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SPATIAL-GRAPH ISOTOPY AND THE REARRANGEMENT THEOREM

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A 1-dimensional finite CW-complex is called a *graph*. The set of all (piecewise linear) embeddings $\Gamma : G \longrightarrow \mathbf{R}^3$ of G is denoted by $\mathcal{S}(G)$. In this paper, we will study spatial-graph isotopy and cobordism, equivalence relations on $\mathcal{S}(G)$ introduced by Taniyama [6], and obtain interaction between them. The subset of $\mathcal{S}(G)$ consisting of all elements isotopic to (resp. cobordant to) $\Gamma \in \mathcal{S}(G)$ is denoted by $[\Gamma]_{\text{isotopy}}$ (resp. by $[\Gamma]_{\text{cobor}}$), and called the *isotopy class* (resp. the *cobordism class*) of Γ. Here, we note that any isotopy between two embeddings $\Gamma, \Gamma' \in \mathcal{S}(G)$ is realized by a finite sequence of blowing-downs $\searrow$ and ups $\nearrow$. In Soma [5] and Inaba-Soma [2], we saw that it is useful for the study of spatial-graph isotopy to rearrange the order of blowing-ups and downs, and presented a rearrangement theorem valid for trivalent graphs, [5, Theorem 2], and that for connected graphs without cut vertices, [2, Theorem 3]. The following shows that such a rearrangement theorem holds for any graphs.

Theorem 1 (The Rearrangement Theorem on Spatial-Graph Isotopy). *For any graph G, let $\Gamma_1, \Gamma_2 : G \longrightarrow \mathbf{R}^3$ be embeddings isotopic to each other. Then, there exists an embedding $\Gamma_3 : G \longrightarrow \mathbf{R}^3$ and a sequence of blowing-downs followed by blowing-ups such that $\Gamma_1 \searrow \cdots \searrow \Gamma_3 \nearrow \cdots \nearrow \Gamma_2$.*

Our proof of Theorem 1 is based on arguments in [2]. However, for the completion of the proof, we must clear the hurdle which the author could not there.

An element $\Gamma^{\text{red}} \in \mathcal{S}(G)$ is said to be *isotopically reduced* if the ambient-isotopy type of Γ^{red} can not be changed by any blowing-down of Γ^{red}. We note that the isotopy class $[\Gamma]_{\text{isotopy}}$ of any $\Gamma \in \mathcal{S}(G)$ contains an isotopically reduced element, see [5, §3, Proposition 1]. Corollary 1 is proved by the argument quite similar to that in [5, Corollary 1] which was effective only for trivalent graphs.

Corollary 1. *Let $\Gamma_1, \Gamma_2 : G \longrightarrow \mathbf{R}^3$ be embeddings of any graph G. Suppose that Γ_i^{red} is any isotopically reduced element in $[\Gamma_i]_{\text{isotopy}}$ for $i = 1, 2$. Then, Γ_1 is isotopic to Γ_2 if and only if Γ_1^{red} is ambient isotopic to Γ_2^{red}.*

The following corollary is a restatement of Corollary 1.

Corollary 2. *For any embedding* $\Gamma : G \longrightarrow \mathbf{R}^3$ *of a graph* G*, the isotopy class* $[\Gamma]_{\text{isotopy}}$ *contains a unique isotopically reduced element up to ambient isotopy.*

This corollary suggests the other question whether the cobordism class $[\Gamma]_{\text{cobor}}$ contains an isotopically reduced element. A graph G is called a *generalized bouquet* if G contains a vertex v such that $G-\{v\}$ is acyclic. According to Taniyama [6, Theorem A], if G is a generalized bouquet, then any embedding $\Gamma : G \longrightarrow \mathbf{R}^3$ is isotopic to a planar embedding $\Gamma_0 : G \longrightarrow \mathbf{R}^2 \subset \mathbf{R}^3$, so the quotient set $\mathcal{S}(G)/\text{isotopy}$ consists of a single element. If the graph G is non-acyclic, then $\mathcal{S}(G)$ has infinitely many cobordism classes. However, except the unknotted class $[\Gamma_0]_{\text{cobor}}$, any other classes $[\Gamma]_{\text{cobor}}$ contain no isotopically reduced elements. For non-generalized-bouquet graphs, we have the following theorem in contrast to Corollary 2.

Theorem 2. *Suppose that* G *is any graph other than a generalized bouquet. Then, for any embedding* $\Gamma \in \mathcal{S}(G)$*, the cobordism class* $[\Gamma]_{\text{cobor}}$ *contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.*

Note that an embedding $\Gamma' \in \mathcal{S}(G)$ obtained by blowing-downs of Γ is, in general, not cobordant to Γ. Thus, the blowing-down method is not applicable to construct isotopically reduced elements in $[\Gamma]_{\text{cobor}}$. In §3, we will construct such embeddings by replacing mutually disjoint, trivial tangles $(B_1, B_1 \cap \Gamma(G)), \ldots, (B_m, B_m \cap \Gamma(G))$ in $(S^3, \Gamma(G))$ by certain simple tangles.

Corollary 3 follows immediately from Theorems 1 and 2.

Corollary 3. *For any graph* G*, let* $\varphi : \mathcal{S}(G) \longrightarrow \mathcal{S}(G)/\text{isotopy}$ *be the natural quotient map. If* G *is not a generalized bouquet, then for any element* $\Gamma \in \mathcal{S}(G)$*, the image* $\varphi([\Gamma]_{\text{cobor}})$ *is an infinite subset of* $\mathcal{S}(G)/\text{isotopy}$.

The referee suggested that it is not hard to prove the following proposition where the positions of isotopy and cobordism in Corollary 3 are exchanged.

Proposition 1. *For any graph* G*, let* $\psi : \mathcal{S}(G) \longrightarrow \mathcal{S}(G)/\text{cobor}$ *be the natural quotient map. If* G *is not acyclic, then for any element* $\Gamma \in \mathcal{S}(G)$*, the image* $\psi([\Gamma]_{\text{isotopy}})$ *is an infinite subset of* $\mathcal{S}(G)/\text{cobor}$.

1. Preliminaries

Let G be a graph, and I the closed interval $[0,1]$. Consider a pair of elements Γ, $\Gamma' \in \mathcal{S}(G)$ admitting a PL-embedding $\Phi : G \times I \longrightarrow \mathbf{R}^3 \times I$ such that, for some $0 < \varepsilon < 1/2$, $\Phi(x,t) = (\Gamma(x),t)$ if $(x,t) \in G \times [0,\varepsilon]$, $\Phi(x,t) = (\Gamma'(x),t)$ if $(x,t) \in G \times [1-\varepsilon,1]$, and $\Phi(G \times [\varepsilon, 1-\varepsilon]) \subset \mathbf{R}^3 \times [\varepsilon, 1-\varepsilon]$. We say that (i) Γ is *ambient isotopic* to Γ' if Φ is locally flat and level-preserving, (ii) Γ is *cobordant* to Γ' if Φ is locally flat, and (iii) Γ is *isotopic* to Γ' if Φ is level-preserving.

