α_X and m_d are non-negative integers. Set $P^+ = \prod_{X\in\Delta'} (X)^{\alpha_X}$ and $P^* = \prod_{d=0}^{s} (A_d)^{m_d}$. Let U_X be the underlying set of $X \in \Delta'$ such that $U_X = [U_X, X]$, and $N_X = N_{\mathbf{S}(U_X)}(X)$.

Suppose $C \in \mathcal{QR}^+$ is given by (2.3). Denote by $C_V(C)$ the fixed-point set $C_V(P_w)$ of the final subgroup P_w of C. Let $\ell = \ell(C)$ be the largest integer such that P_ℓ has a decomposition (3.3), and let $\mathcal{QR}^+(P)$ be the subset of $\mathcal{QR}^+$ consisting of all chains C given by (2.3) such that $P_\ell = P$. Then

$$\mathcal{QR}^+ = \bigcup_P \mathcal{QR}^+(P) \qquad \text{(disjoint)},$$

where P runs over subgroups of G with a decomposition (3.3). Thus

$$N(C_\ell) \simeq \mathbf{S}(V(0)) \times \prod_{X\in\Delta'} (N_X)^{\alpha_X} \times \prod_{d=1}^{s} \left(\prod_{j=1}^{h_d} N_{\mathbf{S}(2^d)}(A_d) \wr \mathbf{S}(\lambda_{d,j}) \right), \tag{3.4}$$

where $(\lambda_{d,1}, \ldots, \lambda_{d,h_d})$ is a partition of m_d and $V(0) = C_V(P)$.

Fix partitions $\lambda_d = (\lambda_{d,1}, \ldots, \lambda_{d,h_d})$ of m_d, and set $\lambda = (\lambda_1, \ldots, \lambda_s)$. Let $\mathcal{QR}^+(P, \lambda)$ be the subset of $\mathcal{QR}^+(P)$ consisting of all chains C such that $N(C_\ell)$ is given by (3.4). Then

$$\mathcal{QR}^+(P) = \bigcup_\lambda \mathcal{QR}^+(P, \lambda) \qquad \text{(disjoint)},$$

where λ runs over all s-tuple partitions λ_d of m_d.

Suppose W is a G-conjugate of P. Then $W^g = P$ for some $g \in G$, and $C^g \in \mathcal{QR}^+(P)$ for each $C \in \mathcal{QR}^+(W)$. Thus a set of representatives for the $N(P)$-conjugacy classes of $\mathcal{QR}^+(P)$ can be regarded as a set of representatives for the G-conjugacy classes of the G-orbit containing $\mathcal{QR}^+(P)$. It is clear that $\mathcal{QR}^+(P)$ and $\mathcal{QR}^+(P, \lambda)$ both are $N(P)$-invariant.

Let $\mathcal{QR}'(P, \lambda) = \{C \in \mathcal{QR}^+(P, \lambda) : C_V(C) = C_V(P)\}$, and let $\mathcal{QR}''(P, \lambda)$ be the complement of $\mathcal{QR}'(P, \lambda)$ in $\mathcal{QR}^+(P, \lambda)$.

(3C). *In the notation above,*

$$\sum_{C\in\mathcal{QR}''(P,\lambda)/N(P)} (-1)^{|C|}\,\mathrm{k}(N(C),B,u)=0$$

for all 2*-blocks* B *and integers* $u\geq 0$.

Proof. Let $C: 1<P_1<\ldots<P_\ell=P<P_{\ell+1}<\ldots<P_w$ be a chain of $\mathcal{M}=\mathcal{QR}''(P,\lambda)$. Then $C_V(P_w)\neq C_V(P)$. Let $m=m(C)$ be the smallest integer such that $C_V(P_m)\neq C_V(P)=V(0)$. Then $\ell+1\leq m\leq w$.