A graph H is a *star* of degree $n \in \mathbf{N}$ and centered at v if H is a tree consisting of n edges which have v as a common vertex. For a given 3-ball B in $\mathbf{R}^3$, we fix a point $v \in \mathrm{int}B$, called the *center* of B. For an element $\Gamma \in \mathcal{S}(G)$, the pair $(B, B \cap \Gamma(G))$ is called a *ball-star* pair if $B \cap \Gamma(G)$ is a star centered at v and with $\partial\varepsilon \subset \partial B \cup \{v\}$ for each edge ε of $B \cap \Gamma(G)$. When $\alpha = B \cap \Gamma(G)$ is a proper arc in B, (B, α) is regarded as a ball-star pair of degree two even if α contains no vertices of $\Gamma(G)$. A ball-star pair $(B, B \cap \Gamma(G))$ is *standard* if there exists a properly embedded disk D in B with $D \supset B \cap \Gamma(G)$. For an embedding $\Gamma : G \longrightarrow \mathbf{R}^3$ with a ball-star pair $(B, B \cap \Gamma(G))$, set $J = G - \Gamma^{-1}(\mathrm{int}B)$. Then, we say that $\Gamma' : G \longrightarrow \mathbf{R}^3$ is obtained from Γ by a *blowing-down* in B and denote it by $\Gamma \searrow_B \Gamma'$ (or shortly $\Gamma \searrow \Gamma'$) if Γ' is ambient isotopic to an embedding $\Gamma'' : G \longrightarrow \mathbf{R}^3$ such that $\Gamma''|_J = \Gamma|_J$ and $(B, B \cap \Gamma''(G))$ is a standard ball-star pair. Conversely, Γ is said to be obtained from Γ' by a *blowing-up* occurring in B and denote it by $\Gamma' \nearrow_B \Gamma$ (or $\Gamma' \nearrow \Gamma$). As was pointed out in [6, §2], for two elements $\Gamma, \Gamma' \in \mathcal{S}(G)$, Γ is isotopic to Γ' if and only if Γ' is obtained from Γ by a finite sequence of blowing-downs and ups. Consider double blowing-ups $\Gamma \nearrow_{B_1} \Gamma' \nearrow_{B_2} \Gamma''$ for $\Gamma \in \mathcal{S}(G)$. Since $(B_2, B_2 \cap \Gamma'(G))$ is a standard pair, one can shrink B_2 by an ambient isotopy of $\mathbf{R}^3$ fixing $\Gamma'(G)$ as a set so that either $B_1 \cap B_2 = \emptyset$ or $B_2 \subset \mathrm{int}B_1$. If $B_2 \subset \mathrm{int}B_1$, then the double blowing-ups can be replaced by a single blowing-up $\Gamma \nearrow_{B_1} \Gamma''$, see Fig. 3 in [5].

First of all, we will give the proof of Proposition 1.

Proof of Proposition 1. Any non-acyclic graph G contains a cycle l. For any embedding $\Gamma \in \mathcal{S}(G)$ and any $n \in \mathbf{N}$, let $\mathcal{B}_n = B_1 \cup \cdots \cup B_n$ be a disjoint union of 3-balls in $\mathbf{R}^3$ such that each $B_i \cap \Gamma(G)$ is an unknotted, proper arc in B_i with $\alpha_i = \Gamma^{-1}(B_i) \subset l$. Consider an embedding $\Gamma_n \in \mathcal{S}(G)$ such that each $\Gamma_n(\alpha_i)$ is a left-handed trefoil in B_i and $\Gamma_n|_{H_n} = \Gamma|_{H_n}$ for $H_n = G - \mathrm{int}(\alpha_1 \cup \cdots \cup \alpha_n)$. Since $\Gamma_n \searrow \cdots \searrow_{\mathcal{B}_n} \Gamma$, Γ_n is contained in $[\Gamma]_{\mathrm{isotopy}}$. Since $\mathrm{sign}(\Gamma_n(l)) = \mathrm{sign}(\Gamma(l)) + 2n$ and since the knot signature is well known to be a cobordism invariant, $\psi(\Gamma_n)$ $(n = 1, 2, \ldots)$ are mutually distinct points of $\mathcal{S}(G)/\mathrm{cobor}$. This completes the proof. □

We identify the 3-sphere S^3 with $\mathbf{R}^3 \cup \{\infty\}$. So, any element $\Gamma \in \mathcal{S}(G)$ can be regarded as an embedding of G into S^3. For any subset X of S^3, an *ambient isotopy* of (S^3, X) means an ambient isotopy of S^3 fixing X as a set.

2. Proof of the rearrangement theorem

Throughout this section, fix a graph G and a pair of blowing-up and down $\Gamma_1 \nearrow_B \Gamma_2 \searrow_C \Gamma_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are elements of $\mathcal{S}(G)$ and B, C are 3-balls with centers v_B, v_C. Note that isolated vertices and free edges of a graph G do not affect equivalence relations on $\mathcal{S}(G)$ such as ambient isotopy, isotopy and cobordism. Thus, we may always assume without loss of generality that G contains no isolated vertices and free edges, that is, the degree of each vertex of G is at least two. If necessary adding extra vertices to G, we may also assume that any cycle in G contains at least two vertices of G. In particular, G satisfies the condition $(**)$ in [2, §2].

It is easily seen that the following proposition implies Theorem 1.

Proposition 2. *With the notation as above, there exist embeddings* $\Gamma_2', \Gamma_3' \in \mathcal{S}(G)$ *and a sequence* $\Gamma_1 \searrow_{C'} \Gamma_2' \nearrow_{B'} \Gamma_3' \nearrow_{C''} \Gamma_3$, *where* B', C', C'' *are* 3*-balls with centers* v_B, v_C, v_C *respectively.*

Note that, in the case of $v_B = v_C$, the double blowing-ups $\Gamma_2' \nearrow_{B'} \Gamma_3' \nearrow_{C''} \Gamma_3$ in Proposition 2 are replaced by a single blowing up $\Gamma_2' \nearrow_{B''} \Gamma_3$. From now on, for any proper subset X of S^3, we set $X^\circ = X - X \cap \Gamma_2(G)$. By [2, Lemma 3], we may assume that each component of $\partial B^\circ \cap \partial C^\circ$ is a loop non-contractible both in ∂B° and ∂C° (even in the case where ∂B°, ∂C° are compressible in $S^3 - \Gamma_2(G)$). For each component R of $B \cap \partial C$, let W_R denote the closure in B of a component of $B - R$ disjoint from v_B. A closure W_R is said to be *innermost* among these closures if $\mathrm{int} W_R \cap \partial C = \emptyset$. According to [2, Lemma 2], if $F_R = W_R \cap \partial B$ is connected for an innermost closure W_R, then we have a sequence $\Gamma_1 \nearrow_B \Gamma_2' \searrow_{C'} \Gamma_3' \nearrow_{C'} \Gamma_3$ with $|\partial B \cap \partial C'| < |\partial B \cap \partial C|$, where $|Y|$ denotes the number of connected components of a compact set Y. In fact, when $v_B \neq v_C$, we showed in [2, Lemma 4] that, for any component R of $B \cap \partial C$, F_R is connected (even if W_R is not innermost), and hence Proposition 2 was proved inductively. So, it suffices to consider the case of $v_B = v_C = v$. Remark that, in this case, the result corresponding to [2, Lemma 4] does not hold in general. We will complete the proof of Proposition 2 by showing that either F_R is connected for at least one innermost W_R or each component of $S^3 - \mathrm{int}(B \cup C)$ is a 3-ball.