Let $V(+)=[V,P]$ and $P(+)=P^+\times\prod_{d\geq 1}(A_d)^{m_d}$, where P^+ is defined after (3.3). Then $P=P(0)\times P(+)$ and $N(P)=\mathbf{S}(V(0))\times N(P)(+)$, where $P(0)=\langle 1_{V(0)}\rangle$ and $N(P)(+)=N_{\mathbf{S}(V(+))}(P(+))$. Thus $N(C_{m-1})=\mathbf{S}(V(0))\times N(C_{m-1})(+)$, where $N(C_{m-1})(+)\leq\mathbf{S}(V(+))$. So $W=P_m$ decomposes as $W=W_0\times W_+$, where W_0 is a radical subgroup of $\mathbf{S}(V(0))$ and $W_+\leq\mathbf{S}(V(+))$. In particular, W_0 is a non-trivial subgroup with a decomposition (3.3). By definition, P_m has no decompositions as that of (3.3), so that $m_{W_+}(Z)\neq 0$ for some $Z\in\{Q_{\mathbf{c}},S_d\wr A_1,S_d\wr A_2\}$, where $d\geq 2$ and $\mathbf{c}$ is a sequence of positive integers. Let $D=\langle 1_{V(0)}\rangle\times W_+$ and $R=P_{m-1}$. Then $D<P_m$, $R=R(0)\times R(+)\leq D$, and D is radical in $N(C_{m-1})$, where $R(0)=\langle 1_{V(0)}\rangle$ and $R(+)=O_2(N(C_{m-1})(+))$. If $R(+)=W_+$, then $m\geq\ell+2$ and $P_m=W_0\times W_+$ is radical in $N(C_{m-2})$. Let $\varphi(C)\in\mathcal{QR}^+$ such that

$$\varphi(C):\begin{cases}1<P_1<\ldots<P_{m-2}<P_m<\ldots<P_w & \text{if } P_{m-1}=D,\\ 1<P_1<\ldots<P_{m-1}<D<P_m<\ldots<P_w & \text{if } P_{m-1}<D.\end{cases}$$

Then $\varphi(C)\in\mathcal{M}$, $N(C)=N(\varphi(C))$, $|\varphi(C)|=|C|\pm 1$ and $\varphi(\varphi(C))=C$. Thus φ is a permutation of $\mathcal{M}$ and preserves $N(P)$-classes in $\mathcal{M}$. This implies (3C). □

Let $\mathcal{QR}'_1(P,\lambda)$ be the subset of $\mathcal{QR}'(P,\lambda)$ consisting of all the chains whose final subgroup is P. For any $C(0)\in\mathcal{QR}'_1(P,\lambda)$ with length $|C(0)|=\ell$, let $\mathcal{QR}'(C(0),\lambda)$ denote the subset of $\mathcal{QR}'(P,\lambda)$ consisting of all the chains C such that $C_\ell=C(0)$. Thus

$$\mathcal{QR}'(P,\lambda)=\bigcup_{C(0)\in\mathcal{QR}'_1(P,\lambda)}\mathcal{QR}'(C(0),\lambda)\qquad\text{(disjoint)}.$$

In addition, two chains $C(0)$ and $C(0)'$ of $\mathcal{QR}'_1(P,\lambda)$ are $N(P)$-conjugate if and only if $\mathcal{QR}'(C(0),\lambda)$ and $\mathcal{QR}'(C(0)',\lambda)$ are $N(P)$-conjugate.

Now we can prove the main result of this paper.

(3D). *Dade's ordinary conjecture holds for any positive defect* 2*-block of the symmetric groups* $\mathbf{S}(n)$ *with* $O_2(\mathbf{S}(n))=1$.

Proof. (1) First of all, we show that if $m_P(A_d) \neq 0$ for some $d \geq 2$, then

$$\sum_{C \in \mathcal{QR}'(C(0),\lambda)/N(C(0))} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0$$

for all 2-blocks B and integers $u \geq 0$.

Let $K = \prod_{d=2}^{s} \prod_{j=1}^{h_d} \mathrm{GL}(d,2)^{\lambda_{d,j}}$ and let $\mathcal{R}(K)$ be the set of all radical 2-chains of K. In addition, let $\mathcal{S} = \mathcal{S}(C(0), \lambda)$ be the set of all chains

$$C : 1 < P_1 < \ldots < P_\ell = P < P_{\ell+1} < \ldots < P_w$$

of G such that $C_\ell = C(0)$ and $C/P : P_\ell/P < P_{\ell+1}/P < \ldots < P_w/P$ is a chain of $\mathcal{R}(K)$. The map $\varphi : \mathcal{S}(C(0), \lambda) \to \mathcal{R}(K)$ given by $\varphi(C) = C/P$ is a bijection (see [6, (5.7)]).