For unoriented loops l, l' in S^3 with $l \cap l' = \emptyset$, $\mathrm{lk}(l, l')$ is the absolute linking number of l and l' in S^3. For a loop l in the punctured surface $\partial(B \cup C)^\circ$, l^+ represents a loop in $S^3 - \Gamma_2(G) \cup B \cup C$ isotopic to l in $S^3 - \Gamma_2(G) \cup \mathrm{int}(B \cup C)$. Intuitively, l^+ is obtained by pushing l outside of $B \cup C$ slightly.

Lemma 1. *With the notation and assumptions as above, suppose that* X *is a connected component of* $S^3 - \mathrm{int}(B \cup C)$. *Then, one of the following* (i) *and* (ii) *holds.*

(i) *X is homeomorphic to a* 3*-ball.*

(ii) *There exists a simple proper arc α in $Q = X \cap \partial C$ connecting distinct components l, l' of ∂Q such that* $\mathrm{lk}(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1$, *where β_1, β_2 are simple arcs in B connecting the end points of α with v and satisfying $\beta_1 \cap \beta_2 = \{v\}$.*

Proof. We assume that the conclusion (ii) does not hold and will show then that the conclusion (i) holds. Let $Y_1, \ldots, Y_m$ be the components of Q. For each Y_i, there exist mutually disjoint disks $D_1^{(i)}, \ldots, D_{r_i}^{(i)}$ in ∂B such that $\partial \mathcal{D}^{(i)} \subset \partial Y_i$ and $X \cap \partial B \subset \mathcal{D}^{(i)}$, where $\mathcal{D}^{(i)}$ is the union $D_1^{(i)} \cup \cdots \cup D_{r_i}^{(i)}$. When $\partial Y_i \cap \mathrm{int} D_j^{(i)} \neq \emptyset$, consider a component l of $\partial Y_i \cap \mathrm{int} D_j^{(i)}$ which is not disconnected from $\partial D_j^{(i)}$ by any other components of $\partial Y_i \cap \mathrm{int} D_j^{(i)}$. Then the triad of l, $l' = \partial D_j^{(i)}$ and any simple arc α in Y_i connecting l with l' would satisfy (ii), a contradiction. Thus, we have $\partial Y_i \cap \mathrm{int}\mathcal{D}^{(i)} = \emptyset$. Then, the union $S_i = Y_i \cup \mathcal{D}^{(i)}$ is a 2-sphere bounding a 3-ball B_i in $S^3 - \mathrm{int} B$ with $B_i \supset X$. Our X coincides with the intersection $B_1 \cap \cdots \cap B_m$.

We set $W_i = S^3 - \mathrm{int}(B \cup B_i)$ and $Z_i = \partial W_i - \mathrm{int} Y_i$. Note that Z_i is a connected surface in ∂B homeomorphic to Y_i. For any distinct $i, j \in \{1, \ldots, m\}$, since $Y_i \subset X$ is disjoint from $\mathrm{int} W_j$, W_i is either contained in W_j or disjoint from W_j. If $W_i \subset W_j$, then X would meet $\mathrm{int} W_j$ non-trivially, a contradiction. It follows that $W_i \cap W_j = \emptyset$. Thus, the boundary $\partial X = (\partial B - Z_1 \cup \cdots \cup Z_m) \cup (Y_1 \cup \cdots \cup Y_m)$ is homeomorphic to the 2-sphere $\partial B = (\partial X - Y_1 \cup \cdots \cup Y_m) \cup (Z_1 \cup \cdots \cup Z_m)$. This shows that X is homeomorphic to a 3-ball. □

Proof of Proposition 2 (and Theorem 1). As was seen above, we may assume that $v_B = v_C = v$.

First, we consider the case where all components $X_1, \ldots, X_m$ of $N_0 = S^3 - \mathrm{int}(B \cup C)$ are 3-balls. Note that $N_0 \cap \Gamma_1(G) = N_0 \cap \Gamma_2(G) = N_0 \cap \Gamma_3(G)$ and the graph $(B \cup C) \cap \Gamma_i(G)$ is a star centered at v for $i = 1, 2, 3$. Take mutually disjoint, simple proper arcs $\alpha_1, \ldots, \alpha_{m-1}$ in $B \cup C$ such that each α_j connects ∂X_j with ∂X_{j+1} and

$$(\alpha_1 \cup \cdots \cup \alpha_{m-1}) \cap (\Gamma_1(G) \cup \Gamma_2(G) \cup \Gamma_3(G)) = \emptyset.$$

The union N_1 of a small regular neighborhood of $\alpha_1 \cup \cdots \cup \alpha_{m-1}$ in $B \cup C$ and N_0 is a 3-ball with $N_1 \cap \Gamma_1(G) = N_1 \cap \Gamma_2(G) = N_1 \cap \Gamma_3(G)$ and, for the 3-ball $\widehat{B} = S^3 - \mathrm{int} N_1$ and $i = 1, 2, 3$, $(\widehat{B}, \widehat{B} \cap \Gamma_i(G)) = (\widehat{B}, (B \cup C) \cap \Gamma_i(G))$ is a ball-star pair. This shows that there exists a (common) embedding $\Gamma_2' \in \mathcal{S}(G)$ admitting blowing-downs $\Gamma_1 \searrow_{\widehat{B}} \Gamma_2'$, $\Gamma_2 \searrow_{\widehat{B}} \Gamma_2'$ and $\Gamma_3 \searrow_{\widehat{B}} \Gamma_2'$. Thus, we have the pair of blowing-down and up $\Gamma_1 \searrow_{\widehat{B}} \Gamma_2' \nearrow_{\widehat{B}} \Gamma_3$ from Γ_1 to Γ_3.

Next, we suppose that $S^3 - \mathrm{int}(B \cup C)$ contains a component X not homeomorphic to a 3-ball. By Lemma 1, there exists a simple proper arc α in $Q = X \cap \partial C$, simple arcs β_1, β_2 in B as in Lemma 1 (ii) and a component l of ∂Q with $\mathrm{lk}(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1$. Consider the 2-fold branched covering $p : S^3 \longrightarrow S^3$ branched over l^+, and set $p^{-1}(v) = \{\widetilde{v}_1, \widetilde{v}_2\}$. The preimage $p^{-1}(B)$ (resp. $p^{-1}(C)$)

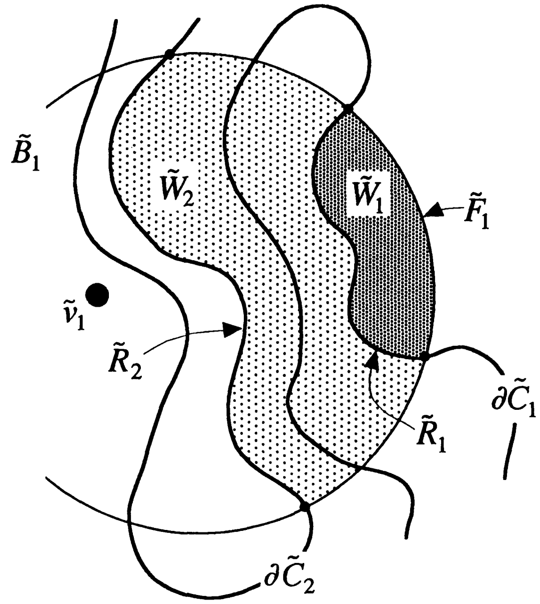

Fig. 1.

is a union of mutually disjoint 3-balls $\widetilde{B}_1$, $\widetilde{B}_2$ with $\widetilde{v}_1 \in \widetilde{B}_1$, $\widetilde{v}_2 \in \widetilde{B}_2$ (resp. $\widetilde{C}_1$, $\widetilde{C}_2$ with $\widetilde{v}_1 \in \widetilde{C}_1$, $\widetilde{v}_2 \in \widetilde{C}_2$). Let $\widetilde{\alpha}$ be the lift of α contained in $\partial\widetilde{C}_2$. Since $\widetilde{\alpha}$ connects $\widetilde{B}_1$ with $\widetilde{B}_2$, $\widetilde{C}_2$ meets both $\widetilde{B}_1$ and $\widetilde{B}_2$. Note that $\widetilde{\Gamma} = p^{-1}(\Gamma_2(G))$ is a spatial graph, and $(\widetilde{B}_j, \widetilde{B}_j \cap \widetilde{\Gamma})$, $(\widetilde{C}_j, \widetilde{C}_j \cap \widetilde{\Gamma})$ are ball-star pairs for $j = 1, 2$. Since $\widetilde{v}_1 \neq \widetilde{v}_2$, Lemma 4 in [2] implies that, for any component $\widetilde{R}_2$ of $\widetilde{B}_1 \cap \partial\widetilde{C}_2$, $\widetilde{F}_2 = \widetilde{W_2} \cap \partial\widetilde{B}_1$ is connected and hence homeomorphic to $\widetilde{R}_2$, where $\widetilde{W_2}$ is the closure in $\widetilde{B}_1$ of a component of $\widetilde{B}_1 - \widetilde{R}_2$ disjoint from $\widetilde{v}_1$. When $\widetilde{W_2} \cap \partial\widetilde{C}_1 = \emptyset$ for a closure $\widetilde{W_2}$ with $\mathrm{int}\widetilde{W_2} \cap \partial\widetilde{C}_2 = \emptyset$, we set $W = p(\widetilde{W_2})$. Otherwise, consider the closure $\widetilde{W_1}$ in $\widetilde{W_2}$ of a component $\widetilde{W_2} - \partial\widetilde{C}_1$ with $\mathrm{int}\widetilde{W_1} \cap \partial\widetilde{C}_1 = \emptyset$ and $\widetilde{W_1} \cap \widetilde{R}_2 = \emptyset$. Note that $\widetilde{R}_1 = \widetilde{W_1} \cap \partial\widetilde{C}_1$ is a connected surface, see Fig. 1. If $\widetilde{C}_2 \cap \mathrm{int}\widetilde{W_2} \neq \emptyset$, then $\widetilde{C}_2$ would contain $\widetilde{W_2} \supset \widetilde{R}_1$, and hence $\widetilde{C}_1 \cap \widetilde{C}_2 \neq \emptyset$, a contradiction. This implies that $\widetilde{C}_2' = \widetilde{C}_2 \cup \widetilde{W_2}$ is a 3-ball. If $\widetilde{W_2} \cap \widetilde{\Gamma} \neq \emptyset$, then any edge e of the star $\widetilde{\Gamma} \cap \widetilde{B}_1$ connecting a point of $\widetilde{F}_2 \cap \widetilde{\Gamma}$ with $\widetilde{v}_1$ would meet $\widetilde{R}_2$, so e would tend toward $\widetilde{v}_2$. This contradicts that $\widetilde{v}_1 \neq \widetilde{v}_2$. It follows that $(\widetilde{C}_2', \widetilde{C}_2' \cap \widetilde{\Gamma}) = (\widetilde{C}_2', \widetilde{C}_2 \cap \widetilde{\Gamma})$ is a ball-star pair centered at $\widetilde{v}_2$. By applying Lemma 4 in [2] to the pair of the 3-balls $\widetilde{C}_1$, $\widetilde{C}_2'$ with distinct centers, one can show that $\widetilde{F}_1 = \widetilde{W_1} \cap \partial\widetilde{C}_2' = \widetilde{W_1} \cap \partial\widetilde{B}_1$ is connected. Then, we set $W = p(\widetilde{W_1})$. In either case, W is a compact 3-manifold in B bounded by the union of the connected surfaces $R = W \cap \partial C$, $F = W \cap \partial B$ and satisfying $\mathrm{int}W \cap \partial C = \emptyset$. Then, by [2, Lemma 2], we have a sequence $\Gamma_1 \nearrow_B \Gamma_2^{(1)} \searrow_{C^{(1)}} \Gamma_3^{(1)} \nearrow_{C^{(1)}} \Gamma_3$ with $|\partial B \cap \partial C^{(1)}| < |\partial B \cap \partial C|$. Repeating the same process finitely many times, we have a sequence $\Gamma_1 \nearrow_{B'} \Gamma_2^{(r)} \searrow_{C'} \Gamma_3^{(r)} \nearrow_{C''} \Gamma_3$ such that each component of

$S^3 - \mathrm{int}(B' \cup C')$ is a 3-ball. As was seen in the previous case, one can then exchange the blowing-up and down of $\Gamma_1 \nearrow_{B'} \Gamma_2^{(r)} \searrow_{C'} \Gamma_3^{(r)}$ and obtain our desired sequence. □

3. Construction of isotopically reduced embeddings

In this section, we will prove that, if a graph G is not a generalized bouquet, then for any embedding $\Gamma \in \mathcal{S}(G)$, the cobordism class $[\Gamma]_{\mathrm{cobor}}$ contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.

Our proof here is based on arguments in Soma [3] and [4], where the author constructed simple links cobordant to given links in S^3 and closed 3-manifolds by using certain simple tangles. Here, a (2-string) *tangle* $(B, t_1 \cup t_2)$ is a pair of a 3-ball B and a disjoint union $t_1 \cup t_2$ of two simple proper arcs in B. A tangle $(B, t_1 \cup t_2)$ is *trivial* if there exists a properly embedded disk in B containing $t_1 \cup t_2$. A tangle $(B, t_1 \cup t_2)$ is *simple* if $\partial B - \partial t_1 \cup \partial t_2$ is incompressible in $B - t_1 \cup t_2$ and if $B - t_1 \cup t_2$ contains no incompressible tori. We refer to [3, §2] for examples of simple tangles. In particular, a *clasp tangle* $(B, t_1 \cup t_2)$ as in Fig. 2 is simple. Let A be a properly embedded annulus in the complement $B - t_1 \cup t_2$ of a simple tangle such that ∂A bounds an annulus A' in $\partial B - \partial t_1 \cup \partial t_2$. If A is incompressible in $B - t_1 \cup t_2$, then any compressing disk Δ for the torus $T = A \cup A'$ is contained in the compact 3-manifold V in $B - t_1 \cup t_2$ bounded by T. Since V is a solid torus and each component of ∂A is contractible in $B - \mathrm{int}V$, A is parallel to A' in $V \subset B - t_1 \cup t_2$. Here, we say that a compact surface F properly embedded in a 3-manifold X is *parallel* in X to a surface F' in ∂X if there exists an embedding $h : F \times I \longrightarrow X$ with $h(F \times \{0\}) = F$ and $h(F \times \{1\} \cup \partial F \times I) = F'$.