The same proof as that after (5.7) of [6] shows that

$$\sum_{C \in \mathcal{S}(C(0),\lambda)/N(C(0))} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0.$$

It suffices to show that there exists a bijective map ψ from $\mathcal{QR}'(C(0), \lambda)$ to $\mathcal{S}(C(0), \lambda)$ such that $N(C) = N(\psi(C))$.

Let C be a chain of $\mathcal{QR}'(C(0), \lambda)$ given by (2.3) and let $N_i = N(C_i)$ for $0 \leq i \leq w$. If $D = P_t$ is the t-th subgroup of C, then

$$D = D(0) \times D(1) \times \prod_{d \geq 2} \left[(S_d \wr A_1)^{\alpha_d} \times (S_d \wr A_2)^{\beta_d} \times D(d) \right], \tag{3.5}$$

such that $D(0) = (A_0)^{m_0}$, $D(1) = (D_8)^{\alpha_1} \times (A_1 \wr A_2)^{\beta_1} \times (A_1)^{t_1}$ and $D(d) = \prod_{\mathbf{c}} (X_{\mathbf{c}})^{t_{\mathbf{c}}}$ with $|\mathbf{c}| = d$, where $X_{\mathbf{c}} \in \{A_d, S_2 = D_8, Q_{\mathbf{c}}\}$, and α_i, β_i, t_1 and $t_{\mathbf{c}}$ are non-negative integers. Let

$$\psi(D) = D(0) \times D(1) \times \prod_{d \geq 2} \left[(S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times D(d) \right]. \tag{3.6}$$

Equivalently, if $D = \prod_i D_i$ such that $D_i \in \Delta^+$ or $D_i \in \{A_d, D_8, Q_{\mathbf{c}}\}$, then $\psi(D) = \prod_k QB(D_k) \times (A_1 \wr A_2)^{\beta_1}$, where $\beta_1 = m_D(A_1 \wr A_2)$ and k runs over the indices such that $D_k \not\simeq A_1 \wr A_2$. Define

$$\psi(C) : 1 < \psi(P_1) < \psi(P_2) < \ldots < \psi(P_w).$$

Then $\psi(C_t) = \psi(C)_t$ for $1 \leq t \leq w$. We shall show that $\psi(C) \in \mathcal{S}$ and ψ is a bijection satisfying $N(C) = N(\psi(C))$. If $\alpha_d = \beta_d = 0$ for $d \geq 2$, then $\psi(D) = D$. In particular, $\psi(P_t) = P_t$, $\psi(C)_\ell = C_\ell = C(0)$ and $N_G(C_t) = N_G(\psi(C_t))$ for $1 \leq t \leq \ell$.

Suppose $\psi(C_t) \in \mathcal{S}$ for some $t \geq \ell$. Then N_t is of the form (2.8), and moreover, if $V(0) = C_V(P_t) = C_V(P)$, then

$$(3.7) \quad N_t = \mathbf{S}(V(0)) \times N_t(1) \times \prod_{d \geq 2} \left[(S_d \wr A_1)^{\alpha_d} \times (S_d \wr \mathbf{S}(4))^{\beta_d} \times N_t(d) \right],$$

where $N_t(1) = (D_8)^{\alpha_1} \times (A_1 \wr \mathbf{S}(4))^{\beta_1} \times A_1 \wr \mathbf{S}(t_1)$ and

$$N_t(d) = \prod_{|\mathbf{c}|=d} N_{N_{\mathbf{S}(2^d)}(A_d)}(X_{\mathbf{c}}) \wr \mathbf{S}(t_{\mathbf{c}}).$$