A compact, connected surface homeomorphic to a closed region in $\mathbf{R}^2$ with n boundary components is called an *n-ply connected disk.* In particular, a doubly connected disk is an annulus.

Lemma 2. *Let $(B, t_1 \cup t_2)$ be a clasp tangle, and let R be an incompressible, triply connected disk properly embedded in $B - t_1 \cup t_2$. Suppose that there exist mutually disjoint disks D_1, D_2, D_3 in ∂B satisfying $\partial(D_1 \cup D_2 \cup D_3) = \partial R$, $D_1 \cap t_1 \neq \emptyset$, $D_2 \cap t_1 \neq \emptyset$ and $(D_1 \cup D_2) \cap t_2 = \emptyset$. Then, R is parallel in $B - t_1 \cup t_2$ to a surface in $\partial B - \partial t_1 \cup \partial t_2$.*

Proof. By the assumptions as above, both $D_1 \cap t_1$ and $D_2 \cap t_1$ consist of single points. Since R is incompressible in $B - t_1 \cup t_2$ and $(D_1 \cup D_2) \cap t_2 = \emptyset$, ∂t_2 is contained in D_3. The 2-sphere $R \cup D_1 \cup D_2 \cup D_3$ bounds a 3-ball C in B containing $t_1 \cup t_2$. Consider the closure W of $B - C$ in B. Note that $F = W \cap \partial B$ is a triply connected disk in $\partial B^\circ = \partial B - \partial t_1 \cup \partial t_2$ with $\partial F = \partial R$. Let Δ be an embedded disk in B as illustrated in Fig. 2 such that $\partial \Delta \supset t_1$ and $\Delta \cap t_2$ is two points in $\mathrm{int}\Delta$. It is easily seen that $\Delta^\circ = \Delta - \Delta \cap (t_1 \cup t_2)$ is incompressible in $B - t_1 \cup t_2$. We

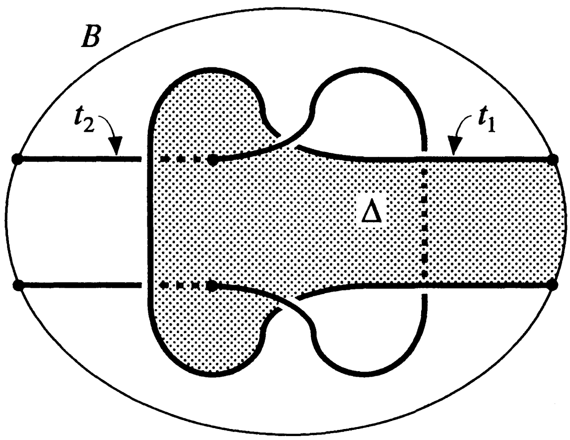

Fig. 2.

may assume that Δ meets R transversely, and each loop component of $\Delta \cap R$ is non-contractible both in Δ° and R. Since the arc $\Delta \cap \partial B$ connects the end points of t_1, the union J of all arc components of $\Delta \cap R$ are non-empty. Let $\Delta_1, \dots, \Delta_n$ $(n \geq 2)$ be the closures in Δ of all components of $\Delta - J$ such that $\Delta_n \supset t_1$ and Δ_1 is *innermost*, that is, $\gamma = \Delta_1 \cap J$ is a single arc. We need to consider the following three cases, though the reader will see that Cases 2 and 3 do not occur really.

CASE 1. $\Delta_1 \cap t_2$ is empty.

In this case, $\text{int}\Delta_1 \cap R$ contains no loop components, so $\text{int}\Delta_1 \cap R = \emptyset$. If $\Delta_1 \subset C$, then for some $j \in \{1,2,3\}$, $D_j \cap \Delta_1$ is an arc separating D_j into two disks D_{j1} and D_{j2} such that $D_{j1} \cap (t_1 \cup t_2) = \emptyset$. The union $\Delta_1 \cup D_{j1}$ is a disk in $C - t_1 \cup t_2$ with $\partial(\Delta_1 \cup D_{j1}) \subset R$. Since R is incompressible in $C - t_1 \cup t_2$, Δ_1 excises a 3-ball C_1 from $C - t_1 \cup t_2$. Deforming Δ in a small neighborhood of C_1 by an ambient isotopy of B rel. $t_1 \cup t_2$, one can reduce the number $|\Delta \cap R|$. Thus, we may assume that $\Delta_1 \cap \text{int}C = \emptyset$. If γ is *inessential* in R, that is, γ excises a disk from R, then one can reduce $|\Delta \cap R|$ as above by invoking the incompressibility of R in W. In the case where γ is essential in R, consider the surface R' obtained by surgery on R along Δ_1. The surface R' consists of at most two annuli which are incompressible in W. Since the boundary of each component A' of R' bounds an annulus in F, A' is parallel to the annulus in W. This implies that R is parallel in $B - t_1 \cup t_2$ to F.

CASE 2. $\Delta_1 \cap t_2$ consists of a single point.

If $\text{int}\Delta_1 \cap R \neq \emptyset$, then there would exist a disk Δ_0 in Δ_1 with $\partial\Delta_0 \subset \text{int}\Delta_1 \cap R$, $\text{int}\Delta_0 \cap R = \emptyset$ and such that $\text{int}\Delta_0 \cap t_2$ is a single point. Since $\Delta_0 \cap D_3 \subset \Delta_0 \cap \partial B = \emptyset$ and $\partial t_2 \subset D_3$, the algebraic intersection number of Δ_0 with t_2 in the 3-ball C would be zero, a contradiction. Thus, $\text{int}\Delta_1 \cap R$ is empty. A similar argument implies that

$\Delta_1 \cap (D_1 \cup D_2) = \emptyset$, and so $\Delta_1 \cap D_3$ is an arc. Since $\Delta_1 \cap t_1 = \emptyset$, γ is inessential in R, and hence Δ_1 excises a 3-ball C_0 from C such that $t_0 = C_0 \cap t_2 = C_0 \cap (t_1 \cup t_2)$ is a proper arc in C_0. Since t_2 is unknotted in B, t_0 is also unknotted in C_0. Thus, there exists a disk E_0 in C_0 bounded by the union of t_0 and an arc u_0 in the disk $\Delta_1 \cup (C_0 \cap D_3)$ with $\partial u_0 = \partial t_0$ and such that $u_0 \cap \Delta_1 \cap D_3$ is a single point. The tangle $(B, t_1 \cup t_2)$ admits an orientation reversing involution h which is the reflection with respect to the horizontal plane containing the barycenter of the 3-ball B in Fig. 2. By using an elementary cut-and-past argument, one can take E_0 so that $E_0 \cap h(E_0) = \emptyset$. Move t_2 in a small neighborhood of $E_0 \cup h(E_0)$ in B by an ambient isotopy of B rel. t_1 so that $t_2 \cap \Delta = \emptyset$. Then, for a regular neighborhood N of Δ in $B - t_2$, $\partial N - \mathrm{int}(N \cap \partial B)$ is a proper disk in B separating t_1 and t_2. This implies that ∂B° is compressible in $B - t_1 \cup t_2$, a contradiction. Thus, Case 2 cannot occur.

CASE 3. $\Delta_1 \cap t_2$ consists of two points.

Since $\Delta_i \cap t_2 = \emptyset$ for $i = 2, \ldots, n$, $\Delta_i \cap J$ consists of two arcs for $i = 2, \ldots, n-1$ and $\Delta_n \cap J$ consists of a single arc. Moreover, Δ_n is a disk in C with $\Delta_n \cap \partial C = \partial \Delta_n - \mathrm{int} t_1$. For the 3-ball C' obtained by cutting C open along Δ_n, $\partial C' - \mathrm{int} D_3$ is a proper disk in B separating t_1 from t_2. This contradiction implies that Case 3 cannot occur. □

Let $\Gamma : G \longrightarrow \mathbf{R}^3 \subset S^3$ be any embedding of a graph G other than a generalized bouquet. As in §2, G can be assumed to contain no isolated vertices and free edges. We denote by $V = \{v_1, \ldots, v_n\}$ the set of all vertices of G. Consider the projection $p : \mathbf{R}^3 \longrightarrow \mathbf{R}^2 (\subset \mathbf{R}^3)$ defined by $p(x, y, z) = (x, y, 0)$. Slightly deforming Γ by an ambient isotopy, we may assume that $p \circ \Gamma$ is a regular projection, that is, (i) the restriction $p \circ \Gamma|_V$ is an embedding, (ii) $p(\Gamma(V)) \cap p(\Gamma(G - V)) = \emptyset$, and (iii) each singular value of $p \circ \Gamma$ is a transversal double point. We regard that the image $\widehat{\Gamma} = p(\Gamma(G))$ is a plane graph, where each double point of $p \circ \Gamma|_G$ is considered to be a vertex of $\widehat{\Gamma}$ of degree four. Let $D_1, \ldots, D_n$ be mutually disjoint disks in $\mathbf{R}^2$ such that $D_i \cap \widehat{\Gamma}$ is a star centered at $\widehat{v}_i = p(\Gamma(v_i))$ for $i = 1, \ldots, n$, and let $\mathcal{D} = D_1 \cup \cdots \cup D_n$. Since G is not a generalized bouquet, for each $v_i \in V$, G contains a cycle l_i disjoint from v_i. We note that l_i may be equal to l_j even if $i \neq j$. Let $\alpha_1, \ldots, \alpha_n$ be mutually disjoint arcs in $\widehat{\Gamma} - \mathcal{D} \cap \widehat{\Gamma}$ disjoint from the set of vertices of $\widehat{\Gamma}$ and with $\alpha_i \subset p(\Gamma(l_i))$. We set $\widetilde{\alpha}_i = (p \circ \Gamma)^{-1}(\alpha_i)$ and $\widetilde{A} = \widetilde{\alpha}_1 \cup \cdots \cup \widetilde{\alpha}_n$. Consider simple arcs $\beta_1, \ldots, \beta_n$ in $\mathbf{R}^2 - \mathrm{int}\mathcal{D}$ meeting each other and $\widehat{\Gamma}$ transversely and such that each β_i connects a point in $\mathrm{int}\alpha_i$ with a point x_i in $\partial D_i - \partial D_i \cap \widehat{\Gamma}$. Let $\Gamma_1 \in \mathcal{S}(G)$ be an embedding ambient isotopic to Γ rel. $G - \widetilde{A}$ such that $p \circ \Gamma_1(\widetilde{\alpha}_i)$ is an arc which tends toward ∂D_i along β_i, and meets β_i at a point y_i near x_i, and then goes round a neighborhood of ∂D_i until meeting y_i again, and finally returns to α_j along β_i as illustrated in Fig. 3. If necessary deforming Γ_1 by an ambient

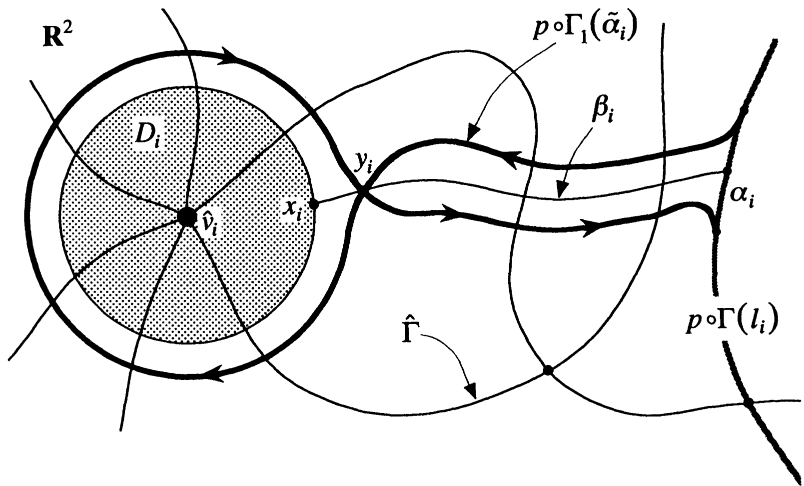

Fig. 3.

isotopy, the plane graph $\widehat{\Gamma}_1 = p \circ \Gamma_1(G)$ can be assumed to satisfy the following (2.1) and (2.2).

(2.1) $\widehat{\Gamma}_1$ is connected.

(2.2) $\widehat{\Gamma}_1$ contains no cut vertices.

Here, a *cut vertex* v of a graph H means a vertex disconnecting the component of H containing v. Let $\{\widehat{w}_1, \ldots, \widehat{w}_m\}$ be the set of the vertices of $\widehat{\Gamma}_1$ corresponding to the double points of $p \circ \Gamma_1$, and let $C_1, \ldots, C_m$ be small regular neighborhoods of $\widehat{w}_1, \ldots, \widehat{w}_m$ in S^3. Note that each $(C_j, C_j \cap \widehat{\Gamma}_1)$ is a standard ball-star pair of degree four centered at $\widehat{w}_j$. Let $\widetilde{\Gamma}_1$ be the regular diagram for Γ_1 obtained by replacing each $C_j \cap \widehat{\Gamma}_1$ by a suitable 2-string trivial tangle in C_j. We set $\mathcal{C} = C_1 \cup \cdots \cup C_m$. Let $\Gamma_2 : G \longrightarrow S^3$ be an embedding such that $\Gamma_2(G) - \mathrm{int}\mathcal{C} = \widehat{\Gamma}_1 - \mathrm{int}\mathcal{C}$, and for each $j = 1, \ldots, m$, $(C_j, C_j \cap \Gamma_2(G))$ is obtained by exchanging each trivial tangle $(C_j, C_j \cap \widetilde{\Gamma}_1)$ by a clasp tangle so that Γ_2 is cobordant to Γ_1 and hence to Γ.