Since P_{t+1} is radical subgroup of N_t and $C \in \mathcal{QR}'(C(0), \lambda)$, it follows that

$$P_{t+1} = D(0) \times D(1) \times \prod_{d \geq 2} \left[(S_d \wr A_1)^{\alpha_d} \times (S_d \wr A_2)^{\beta_d} \times W(d) \right],$$

where $W(d) = \prod_{|\mathbf{c}|=d} W_{\mathbf{c}}$ such that $W_{\mathbf{c}}$ is radical in $H_{\mathbf{c}} = N_{N_{\mathbf{S}(2^d)}(A_d)}(X_{\mathbf{c}}) \wr \mathbf{S}(t_{\mathbf{c}})$. As shown in the proof of (2E) (c) $W_{\mathbf{c}} = \prod_{\mathbf{w}} (Y_{\mathbf{w}})^{m_{\mathbf{w}}} \times (Z)^{\gamma_Z}$, where the $\mathbf{w}$'s are sequences of positive integers such that $|\mathbf{w}| = |\mathbf{c}| = d$, $Z \in \{S_d \wr A_1, S_d \wr A_2\}$, $Y_{\mathbf{w}} \in \{A_d, D_8, Q_{\mathbf{w}}\}$ and $\gamma_Z = 0, 1$. Moreover, $X_{\mathbf{c}} \leq Y_{\mathbf{w}}$ and $(Z)^{\gamma_Z} = S_d \wr A_1$ or $S_d \wr A_2$ according as $m_{QB(W_{\mathbf{c}})}(S_d) = 2$ or 4. So $QB(W_{\mathbf{c}}) = \prod_{\mathbf{w}} (Y_{\mathbf{w}})^{m_{\mathbf{w}}} \times (S_d)^{\eta\gamma_Z}$ and $(X_{\mathbf{c}})^{t_{\mathbf{c}}} \leq QB(W_{\mathbf{c}})$, where $\eta = 2$ or 4 according as $(Z)^{\gamma_Z} = S_d \wr A_1$ or $S_d \wr A_2$. In particular, $N_{H_{\mathbf{c}}}(W_{\mathbf{c}}) = N_{H_{\mathbf{c}}}(QB(W_{\mathbf{c}}))$, and $QB(W_{\mathbf{c}})/(A_d)^{t_{\mathbf{c}}} = \prod_{\mathbf{w}} (Y_{\mathbf{w}}/A_d)^{m_{\mathbf{w}}} \times (S_d/A_d)^{\eta\gamma_Z}$ is a radical subgroup of $\mathrm{GL}(d,2)^{t_{\mathbf{c}}}$. By definition,

$$\psi(P_{t+1}) = D(0) \times D(1) \times \prod_{d \geq 2} \left[(S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times QB(W(d)) \right],$$

so that $N_{N_t}(\psi(P_{t+1})) = N_{t+1}$. By (3.6), $\psi(P_t) \trianglelefteq \psi(P_{t+1})$ and $\psi(P_{t+1})/P$ is a radical subgroup of K. Thus $\psi(C_{t+1})/P \in \mathcal{R}(K)$, and by induction, $\psi(C)/P \in \mathcal{R}(K)$, so that $\psi(C) \in \mathcal{S}$. Since $N_t = N(C_t) = N(\psi(C)_t)$ for $t \geq 1$, it follows that $P_t = O_2(N(\psi(C)_t))$, so that C is determined uniquely by $\psi(C)$. Thus ψ is a bijection if and only if it is onto.

Let $C' : 1 < P'_1 < \ldots < P'_w$ be a chian in $\mathcal{S}$, and let C be the chain of length w such that its t-th non-trivial subgroup P_t is $O_2(N(C'_t))$. Since $C'_\ell = C(0)$ is radical, it follows that $C_\ell = C(0)$, and so $\psi(C_t) = C'_t$ for $0 \leq t \leq \ell$. Suppose $\psi(C_t) = C'_t$ and $C_t \in \mathcal{QR}'(C(0), \lambda)$ for some $\ell \leq t \leq w$. Then $N_t = N(C_t) = N(C'_t)$ is given by (3.7). Since C_t is a radical chain and $P_t = O_2(N_t)$, it follows that $P_t = D$ is given by (3.5) and $P'_t = \psi(D)$ is given by (3.6). Since $P'_t \trianglelefteq P'_{t+1} \leq N_t$ and P'_{t+1}/P is radical in K, it follows that

$$P'_{t+1} = D(0) \times D(1) \times \prod_{d \geq 2} \left[(S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times T(d) \right]$$

such that $T(d) = \prod_{|\mathbf{c}|=d} T_{\mathbf{c}}$, where $T_{\mathbf{c}}/(A_d)^{t_{\mathbf{c}}}$ is a radical subgroup of $\mathrm{GL}(d,2)^{t_{\mathbf{c}}}$.