Now, we will prove the following lemma which is crucial in the proof of Theorem 2.

Lemma 3. *With the notation as above, any ball-star pair* $(B, B \cap \Gamma_2(G))$ *in* $(S^3, \Gamma_2(G))$ *is standard. In particular,* Γ_2 *is isotopically reduced.*

Proof. The argument quite similar to that in Assertion 1 of [3, Theorem 3] implies that $\Gamma_2(G)$ is "prime", that is, any 2-sphere in S^3 meeting $\Gamma_2(G)$ transversely in two points bounds a 3-ball B_0 in S^3 such that $B_0 \cap \Gamma_2(G)$ is an unknotted arc in B_0. In particular, any ball-arc pair in $(S^3, \Gamma_2(G))$ is standard. Thus, we may assume that B contains a vertex of $\Gamma_2(G)$, say $\widehat{v}_1$. As in §2, for a proper subset X

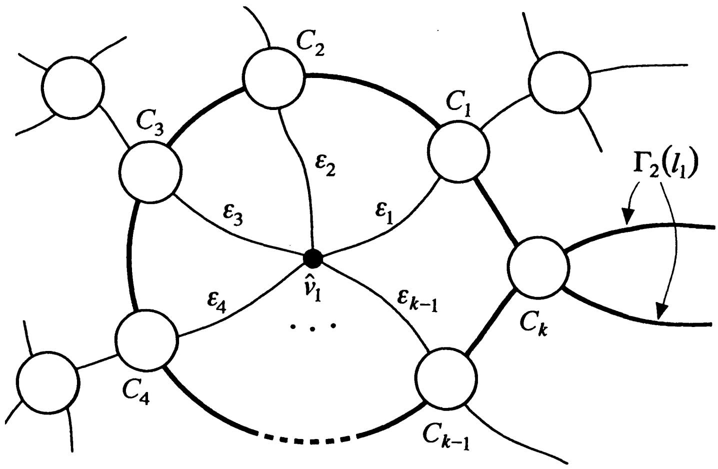

Fig. 4.

of S^3, we set $X^\circ = X - X \cap \Gamma_2(G)$. By (2.1), $S^3 - \mathcal{C} \cup \Gamma_2(G) = S^3 - \mathcal{C} \cup \widehat{\Gamma}_1$ is irreducible. Since a clasp tangle is simple, ∂C_j° is incompressible in C_j°. By (2.2), each ∂C_j° is also incompressible in $S^3 - \mathrm{int}\mathcal{C} \cup \Gamma_2(G)$. This shows that ∂C_j° is incompressible in $S^3 - \Gamma_2(G)$ and $S^3 - \Gamma_2(G)$ is irreducible. Set $B = \widehat{B}$ if ∂B° is incompressible in $S^3 - \Gamma_2(G)$. If ∂B° is compressible in $S^3 - \Gamma_2(G)$, then we consider mutually disjoint, compressing disks $\Delta_1, \ldots, \Delta_r$ for ∂B° in $S^3 - \mathrm{int}B \cup \Gamma_2(G)$ and 3-balls $B_1, \ldots, B_r$ in S^3 with $\partial B_i \subset \Delta_i \cup \partial B$ and $B_i \cap \Gamma_2(l_1) = \emptyset$. Then, the union $\widehat{B} = B \cup B_1 \cup \cdots \cup B_r$ is a 3-ball disjoint from $\Gamma_2(l_1)$ and such that $\partial \widehat{B}^\circ$ is incompressible in $S^3 - \Gamma_2(G)$. Note that $\partial \widehat{B} \cap \Gamma_2(G) \subset \partial B \cap \Gamma_2(G)$, and $\partial \widehat{B} \cap \Gamma_2(G) = \partial B \cap \Gamma_2(G)$ if and only if ∂B° is incompressible in $S^3 - \Gamma_2(G)$. Since $S^3 - \Gamma_2(G)$ is irreducible, $\partial \widehat{B} \cap \Gamma_2(G)$ is non-empty. If necessary deforming $\partial \widehat{B}$ by an ambient isotopy of $(S^3, \Gamma_2(G))$, one can assume that $\partial \widehat{B}$ meets $\partial \mathcal{C}$ transversely and each component of $\partial \widehat{B} \cap \partial \mathcal{C}$ is non-contractible both in $\partial \widehat{B}^\circ$ and $\partial \mathcal{C}^\circ$. Renumber $\widehat{w}_j$'s so that the subset $\{\widehat{w}_1, \ldots, \widehat{w}_k\}$ of $\{\widehat{w}_1, \ldots, \widehat{w}_m\}$ consists of the double points of $\widehat{\Gamma}_1$ surrounding $\widehat{v}_1$, and $\widehat{w}_k$ corresponds to the double point y_1 of $p \circ \Gamma_1(\widetilde{\alpha}_1)$. Let ε_j $(j = 1, \ldots, k-1)$ be the edge of $\Gamma_2(G)$ meeting both $\widehat{v}_1$ and C_j, see Fig. 4. Since C_j meets $\Gamma_2(l_1)$ non-trivially for any $j = 1, \ldots, k$, C_j is not contained in $\widehat{B}$. If there existed a disk Δ in ∂C_j with $\partial \Delta \subset \partial \widehat{B} \cap \partial C_j$, $\mathrm{int}\Delta \cap \partial \widehat{B} = \emptyset$ and such that $\Delta \cap \Gamma_2(l_1)$ is a single point, then Δ would be a non-separating proper disk in the 3-ball $S^3 - \mathrm{int}\widehat{B}$, a contradiction. Thus, in the case of $\partial \widehat{B} \cap \partial C_j \neq \emptyset$, the closure F in ∂C_j of any connected component of $\partial C_j - \partial \widehat{B} \cap \partial C_j$ is either a disk with $1 \leq \#(F \cap \Gamma_2(G)) \leq 3$, or an annulus with $0 \leq \#(F \cap \Gamma_2(G)) \leq 2$, or a triply connected disk with $F \cap \Gamma_2(G) = \emptyset$, where $\#(X)$ denotes the number of elements of a finite set X. If F is either a disk with $\#(F \cap \Gamma_2(G)) = 3$ or an annulus with $\#(F \cap \Gamma_2(G)) = 2$, then $F \cap \Gamma_2(l_2) \neq \emptyset$, and hence F is not contained in $\widehat{B}$.

We set $\mathcal{C}(k) = C_1 \cup \cdots \cup C_k \subset \mathcal{C}$. If $\partial\mathcal{C}(k) \cap \widehat{B}$ contains a disk component F with $\#(F \cap \Gamma_2(G)) = 1$, then ∂F bounds a disk F' in $\partial\widehat{B}$ with $\#(F' \cap \Gamma_2(G)) = 1$. Since $\Gamma_2(G)$ is prime, the 2-sphere $F \cup F'$ bounds a 3-ball B' in $\widehat{B}$ such that $B' \cap \Gamma_2(G)$ is an unknotted arc in B'. This enables us to reduce the number $|\partial\mathcal{C}(k) \cap \partial\widehat{B}|$ by deforming $\partial\mathcal{C}(k)$ in a small neighborhood of B'. Similarly, if $\partial\mathcal{C}(k) \cap \widehat{B}$ contains an annulus component F with $F \cap \Gamma_2(G) = \emptyset$, then one can reduce the number $|\partial\mathcal{C}(k) \cap \partial\widehat{B}|$, for example see Assertion 2 in the proof of [3, Theorem 3]. Thus, we may assume that, for each C_j $(j = 1, \ldots, k)$ with $\partial\widehat{B} \cap \partial C_j \neq \emptyset$, $F_j = \partial C_j \cap \widehat{B}$ is a connected surface which is either a disk with $\#(F_j \cap \Gamma_2(G)) = 2$, or an annulus with $\#(F_j \cap \Gamma_2(G)) = 1$, or a triply connected disk with $F_j \cap \Gamma_2(G) = \emptyset$. One can reduce the former two cases to the latter case, by pushing a small neighborhood of $F_j \cap \Gamma_2(G)$ toward the outside of $\widehat{B}$ along the edges of $\Gamma_2(G)$ meeting F_j. So, it suffices to consider the case where F_j is a triply connected disk disjoint from $\Gamma_2(G)$. Let W_j be the closure in $\widehat{B}$ of a component of $\widehat{B} - F_j$ disjoint from $\widehat{v}_1$. It is easy to see that $R_j = \partial W_j \cap \partial\widehat{B}$ is also a triply connected disk. We assume that W_1 is innermost among all W_j's, that is, $\mathrm{int}W_1 \cap \partial\mathcal{C}(k) = \emptyset$. If W_1 were not contained in C_1, then C_1 would contain $\widehat{v}_1$, a contradiction. It follows that W_1 is contained in C_1, and hence Lemma 2 shows that R_1 is parallel to F_1 in C_1°. This implies that one can reduce the number $|\partial\widehat{B} \cap \partial\mathcal{C}(k)|$, and finally get the situation of $\widehat{B} \cap \mathcal{C}(k) = \emptyset$.

Since $\partial B \cap \Gamma_2(G) \supset \partial\widehat{B} \cap \Gamma_2(G) \neq \emptyset$, at least one of $\varepsilon_1, \ldots, \varepsilon_{k-1}$, say ε_1, meets $\partial\widehat{B}$ non-trivially. Let α be the subarc of ε_1 connecting $\widehat{v}_1$ with $\varepsilon_1 \cap \partial\widehat{B}$. If $\alpha \cap C_1 \neq \emptyset$,

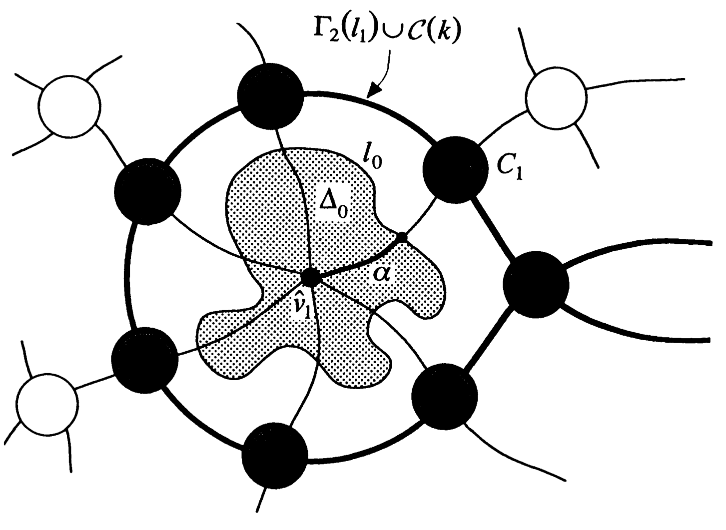

Fig. 5.

then C_1 would meet $\widehat{B} \supset \alpha$ non-trivially. This contradiction implies $\alpha \cap C_1 = \emptyset$. Since $\partial\widehat{B} \cap (\Gamma_2(l_1) \cup \mathcal{C}(k)) = \emptyset$ and since $\partial\widehat{B}$ meets any ε_j $(j = 1, \dots, k-1)$ at most one point, the component l_0 of $\partial\widehat{B} \cap (\mathbf{R}^2 - \mathrm{int}\,\mathcal{C} \cap \mathbf{R}^2)$ containing $\varepsilon_1 \cap \partial\widehat{B}$ is a loop in $\mathbf{R}^2 - \mathcal{C} \cap \mathbf{R}^2$ bounding a disk Δ_0 such that $\Delta_0 \cap \Gamma_2(G)$ is a star of degree $k-1$ centered at $\widehat{v}_1$, see Fig. 5. Since $\#(\partial\widehat{B} \cap \Gamma_2(G))$ is equal to the degree of $\widehat{v}_1$ in $\Gamma_2(G)$, we have $B = \widehat{B}$ or equivalently that ∂B° is incompressible in $S^3 - \Gamma_2(G)$. Since each component of $\partial B - l_0$ is an open disk disjoint from $\Gamma_2(G)$, one can deform ∂B by an ambient isotopy of $(S^3, \Gamma_2(G))$ rel. l_0 so that $\partial B \cap (\mathbf{R}^2 \cup \mathcal{C}) = l_0$. In particular, $(B, B \cap \Gamma_2(G))$ is a standard ball-star pair. This shows that Γ_2 is isotopically reduced. □

Proof of Theorem 2. For any positive integer m, choose the regular projection $\widehat{\Gamma}_1 = p(\Gamma_1(G))$ as above so that $\widehat{\Gamma}_1$ has at least m double points. Then, for the isotopically reduced embedding $\Gamma_2 \in [\Gamma]_{\mathrm{cobor}}$ given in Lemma 3, the complement $S^3 - \Gamma_2(G)$ contains at least m mutually disjoint and non-parallel, incompressible, four-punctured 2-spheres. On the other hand, by Haken's Finiteness Theorem [1], there exists a positive integer $n(\Gamma_2)$ depending only on the ambient isotopy type of Γ_2 so that the number of such four-punctured 2-spheres in $S^3 - \Gamma_2(G)$ is not greater than $n(\Gamma_2)$. This observation implies that one can construct infinitely many isotopically reduced elements of $[\Gamma]_{\mathrm{cobor}}$ which are not ambient isotopic to each other. □

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References

[1] W. Haken: *Some results on surfaces in 3-manifolds*, in Studies in Modern Topology, (P.J. Hilton ed.), Math. Assoc. Amer. Studies in Math. **5** Prentice-Hall, (1968), 39–98.

[2] H. Inaba and T. Soma: *On spatial graphs isotopic to planar embeddings*, in Proceedings of Knots 96 (S. Suzuki ed.), World Scientific Publ. Co, 1997, 1–22.

[3] T. Soma: *Simple links and tangles*, Tokyo J. Math. **6** (1983), 65–73.

[4] T. Soma: *Hyperbolic, fibred links and fibre-concordances*, Math. Proc. Camb. Phil. Soc. **96** (1984), 283–294.

[5] T. Soma: *Spatial-graph isotopy for trivalent graphs and minimally knotted embeddings*, Topology Appl. **73** (1996), 23–41.

[6] K. Taniyama: *Cobordism, homotopy and homology of graphs in* R^3, Topology **33** (1994), 509–523.

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