By (2A) (b), $T_{\mathbf{c}} = \prod_{\mathbf{w}} (Y_{\mathbf{w}})^{m_{\mathbf{w}}}$, where $|\mathbf{w}| = |\mathbf{c}| = d$ and $Y_{\mathbf{w}} \in \{A_d, D_8, Q_{\mathbf{w}}\}$. Thus

$$N_{H_{\mathbf{c}}}(T_{\mathbf{c}}) = \prod_{\mathbf{w}} N_{N_{\mathbf{S}(2^d)}(A_d)}(Y_{\mathbf{w}}) \wr \mathbf{S}(m_{\mathbf{w}}).$$

Since $Y_{\mathbf{w}}$ is self-normalizing if and only if $Y_{\mathbf{w}} = S_d$, it follows that $O_2(N_{H_{\mathbf{c}}}(T_{\mathbf{c}})) = T_{\mathbf{c}}$ except when $m_{T_{\mathbf{c}}}(S_d) \in \{2, 4\}$, in which case $O_2(N_{H_{\mathbf{c}}}(T_{\mathbf{c}})) = \prod_{Y_{\mathbf{w}} \neq S_d} (Y_{\mathbf{w}})^{m_{\mathbf{w}}} \times Z$, where $Z = S_d \wr A_1$ or $S_d \wr A_2$ according as $m_{T_{\mathbf{c}}}(S_d) = 2$ or 4. Thus $\psi(O_2(N_{H_{\mathbf{c}}}(T_{\mathbf{c}}))) = T_{\mathbf{c}}$ and $\psi(P_{t+1}) = P'_{t+1}$. By induction, $\psi(C) = C'$ and ψ is onto. Thus ψ is a bijection.

(2) In order to complete the proof, it suffices to consider chains $C \in \mathcal{QR}'(P, \lambda)$ such that

$$P = (A_0)^{m_0} \times (D_8)^{\alpha} \times (A_1 \wr A_2)^{\beta} \times (A_1)^{m_1}, \tag{3.8}$$

where α, β, m_0 and m_1 are non-negative integers. It follows by (3B) and (3C) that P is the final subgroup for each chain $C \in \mathcal{QR}'(P, \lambda)$. Let $\mathcal{QR}^*(G) = \cup_{P,\lambda} \mathcal{QR}'(P, \lambda)$, where P runs over subgroups of form (3.8) and λ runs over partitions of m_1. It suffices to show that

$$\sum_{C \in \mathcal{QR}^*(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0 \tag{3.9}$$

for all positive defect 2-blocks B and integers $u \geq 0$.

Now each subgroup of a chain $C \in \mathcal{QR}^*(G)$ is of the form (3.8). Let $\phi(P) = (A_0)^{m_0} \times (A_1)^{2\alpha} \times (A_1)^{4\beta} \times (A_1)^{m_1}$ and let

$$\phi(C) : 1 < \phi(P_1) < \phi(P_2) < \ldots < \phi(P_w)$$

for chain $C \in \mathcal{QR}^*(G)$ given by (2.3). In addition, let $\mathcal{S}(G) = \{\phi(C) : C \in \mathcal{QR}^*(G)\}$. A proof similar to that of (1) above shows that ϕ is a bijection between $\mathcal{QR}^*(G)$ and $\mathcal{S}(G)$, and $N(C) = N(\phi(C))$.

The same proof as that of [6, Proposition (6.1)] shows that

$$\sum_{C \in \mathcal{S}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, u) = 0,$$

which implies (3.9). This completes the proof. □

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